GRESHAM COLLEGE

Founded 1597

Shaping Modern Mathematics: From One to Many Geometries

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Gresham Professor of Geometry

OVERVIEW

- What is Proof?
- Euclid’s *Elements*
- The Parallel Postulate – a Blot on Euclid?
- Attempts to prove the Parallel Postulate
- Non-Euclidean geometry: János Bolyai and Nikolai Lobachevsky
- The Poincaré disc
- Infinitely many different geometries

The Greek Empire

Euclid's *Elements*

BOOK I	Triangles, parallels, and area
BOOK II	Geometric algebra
BOOK III	Circles
BOOK IV	Constructions for inscribed and circumscribed figures
BOOK V	Theory of proportions
BOOK VI	Similar figures and proportions
BOOK VII	Fundamentals of number theory
BOOK VIII	Continued proportions in number theory
BOOK IX	Number theory
BOOK X	Classification of incommensurables
BOOK XI	Solid geometry
BOOK XII	Measurement of figures
BOOK XIII	Regular solids

Books on Plane Geometry

BOOK I	**Triangles, parallels, and area**
BOOK II	**Geometric algebra**
BOOK III	**Circles**
BOOK IV	**Constructions for inscribed and circumscribed figures**
BOOK V	**Theory of proportions**
BOOK VI	**Similar figures and proportions**
BOOK VII	Fundamentals of number theory
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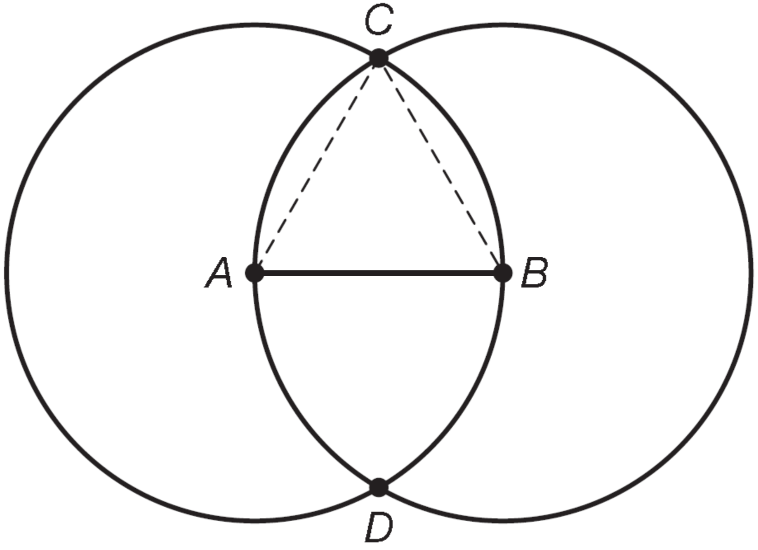
Book 1, Proposition 1

Given a straight line AB, construct an equilateral triangle with AB as its base.

Two Postulates. We can

- draw a straight line from any given point to any other,
- draw a circle with any given centre and radius.



Draw two circles, one with centre A and radius AB, and the other with centre B and radius AB

These circles meet at two points C and D, and the triangle ABC (or ABD) is then the required equilateral triangle.

At each stage of his proof he makes reference to an appropriate definition or postulate.

Books on Arithmetic

BOOK I	Triangles, parallels, and area
BOOK II	Geometric algebra
BOOK III	Circles
BOOK IV	Constructions for inscribed and circumscribed figures
BOOK V	Theory of proportions
BOOK VI	Similar figures and proportions
BOOK VII	**Fundamentals of number theory**
BOOK VIII	**Continued proportions in number theory**
BOOK IX	**Number theory**
BOOK X	**Classification of incommensurables**
BOOK XI	Solid geometry
BOOK XII	Measurement of figures
BOOK XIII	Regular solids

Arithmetic

- odd and even numbers
- what it means for one number to be a factor of another
- Euclidean algorithm, a systematic method for finding the highest common factor of two numbers
- prime numbers

Books on Solid Geometry

BOOK I	Triangles, parallels, and area
BOOK II	Geometric algebra
BOOK III	Circles
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BOOK VII	Fundamentals of number theory
BOOK VIII	Continued proportions in number theory
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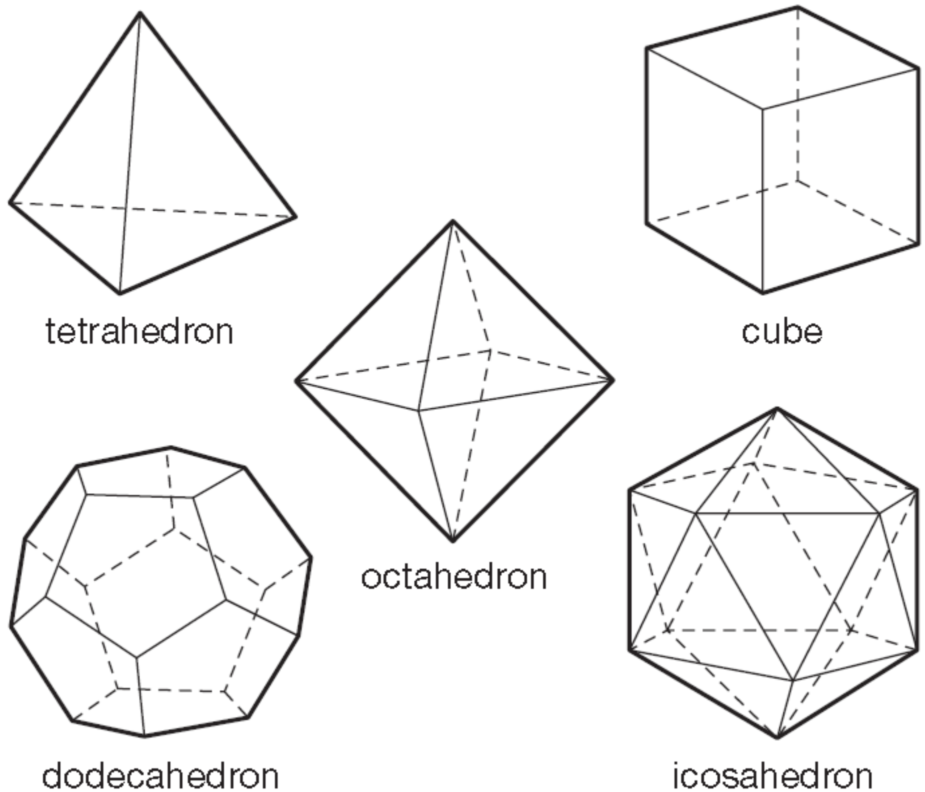
Regular or Platonic solids

A regular solid is one whose faces all have the same shape and size, and whose vertices all look the same.



