The Art and Science of Tuning

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“There is geometry in the humming of the strings, there is music in the spacing of the spheres.”

(attributed to both Pythagoras and Aristotle, source unknown)

An Ocean of Waves

We inhabit a world full of waves, constantly bombarded by cycles of electromagnetism (from visible light down to Wi-Fi signals, radio waves and radiation), gravity, and the movement of countless molecules. Our bodies and brains oscillate with electrical and chemical waves, and we are carried and swayed by the astronomical cycles of the Earth, Moon and Sun. Within this furious ocean of superimposed waves, exists the *audible spectrum*, made up of miniscule movements of (usually airborne) particles to which we have evolved to perceive as sound. This lecture examines how we recognise pitch within this spectrum, and how music – in its various historical and stylistic forms - selects, tunes and organises thispitch spectrum, in order to express and communicate to others our complex emotions through this strange medium.

Our ears and listening faculties are extraordinarily sensitive to oscillations in the air, even when surrounding molecules are displaced by movements on the nanoscale, and capture frequencies in the range of about 20Hz (waves per second) up to 20kHz (20,000 waves per second). Although this upper limit is quite aspirational and descends as we age. The 15kHz tone John Lennon places at the end of The Beatles *A Day in the Life* “to annoy the dog” is inaudible to most adult listeners but clearly visible on a spectrogram (Figure 1).

This narrow and closing window is older than our species, but prior to any absolute reference point we really had no way to map its various landmarks – we could say that one pitch was higher, lower or about the same as another, but this range has been mainly uncharted, without any absolute measure. This journey to a precise measure of frequency started with physical objects; the tuning fork (invented in 1711) provided a far more stable reference point than say the church organs with their susceptibility to wear and climate. Ellis’ 1880 survey of surveying tuning forks provide an insight into the wild diversity and staggering complexity of the history of musical pitch. 19th century mechanical, and then 20th century electronic devices were developed not just to emit a stable frequency but to identify an incoming pitch, providing objective insight into pre-existing theoretical models, and listening experiences.

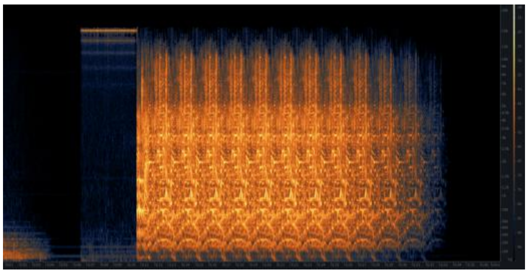

Figure 1: A Spectrogram representation with time running left to right, frequencies from bottom to top (low to high), and amplitude by brightness. This extract - from the final section of The Beatles’ *A Day in The Life*, a clear 15kHz tone (at or beyond the range of most adults’ hearing) is followed by a looped section that fades after a dozen repetitions.

From the Audible to the Pitch Spectrum

Once sound waves have been converted from acoustic to electrical energy, our brain feasts on the waveforms with its various *tonotopic* surfaces, decomposing the waveform into its constituent frequencies, similar to the spectrographic representation in Figure 1. Like a prism refracting light into its constituent colours, our listening mind refracts a beam of sound into a spectrum of component frequencies.

To understand how pitch works we need to address the briefest fundamentals. Any wave can be described in terms of a) amplitude – how much it displaces the medium from its normal position, and b) frequency – how often it completes that cycle of motion. The frequency of a wave is inversely related to its wavelength, the shorter the wavelength the higher the frequency and vice versa) as illustrated in Figure 2. c) *phase* described where the wave is in its cycle, its 'offset'

If we take the analogy of an ocean, waves can be high (and low) to varying degrees (amplitude), and also widely or narrowly spaced apart (with low and high frequencies respectively). Amplitude and frequency are easy to visualise with simple waveforms, but even the most complex repeating waveform which might look (or sound) like noise, can be theoretically deconstructed (and reconstructed) as the superposition of multiple simple waves of varying amplitude, frequency and its position in the cycle (its phase). The simplest wave is the sine (in fact sinusoidal) wave, it is atomic and irreducible, generating one frequency and so – in theory – any sound is a composite of multiple sine waves.