Tetrahedron whose faces are four equilateral triangles
Cube, or hexahedron, whose six faces are squares
Octahedron whose eight faces are equilateral triangles
Dodecahedron whose twelve faces are regular pentagons
Icosahedron whose twenty faces are equilateral triangles

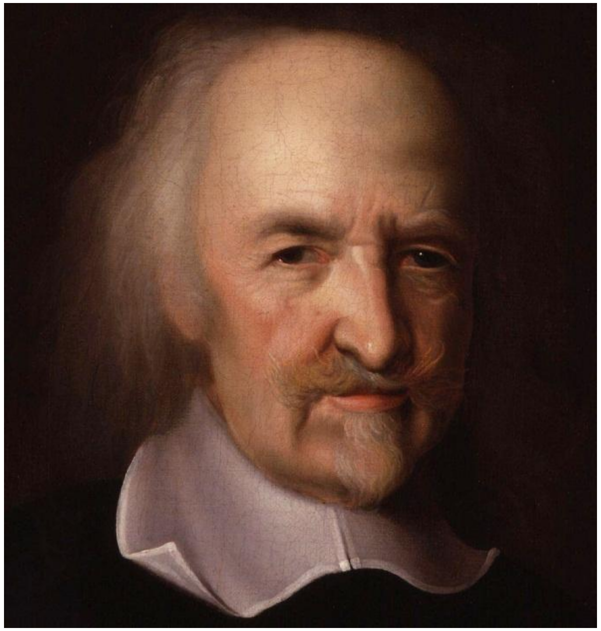
Proclus (5th century AD) on Euclid's *Elements*

- It is a difficult task in any science to select and arrange properly the elements out of which all other matters are produced and into which they can be resolved …

Proclus (5th century AD) on Euclid's *Elements*

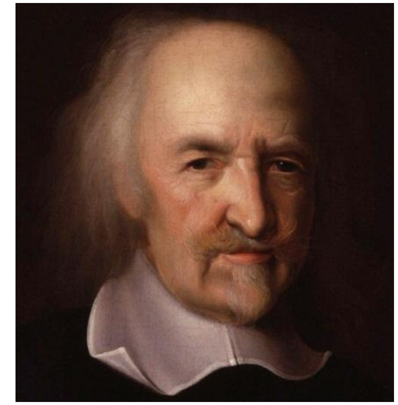
- It is a difficult task in any science to select and arrange properly the elements out of which all other matters are produced and into which they can be resolved …
- Judged by these criteria, you will find Euclid's introduction superior to others.

Thomas Hobbes
1588 - 1679



He was 40 years old before he looked on Geometry; which happened accidentally.

Being in a Gentleman’s Library, Euclid’s *Elements* lay open, and ’twas the *47 El. libri I* [the ‘theorem of Pythagoras’].



He read the Proposition.

By G—, sayd he (he would now and then sweare an emphaticall Oath by way of emphasis) *this is impossible!*

So he reads the Demonstration of it, which referred him back to such a Proposition; which proposition he read.

That referred him back to another, which he also read.

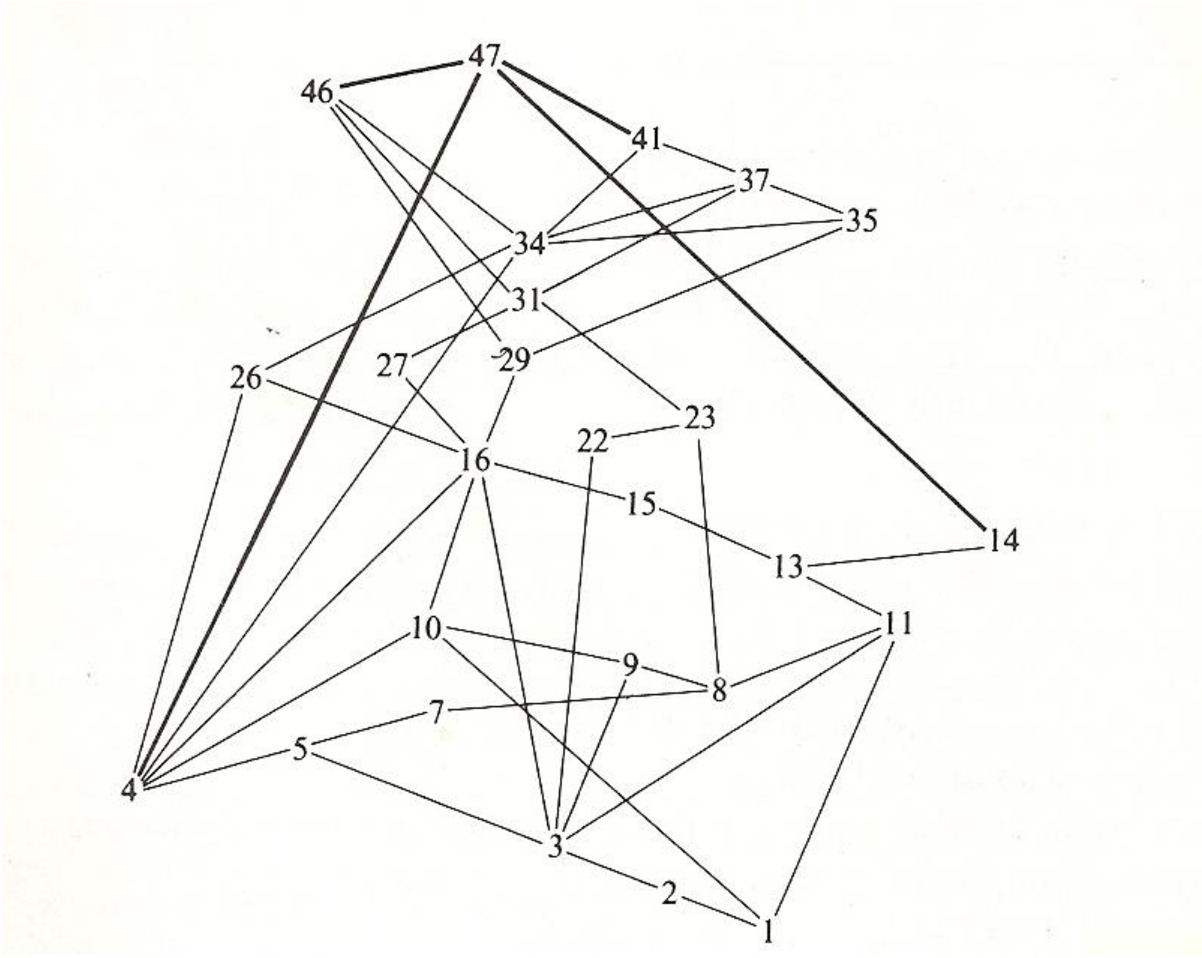
Et sic deinceps ('going thus one after another') that at last he was demonstratively convinced of that trueth.

This made him in love with Geometry.

At the age of eleven, I began Euclid with my brother as my tutor. This was one of the great events of my life, as dazzling as first love.
I had not imagined there was anything so delicious in the world. After I had learned the fifth proposition, my brother told me it was generally considered difficult, but I had found no difficulty whatever.
This was the first time that it dawned on me that I might have some intelligence.
From that moment until … I was thirty-eight, mathematics was my chief interest and my chief source of happiness.

Bertrand Russell
1872 - 1970

Propositions of *Elements*, Book I which enter into the proof of Proposition 47, The theorem of Pythagoras. An upwards directed line indicates that the proof of the higher numbered proposition directly invokes the lower one.



COMMON NOTIONS

1. Things which are equal to the same thing are also equal to one another.
2. If equals be added to equals, the wholes are equal.
3. If equals be subtracted from equals, the remainders are equal.
4. Things which coincide with one another are equal to one another.
5. The whole is greater than the part.

POSTULATES

1. To draw a straight line from any point to any point.
2. To produce a finite straight line continuously in a straight line.
3. To describe a circle with any centre and distance.
4. That all right angles are equal to one another.

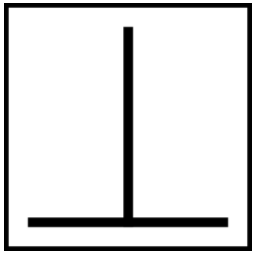
5. That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

Definitions 1 to 6

1. A **point** is that which has no part
2. A **line** is breadthless length.
3. The **extremities of a line** are points.
4. A **straight line** is a line which lies evenly with the points on itself.
5. A **surface** is that which has length and breadth only.
6. The **extremities of a surface** are lines

Definitions 10 and 23

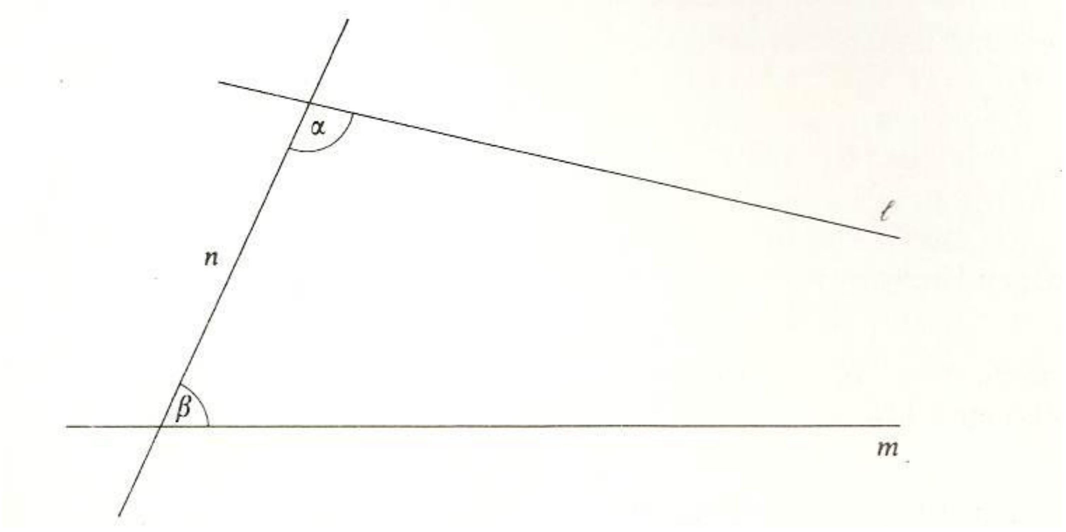
10. When a straight line set up on a straight line makes the adjacent angles equal to one another, each of the equal angles is **right**, and the straight line standing on the other is called a **perpendicular** to that on which it stands.



23. **Parallel** straight lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction.

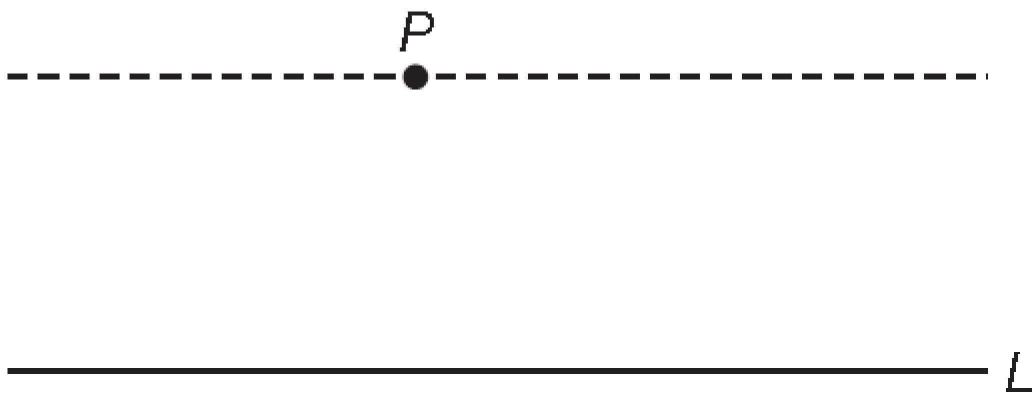
Postulate 5 – A Blot on Euclid?

That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.



The Parallel Postulate

The parallel to any line through a given point (not on the given line) is unique.