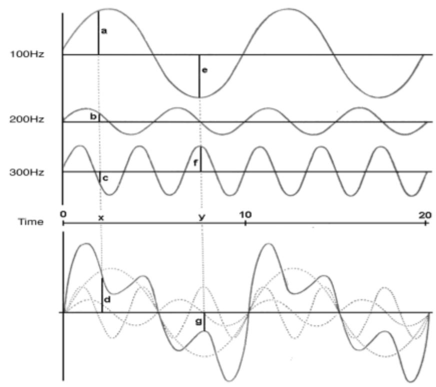

Figure 2: The top three images show sinusoidal of differing frequencies (low frequencies have large wavelengths and vice versa) and varying amplitude (reflecting in the wave height and experienced as loudness). When superimposed they produce a more complex waveform (bottom of image). The process may be reversed, any complex repeating waveform can be theoretically deconstructed into its component frequencies via technological or listening processes. (Image adapted from Ladefoged 1962)

All sound is made in this frequency spectrum, and from it we learn to recognise sound objects in our environment be it wind, waves, animals and humans. However, the recognition of *pitch* requires certain conditions, it is a sort of distilled frequency – it tells us not just what is making the sound but what specific frequency is being articulated. While many musical components; rhythm, texture, dynamics rely on the presence and shape of frequencies, pitch is the stuff of melody and harmony, and the playground of countless music forms.

We extract a sense of pitch from the frequency spectrum when there is a particular focus on a narrow band of frequencies, either with a single sine wave, or a pattern of frequencies that 'point' to some central frequency. For example the waveform at the bottom of Figure 2 would likely be heard as a pitch of 100Hz (its loudest and lowest frequency) with the upper components providing the sound's 'character'

This extraordinary ability to integrate and segregate frequencies so that we can not only recognise the source of a sound but the pitch it is creating is a lifeblood of music. Not only can we recognise the instruments in an ensemble, but their individual pitches and how they interrelate. Once we can perceive these it is like specific colours bursting through a mist, providing a unique vocabulary and language of expression, and allows a range of sound sources to be 'in tune' with each other despite their distinctive sonic characteristics.Figure 3 illustrates the spectrograms of the same melody performed by different instruments. We are able to identify the melodies from the instruments themselves. If played together these would be 'in tune' - playing the 'same' pitches and melody - despite the complex variance of component frequencies.

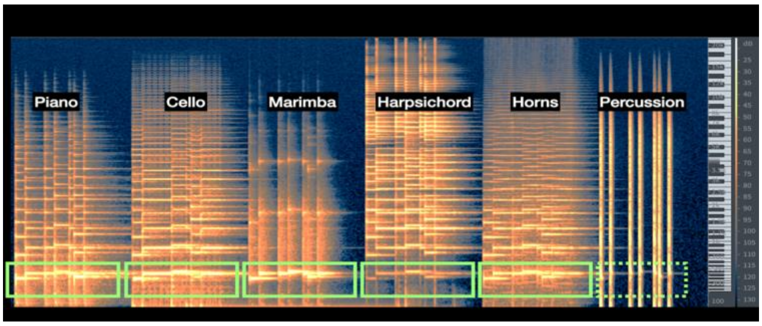

Figure 3: Spectrogram demonstrating sound source,and pitch segregation. We are able to recognise the instruments from the pattern of frequencies, but also - in the case of the pitched instruments - distil the same melody (in green boxes). The unpitched percussion instrument gives rhythmic and dynamic - bot no clear pitch - information. Mermikides (2023)

Still My Beating Tone

So far we have defined being 'in tune' as when important component frequencies (the 'fundamental pitch') of sounds match. But how close must they be? When frequencies are significantly distinct we recognise them as different pitches, however as they get close (roughly a couple of percent depending on context and training) we recognise them as the same (or versions of the same) pitch. We can understand this from a physical perspective, two identical frequencies will blend into each other producing a similar frequency, although the collective amplitude may be reinforced or attenuated depending on their relative positions in the cycle (or 'phase') of the individual waveforms. Similar frequencies that are coordinated – "in phase' – will reinforce each other (the peaks hitting the peaks), while if out of phase (peaks of one wave matched to the trough of another), they will cancel each other out (in fact that is how noise-cancelling headphones work). Perfectly in tune frequencies will in this way meld, becoming a composite force.