Using the Parallel Postulate

- The angles in any triangle add up two right angles i.e. 180 degrees,

  (Book I, Proposition 32)

- Parallel lines are always equidistant

  (Book I, Proposition 33)

- Pythagorean Theorem

  (Book I, Proposition 47)

Other Equivalent Assumptions

- The equidistant curve to a straight line is itself straight

(Ibn Al-Haytham, 965 – c.1040)

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Other Equivalent Assumptions

- The equidistant curve to a straight line is itself straight

(Ibn Al-Haytham, 965 – c.1040)

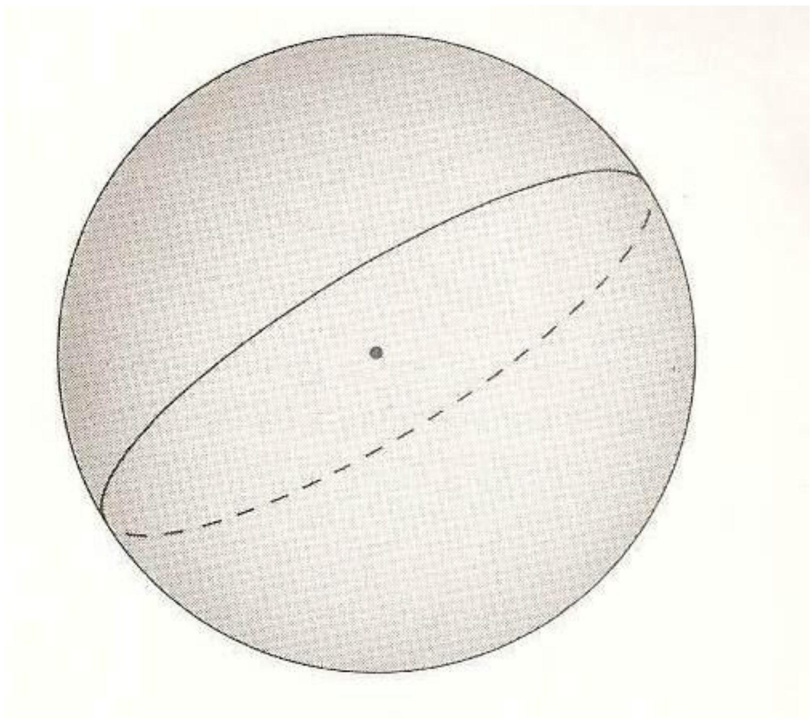
- The angle sum of a triangle is 180 degrees

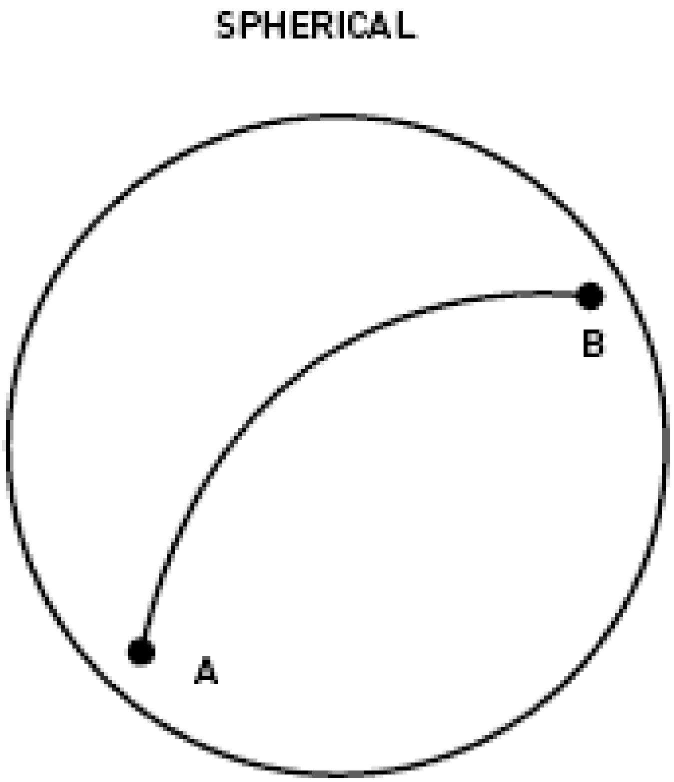
(Nasir Eddin, 1201 – 74)

- There exist triangles of different sizes with the same angles

(John Wallis, 1616 – 1703)

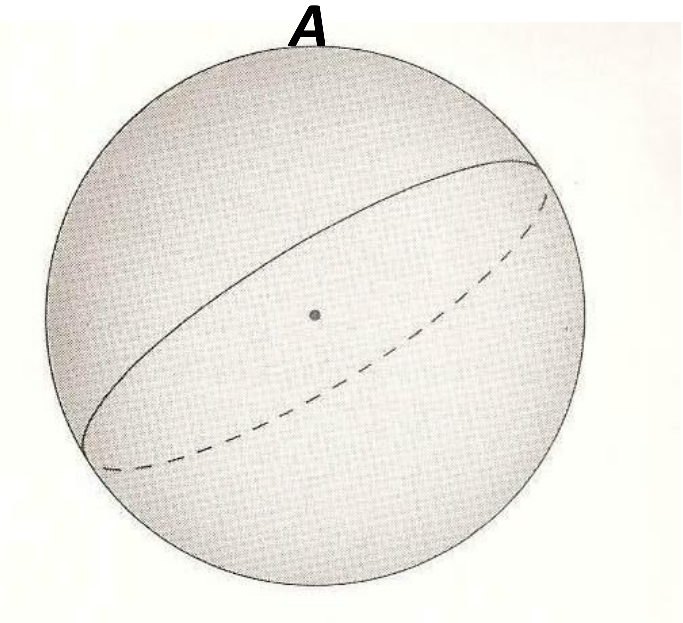
Spherical Geometry





Spherical geometry does not satisfy the first Postulate of Euclid which says that there is one and only one line connecting any two points.

Spherical Geometry

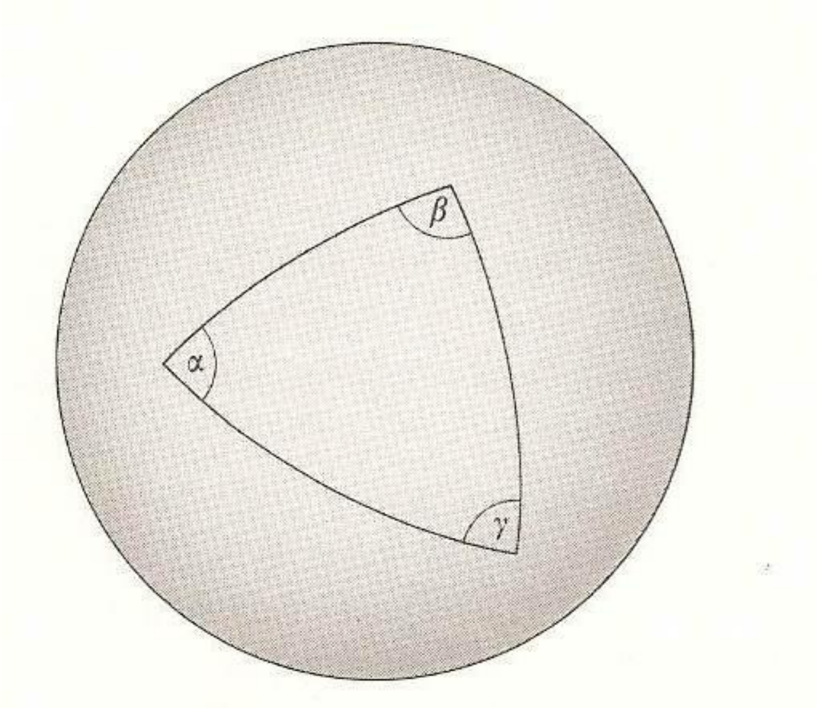


Spherical geometry does not satisfy the Parallel Postulate, for if you take a great circle and a point, ***A***, not on it then every great circle through the point, ***A***, meets the first great circle twice.

There are **no** parallel lines!

Spherical Triangle

$\alpha + \beta + \gamma > \pi$



In Fact:

$$\alpha + \beta + \gamma = \pi + \frac{1}{R^2} \text{Area (Triangle)}$$

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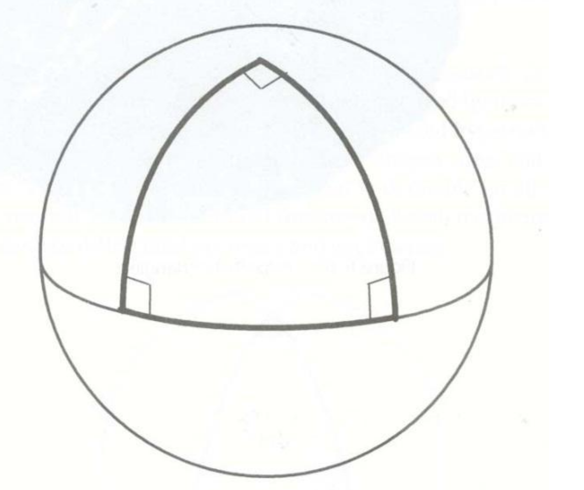
Spherical Triangle with three right angles



$$\alpha + \beta + \gamma = \pi + \frac{1}{R^2}\,\text{Area (Triangle)}$$

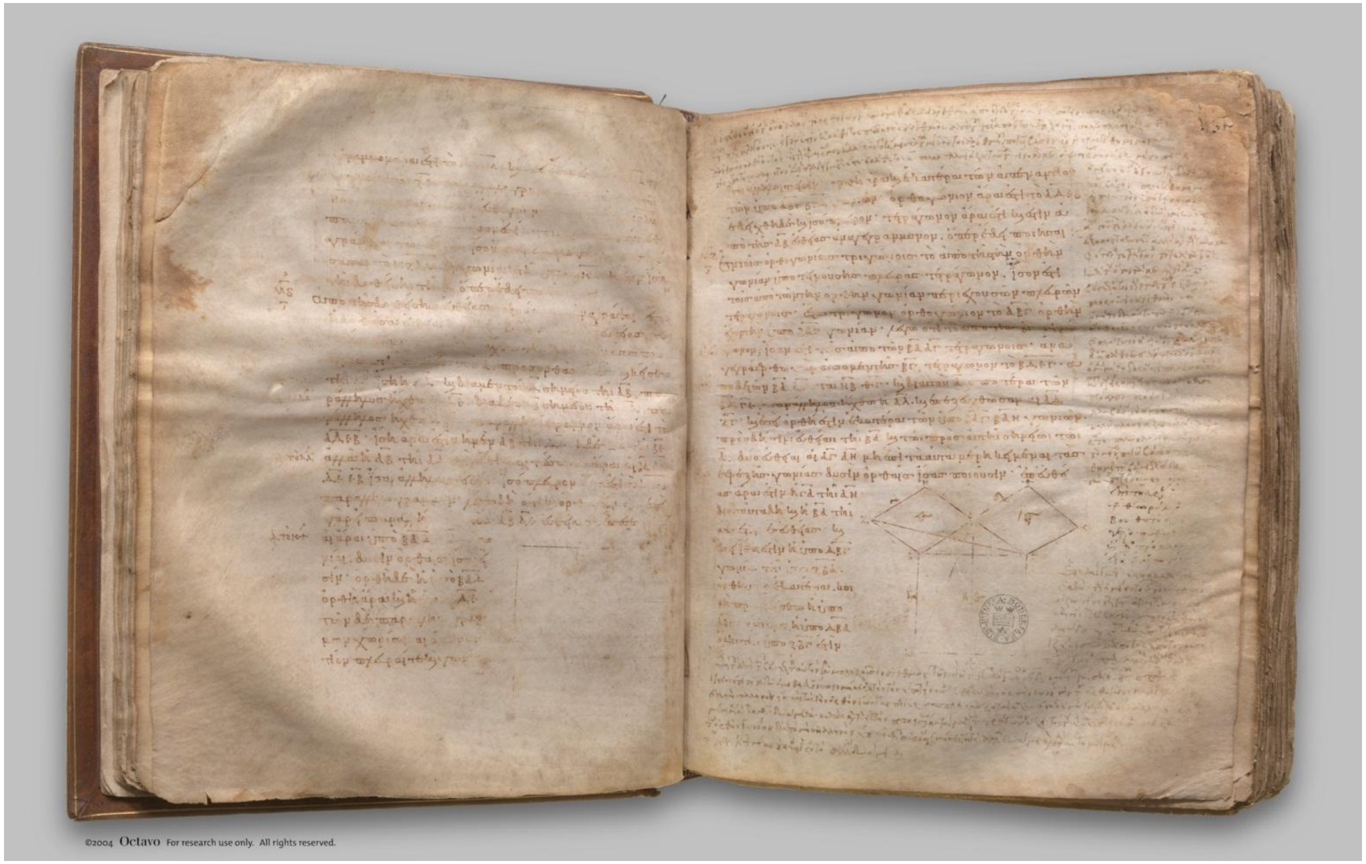
Left-hand side = $\alpha + \beta + \gamma = 3\pi/2$

Because the triangle is one eighth of the surface area of the sphere its area is one eighth of $4\pi R^2 = \frac{1}{2}\pi R^2$.