However, if two frequencies are slightly mismatched, their peaks will slowly drift against each other creating a sour throbbing effect known as 'beats' (illustrated in Figure 4). The more out of tune the two frequencies the faster the beats, and as they become more in tune the beats slow to gentle pulsation before disappearing. When a guitarist tunes two strings by ear they are intuitively – or even actively – listening for and eliminating these beats. Some piano tuners need strings to be precisely out of tune, and will actually count beats to create the desired mismatch. The jarring experience of hearing not-quite-right notes together is entangled with this beating phenomenon.

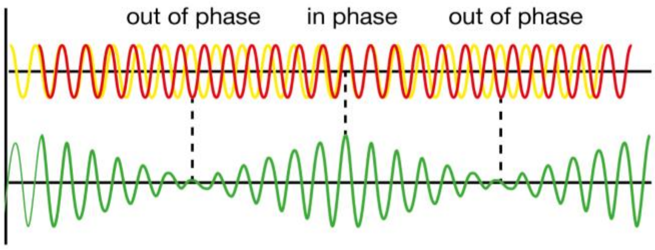

Figure 4: The red and yellow waveforms have slightly differing frequencies, so their peaks slowly drift against each other. When aligned (in phase) they create a high amplitude, and the amplitude is lowered when the waveforms are out of phase. This beating effect occurs at the difference between the frequencies (e.g. 100Hz and 102Hz will create a 2 Hz pulse). The beating phenomenon disappears when the frequencies match. Mermikides (2023)

So far, we have explained one way that two or more pitched instruments may appear in tune – the fundamental pitches (usually the lowest and loudest distinct frequencies) of each instrument are close enough together so that the beating effect is negligible or at least sufficiently tolerable. However, most music involves more than one pitch, with several occurring at the same time, how then might we consider tuning these other notes?

Simplicity Itself: The Harmonic Series

When two frequencies are identical they meld into a sense of stable union. However, it is also possible to create a similar sense of harmonic stability with *differing* frequencies. Picture a single string under tension. When plucked, the string oscillates along its whole length (at its *fundamental* frequency) which is typically experienced as its pitch (the exact pitch depends on how long the string is, its density and the degree of tension. In addition to this fundamental frequency, the string typically resonates at other frequencies simultaneously. It does so at equal divisions of the string length, a waveform that splits the string into two equal divisions, another at three equal divisions and so on. Since wavelength and frequency are inversely related, this means that an oscillating string (or column of air etc) of a pitched instrument, at a fundamental frequency (say 100Hz) will also produce frequencies at integer multiples (1, 2, 3, 4 times) the frequency of the fundamental (at 100Hz, 200Hz, 300Hz etc). Such overtones (tones above the fundamental) are known as *harmonics*, and produce a series of pitches known as the *harmonic series.* Note that various instruments produce harmonics at different relative amplitudes, as well as inharmonic partials (overtones that are not integer multiples) such as the percussive *thwack* at the beginning of a piano tone. This dynamic pattern of overtones gives an instrument its character, but the template of the harmonic series is consistent – a perhaps defining feature – of pitched instruments, as can be clearly seen in some of the instruments in Figure 3. It helps define and reinforce the fundamental, and instruments with harmonic overtones blend comfortably with each other. The harmonic series is an emergent property of physical objects and is encountered by all musical cultures, not only in terms of vocal and instrumental quality, but in terms of selecting and tuning pitches that sound consonant together, melting into each other.