Right-hand side = $\pi + \frac{1}{R^2}\frac{1}{2}\pi R^2 = \pi + \frac{1}{2}\pi = 3\pi/2$

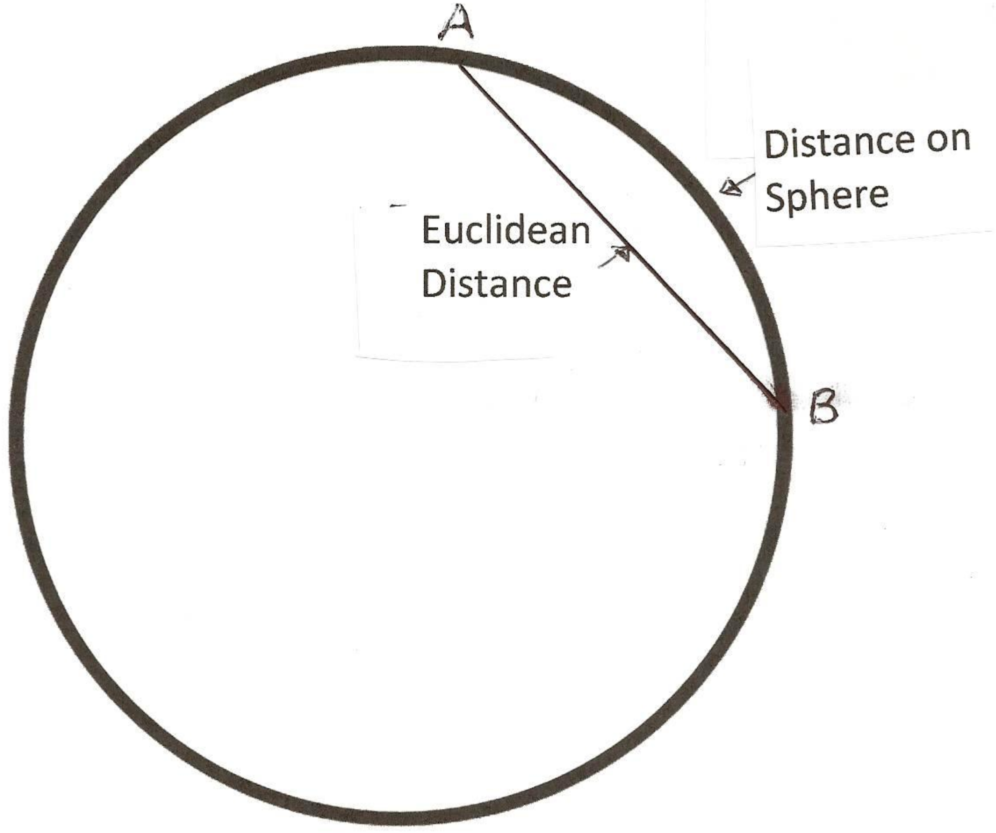
Pythagorean Theorem

In right-angled triangles the square on the side subtending the right angle is equal to the squares on the sides containing the right angle



From MS D'Orville 301, copied by Stephen the Clerk for Arethas of Patras, in Constantinople in 888 AD. The manuscript is now in the Bodleian Library, Oxford University. (Image courtesy of the Clay Mathematics Institute)

Euclidean versus Spherical distance between points A and B on a sphere



Gerolamo Saccheri, 1667 - 1733

Euclid freed of all Blemish

EUCLIDES
AB OMNI NÆVO VINDICATUS:
SIVE
CONATUS GEOMETRICUS
QUO STABILIUNTUR
Prima ipsa universæ Geometriæ Principia.
AUCTORE
HIERONYMO SACCHERIO
SOCIETATIS JESU
In Ticinensi Universitate Matheseos Professore.
OPUSCULUM
EX.$^{\text{MO}}$ SENATUI
MEDIOLANENSI
Ab Auctore Dicatum.

MEDIOLANI, MDCCXXXIII.

Ex Typographia Pauli Antonii Montani. *Superiorum permissi*

Saccheri considered the system of the *Elements* but with the Parallel Postulate removed and one of the following assumptions put in its place

1. **Hypothesis of the Obtuse Angle**

The angle sum of a quadrilateral is greater than 360°

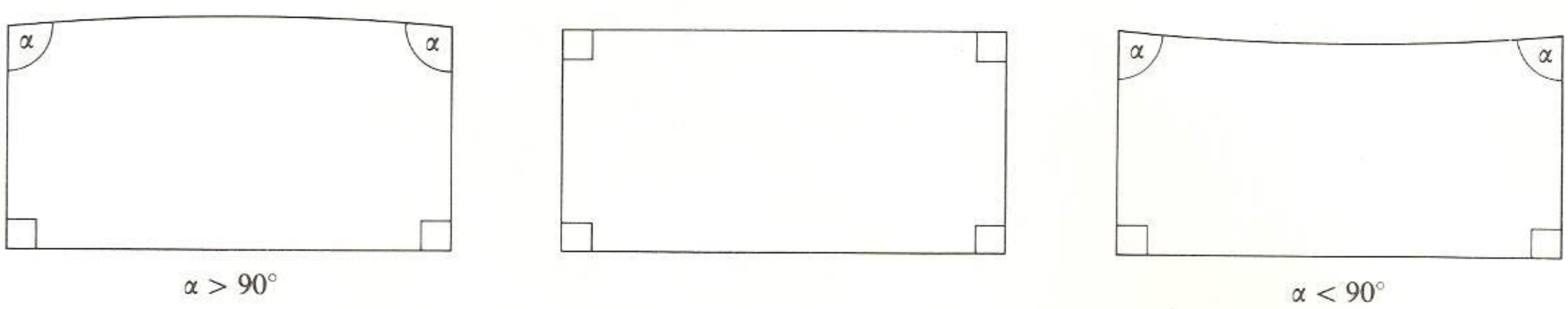
2. **Hypothesis of the Right Angle**

The angle sum of a quadrilateral is exactly 360°

This is equivalent to the Parallel Postulate

3. **Hypothesis of the Acute Angle**

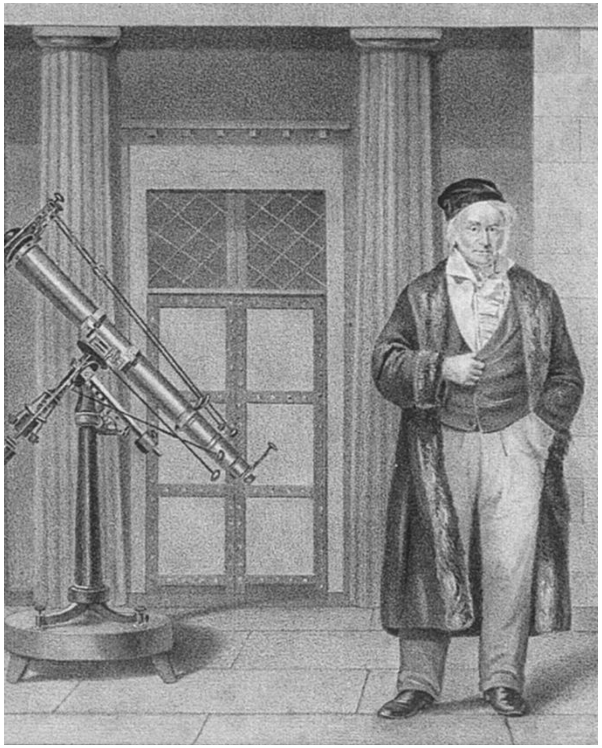
The angle sum of a quadrilateral is less than 360°



There are ‘non-Euclidean’ geometries that satisfy the first four postulates, but not the fifth. Their existence was first published around 1830 by the Transylvanian János Bolyai (1802–1860) and the Russian Nikolai Lobachevsky (1792–1856).