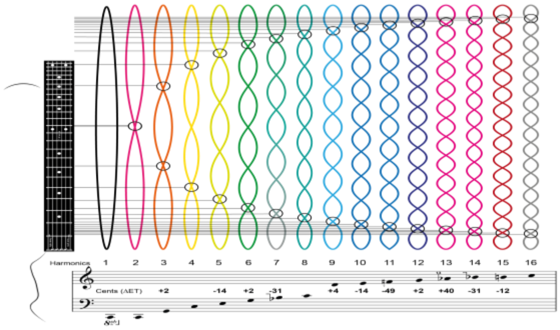

Figure 5: The 1st 16 harmonics of an oscillating string (at low C). The resulting pitches – with how it compares (in cents) to an equal-tempered scale are shown. Some representative nodes are shown with their positions on a guitar fretboard. Mermikides (2023)

The harmonic series emerges from all sorts of instruments and contexts and is easily demonstrated. Touch lightly *exactly halfway* along a string and the 2nd harmonic will emerge, at a third of the string you have two options) and the 3rd harmonic is heard. Some of these 'nodes' are illustrated in Figure 5 with a guitar string but they can also be generated on wind, brass, electronic and other plucked or bowed string instruments.

The pattern that emerges is profoundly familiar: The 2nd harmonic (halfway along the string and double the frequency) produces an *octave*. This interval is near universal in music and is so connected to the fundamental that in almost all cultures: There is a repeating cycle of note names (e.g. C to C in Western Music Notation, Do to Do in Eastern Europe, Sa to Sa in Indian Classical Music, د to د in Arabic music, 도 to 도 in Korea etc). People will naturally sing in their most comfortable octave to a piece of music. In short, the next nearest (and most 'in tune')) note to singing in unison is at octave. There is a sort of *octave equivalence* in many music forms. Note that if we halve a string (double a frequency) we produce an octave, and so this pattern continues: a quarter of a string produces two octaves, a similar string double the length would be an octave lower and so on.

The third harmonic (at one third of the string's length, tripling the frequency) produces an octave and the *perfect 5th*. The 5th is a remarkably important and ubiquitous interval and referenced throughout the lecture series. Note that since a third of a string (3 times the frequency) produces an *octave and a perfect 5th*, *two*-thirds of a string (a 3/2 frequency ratio) will produce a perfect 5th.

The fourth harmonic is simply two octaves (2x2 = 4), but the fifth harmonic – being a prime number – introduces a new tuning flavour into the mix, the *major third* (sometimes referred to as a 'pure' or' just' major third) (two octaves about the fundamental). So by the first 5 harmonics we have spelled out the ubiquitous major triad – as well as the melodic scaffolding for Strauss's 1896 iconic theme from *Also Sprach Zarathustra*. The harmonic series continues upwards in ever narrowing intervals, implying a dominant chord, a pentatonic scale (a recognisable rendition of *Amazing Grace* is possible from the 5th-12th harmonics), and by the 16th harmonic enough notes to construct familiar (if perhaps exotically tuned to the uninitiated) scales. Higher sections of the harmonic series are particularly prevalent in instruments like the bagpipes, overtone singing, didgeridoo, a wah-wah guitar, trumpet but it forms the fabric of countless instruments, a universal physical phenomenon connecting diverse cultures.

Other than the octaves, the notes in the harmonic series do not ever align perfectly with the common even-tempered systems used in common practice and theory. How each differ in cents (100ths of a semitone) from the familiar equal temperament is indicated in Figure 5, and these differences can become startlingly distinct - particularly in the higher primes (0-5 cents is subtle for most listeners, by 14 cents deviations are significant to musicians, and almost all listeners will notice anything above this range).

There is always something inherently simple about the use of the harmonic series: it is a simple pattern of frequencies, and – if slowed down sufficiently – a sequence of rhythms. Once one's ears come attuned to it, it's hard not to appreciate how 'smooth' a major third built of the 5th harmonic sounds as compared to its equal tempered counterpart. It is essentially a simple rhythmic relationship which melts away the beating phenomenon described previously.

This connection between the pitch and the rhythmic domains is apparent in historical treatises such as Hermann von Helmholtz's staggeringly comprehensive and insightful 1875 work *On the Sensation of Tone*. It includes a description of a device the *Siren* where a series of concentric holes in spinning circular discs at various rhythmic relationships are used to control the flow of air to a wind instrument. When accelerated a series of 3 holes and a series of 2 holes – a 3 against 2 rhythm – produces a perfect 5th. Similar approaches using mechanical cogs created the first electronic synthesizer Cahill's 1896 *Telharmonium*. Theremin's *Rhythmicon* (often cited as the world's first drum machine) used a similar mechanism of perforated discs (in this case controlling the passage of light to photoreceptor cells) and could produce complex rhythms which would transform into harmonies when accelerated to the pitch domain.

Despite the simplicity of these harmonic ratios, a huge (in fact infinite) amount of musical pitch construction is possible even with the first few harmonics.

Building Scales with Pythagoras

This link between mathematical simplicity and musical consonance (and objective 'science' with subjective aesthetic experiences in general) is an ancient idea that still continues to inspire musicians and listeners. Its roots may lie in the Babylonian concept of *God Numbers*, and most famously the experiments purportedly conducted by the ancient Greek polymath Pythagoras of Samos (c.570 – c.495).

During a visit to a market, Pythagoras was reportedly intrigued to notice the difference in musical pitches the various size anvils produced when struck, and set about to explore the connection between physical and musical properties. A simple instrument - the *monochord* - allowed him to experiment in a controlled environment the effect of tension, and precise string-length (and thus frequency) on the experience of musical pitch. He noted how simple proportions of the string (one half, three quarters, two thirds, three quarters and other *rational* divisions) produced sweet consonances. The more simple a division (say two thirds of the string) the more stable the more complex ratio (e.g. 16/27 of the string) the more flavoursome and dissonant the tone.

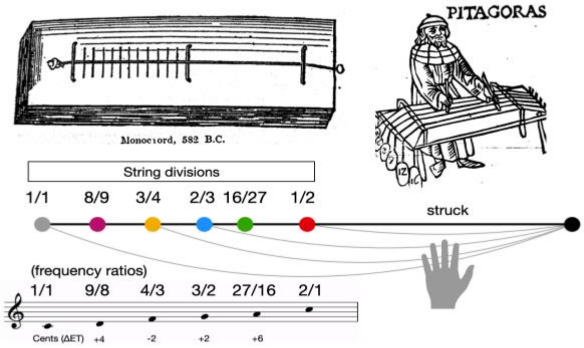

Figure 6: On the right Crotch's 1861 image of a monochord, next to Gafori's *Theorica Musice* (1492) illustration of Pythagoras operating it. Below are some representative rational divisions of the string and the pitches they produce.

From this perspective, selecting musical notes is an exercise in geometry, creating a variety of string divisions and notes simultaneously. This approach is still tied to the harmonic series, creating multiples and divisions of stringlength (and hence divisions and multiples of frequency) is essentially making harmonics of harmonics. We can also use subharmonics, such as creating a third of a string and then *doubling* its length, dropping it down an octave. The so called Pythagorean scale (a selection of which is at the bottom of Figure 6) in fact only uses up to the 3rd harmonic (it is a *3-limit* system) so every frequency ratio will only integers divisible by 2 or 3 (or both) such as 3/2 9/8 27/16 4/3 or 81/64. Despite these limits there are countless pitches available even with one octave (from 1/1 to 2/1). Pythagoras and his advocates believed that not only is music subject to the sublime purity of mathematical laws, but so too does the rest of the physical universe including the planets (The orbital periods of Saturn and Jupiter are close to a 5/2 ratio) - a universal music (*Musica Universalis*), a Music of the Spheres. An early 17th Century Figure 7 depicts this Pythagorean concept with the planets interconnected in a series of musical intervals - you may notice a similar visual language as to Figures 5 and 6.

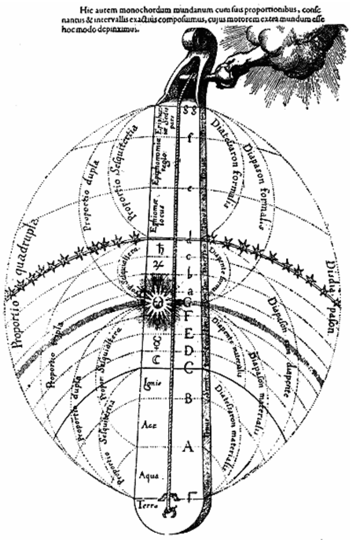

Figure 7: An engraving from Fludd's *Ultriusque Cosmi* (1617-1619) depicting the Pythagorean concept of *Musica Universalis* – a universal music linking music and the physical world.