Nikolai Lobachevsky

The tomb of János Bolyai and his father, Farkos, at Marosvásárhely



Carl Friedrich Gauss (1777–1855) at his astronomical observatory in Göttingen

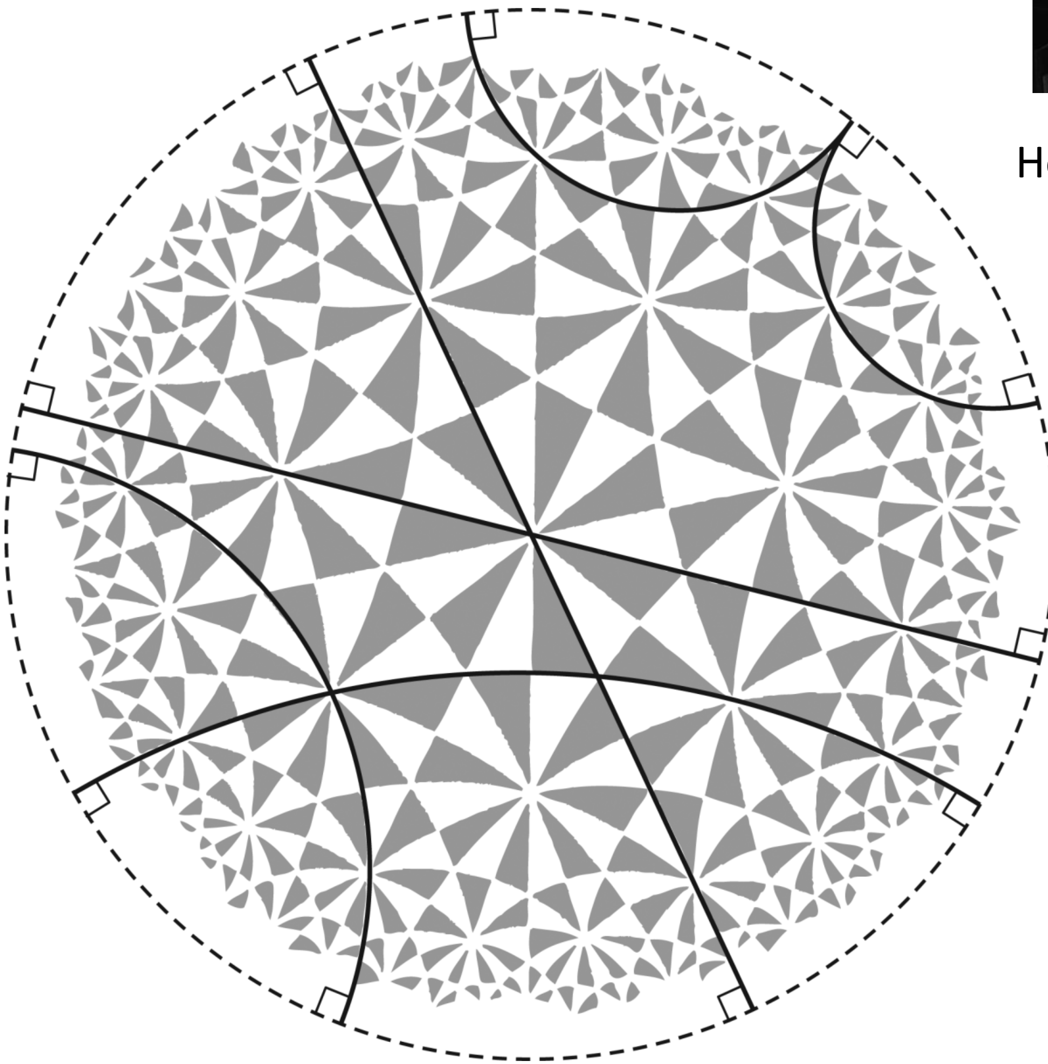
I am becoming more and more convinced that the necessity of our [Euclidean] geometry cannot be proved … Perhaps in another life we will be able to obtain insight into the nature of space, which is now unattainable.

Father's advice to his son

You must not attempt this approach to parallels. I know its way to its very end. I have traversed this bottomless night, which extinguished all light and joy of my life …

I have travelled past all reefs of this infernal Dead Sea and have always come back with broken mast and torn sail.

THE POINCARÉ DISC



Henri Poincaré (1854–1912)

Non – Euclidean Geometry

Given any line L and any point P that does not lie on this line, there are infinitely many lines, parallel to L, that pass through P

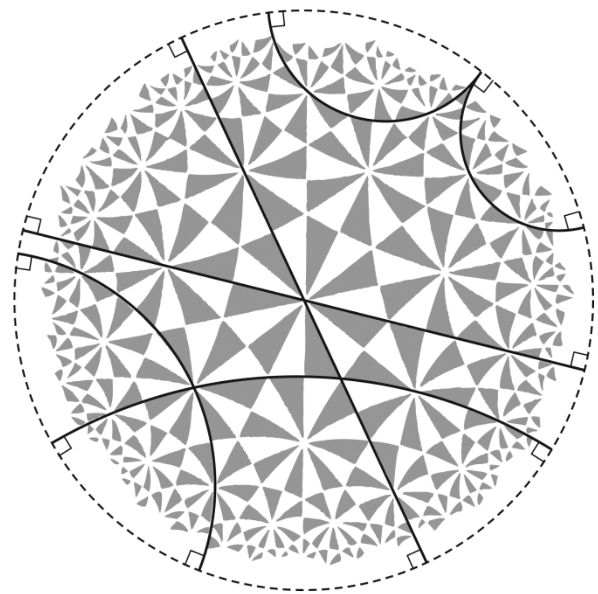
Moreover, if two triangles are similar (they have the same angles), then they must also be congruent (they have the same size)