It's Just Intonation

The harmonic series – and its extrapolation into just intonation – is a simple *lemma,* a cognitive stepping stone, with limitless possibilities. It forms the backbone to the tuning systems of many seminal and historical musical forms. The 2nd Century Greek theorist Cleonides documented aspects of the Greater and Lesser Perfect Systems of Ancient Greek Music. This involved tuning notes in *tetrachords* (groups of four notes) – such as the four strings of a lyre – which could be joined into full octave scales. A tetrachord would typically involve 'immovable' notes - the outer two in a fixed ratio – leaving the two notes within movable to other rational divisions. A tetrachord would be tuned (in fact from high to low notes), so that a total span of 4/3 was preserved. Such *genus* would include the *diatonic* (1/1 9/8 81/64 4/3), the *chromatic* (1 32/27 81/64 4/3) and the scrunchy *enharmonic* (1/1 5/4 9/7 4/3) collectively interlinking to form a stunning array of exotic but geometrically conceived scales.

Just intonation – though typically inspired by notations of harmonic purity and 'musical truth' – is open to many avenues of exploration and interpretations of such purity/ We can for example just use the first 3 harmonics to generate all our pitches. Multiplying frequencies by 3 (and dividing by 2 to bring them down to the same octave) and inverting the process (*dividing* frequencies by 3 and bringing them up by octave), produces an infinite spiral of pitches unified by the first 3 harmonics. Some systems of Turkish *maqam* music do just this, gathering its 24 notes from 12 ascending and 12 descending 'pure' 5ths respectively.

Hindustani music opts to not extend this far into the 3rd harmonic, but to open the doorway to the 5th harmonic, allowing direct access to the sumptuous just major third (5/4) interval. The 22-Shruti system builds simple ratios with relatively low integers (divisible by 2, 3 or 5 so '5-limit'). There are two 'immovable' intervals - the octave and the 5th, but every other of the 12 note names have two options - a bright and dark counterpart. These 22 *shruti* (that which is heard'), in the hands and voice of an expert performer, against a mesmerising raga drone, are impossibly beautiful, and a testament to just intonation's potential. Figure 8, lists these in terms of note names, integer ratios and cents. Note that none other than the octave fall exactly on the even tempered frets.

Note name (in C)	C	Db-	Db	D-	D	Eb	Eb+	E	E+	F	F+	F#-	F#	G	Ab	Ab+	A	A+	Bb	Bb+	B	B+	C
Hindustani note name	S	r1	r2	R1	R2	g1	g2	G1	G2	M1	M2	m1	m2	P	d1	d2	D1	D2	n1	n2	N1	N2	S
Ratio	1/1	256/ 243	16/ 15	10/9	9/8	32/ 27	6/5	5/4	81/ 64	4/3	27/ 20	45/ 32	729/ 512	3/2	128/ 81	8/5	5/3	27/ 16	16/9	9/5	15/8	243/ 128	2/1
Cents	0	90	112	182	204	294	316	386	408	498	519	590	612	702	792	814	884	906	996	1018	1088	1110	1200

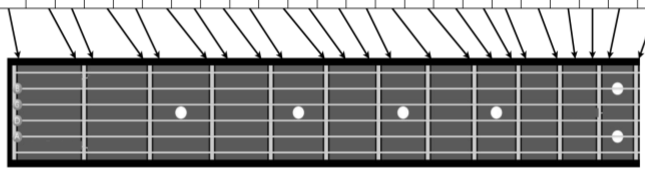

Figure 8: The 22 Shruti of Hindustani music.

As well as embedded in many traditional global practices, there have been a minority but highly dedicated devotees who – disillusioned by a world where music involves the manipulation of 12 equally spaced notes – are enamoured by the potential of just intonation.