Non – Euclidean Geometry

$\alpha + \beta + \gamma < \pi$

In Fact $\alpha + \beta + \gamma = \pi - \frac{1}{R^2}$ Area (Triangle)

$$ds^2 = \frac{dx^2 + dy^2}{(1 - (x^2 + y^2))^2}$$



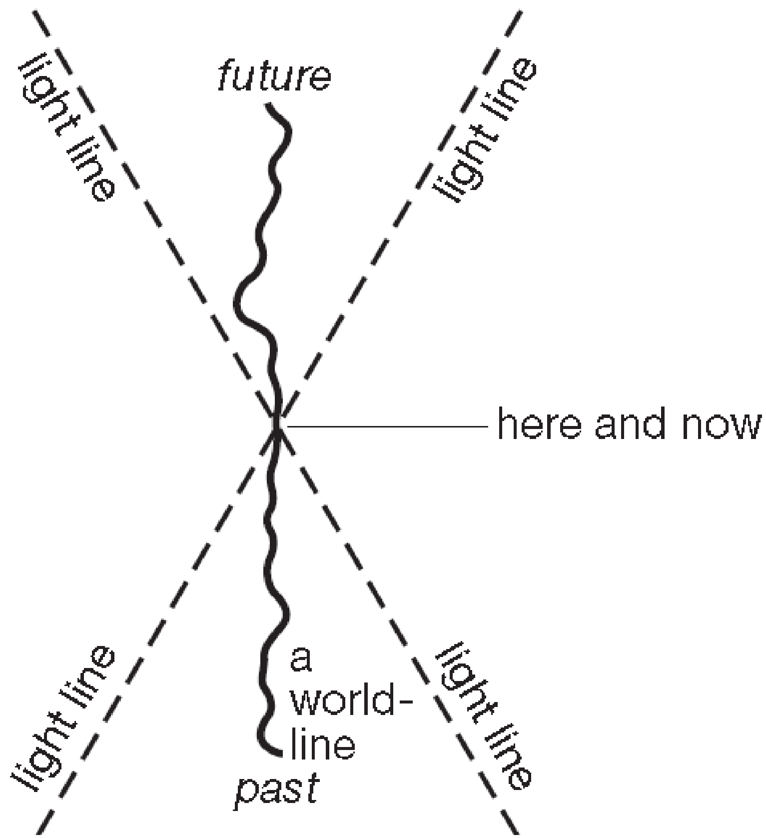
Bernhard Riemann
(1826–1866)

Hermann Minkowski
(1864–1909)

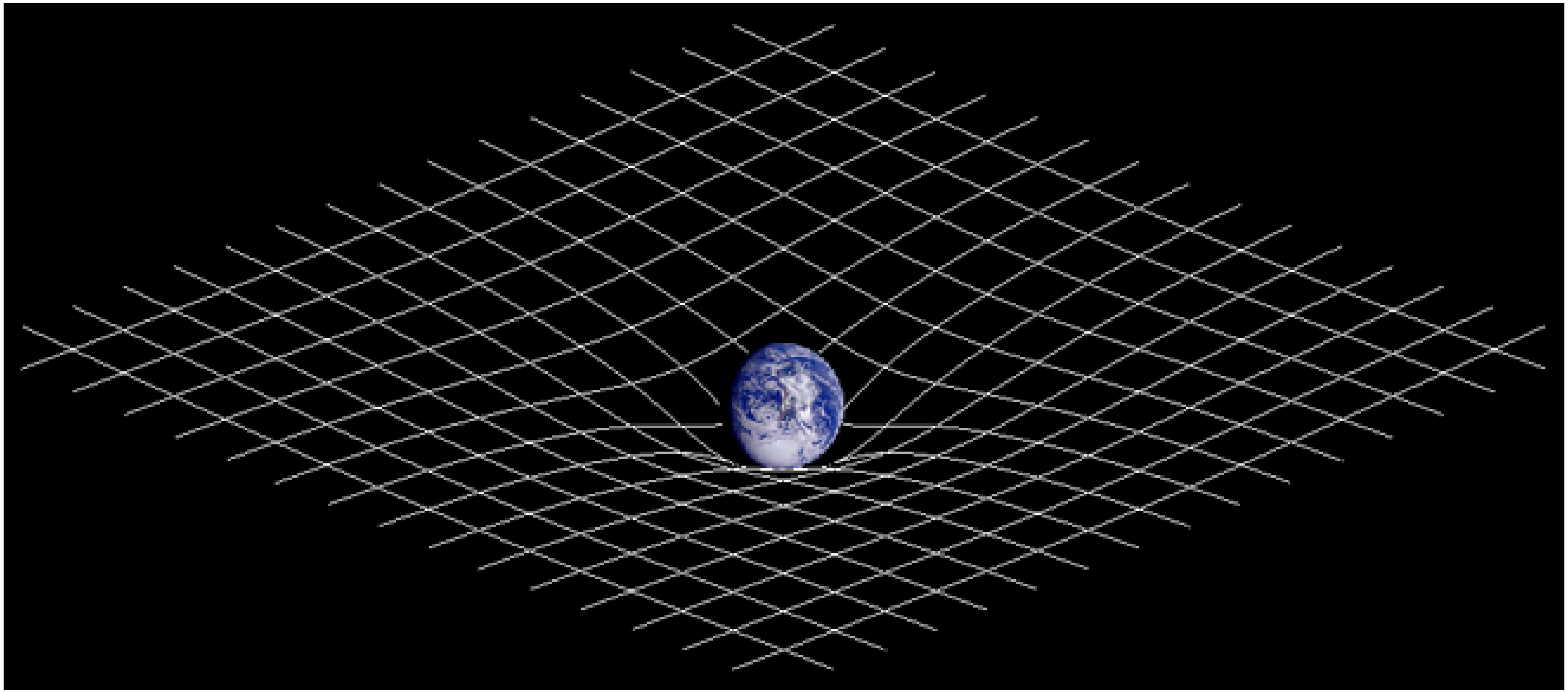
Space-time

- Henceforth space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two preserves an independent reality.
- In Euclidean geometry the distance of each point (x, t) to the origin is $\sqrt{(x^2 + t^2)}$
- but the requirements of relativity replace this in space-time with the distance $\sqrt{(x^2 - c^2t^2)}$

Special Relativity



General Relativity
Space-time Curvature



‘Let no one ignorant of geometry enter these doors’

Lectures
At the Museum of London

- **Ghosts of Departed Quantities: Calculus and its Limits**

  Tuesday 25 September 2012
- **Polynomials and their Roots**

  Tuesday 6 November 2012
- **From One to Many Geometries**

  Tuesday 11 December 2012
- **The Queen of Mathematics**

  Tuesday 22 January 2013
- **Are Averages Typical?**

  Tuesday 19 February 2013
- **Modelling the World**

  Tuesday 19 March 2013