The British musicologist and curator of instruments at the British Museum Kathleen Schlesinger (1862–1953), was an extraordinary scholar who reportedly wrote most of the articles on musical instruments in the 1911 *Encyclopedia Brittanica*. Her 1939 Study on the Greek Aulos (an ancient flute) is an exhaustive overview and theoretical models of Ancient Greek – particularly tuning – music systems. This inspired a generation of composers such as the Australian composer Elsie Hamilton (1880 – 1965) who developed Ancient Greek tuning concepts into elaborate 'planetary modes'. Harry Partch– another friend of Schlesinger's and just intonation composer and pioneer - dedicated his life to the exploration of the craft, and the manufacture of elaborately designed (and named) custom instruments to perform his music. His *Quadrangularis Reversum* was built to execute his 43-tone scale, an 11-limit conceptual 'diamond' of pitches which started as "absolute consonance (1/1)" and through overtone ('otanity') and undertone ('utonaity') harmonic exploration "gradually progresses into an infinitude of dissonance". He in fact invented the consent of a harmonic prime 'limit' and claimed that just intonation could provide sweeter harmonies and more intense dissonance than ever possible by wrangling the 12 tones of equal temperament. Figure 9 displays part of an image of his *Chromolodeon* - note how the prime numbers of the harmonic series are colour coded on some keys (3 is orange, 5 is yellow, 11 is purple).

Figure 9: A portion of the keyboard of Harry Partch's *Chromolodeon* - on some keys the prime numbers of the harmonic series are colour coded (3 is orange, 5 is yellow, 11 is purple)

American composer Ben Johnston (1926 – 2019) composed highly detailed (and disarmingly accessible) just intonation music for traditional instruments and familiar, and devised an extremely elegant series of accidentals to accommodate just intonation tones up to the astronomical 31st harmonic.

The field of microtonality (essentially anything but 12-tone equal temperament), is vibrant and reinforced by robust theoretical, musicianship and technological resources. Though still subject to the contemporary split from common practice music making, the depth to which some musicians choose to dive is quite staggering. Even an overview is beyond the scope of this essay and would in fact leave the vast majority of even professional musicians dumbfounded by its theoretical complexities, However, a peruse of the Xenharmonic wiki page should give an idea of its possibilities of microtonality and the dedication of its practitioners.

Temper, Temper

The harmonic purity of just intonation comes at a cost and limitation. From its simple beginnings unfolds boundless complexity. For example if we want the fundamental to have access to a pure major triad (like two revolving planets) that's easily done, however if we wish to give the 3rd or 5th have *their own* personal pure major triad, 4 more notes must be added (moons to these planets), and then these added notes would require more notes (satellites to these moons) and so the exponential growth continues *ad harmonic infinitum*. If we want to limit the number of notes, create polyphonic music or modulate to other keys, we must somehow close the loop, and provide each note with a similar (if not identical) harmonic vista.

In conventional music theory we are very familiar with such a closed loop: the *cycle of 5ths* which connects all the twelve notes in a circle of ascending and descending fifths starting and ending on the same note. The cycle of 5ths is essential in the construction of keys and of a range of harmonic possibilities. However we run into a problem when constructing it with just intonation. A pure fifth has a frequency ratio of 3/2. If we start at a note (say C) and keep stacking fifths we expect to get back to the same note (7 octaves higher). However we end up overshooting by a fraction (23.46 cents, about a quarter of a semitone). This extra tuning residue is known as a Pythagorean comma (comma from the Ancient Greek to 'cut off'). It comes close enough (suggesting perhaps why 12-note names in the octave (even if subject to tuning discrepancy) are so globally common. But still, pure 5ths just don't fit with pure octaves. This may seem unfortunate but it is in fact mathematically inevitable. No matter how many pure 5ths we ascend or descend it will never exactly match an octave. The same is provably true for any of the prime numbered harmonics: Stacks of pure major 3rds (from the 5th harmonic) don't match an octave (from the 3nd harmonic). In 12-TET they create a closed loop (e.g. C-E-G#/Ab-C) but pure major 3rds fall way short of an octave (creating another comma: the *diesis)*. Neither do the 3rd and 5th harmonic mix, a major third created by stacked pure 5ths (The Pythagorean Major 3rd) is far sharper than the pure major 3rd (they are separated by the *Didymean comma*).

How do we ever resolve this? For example, this 23 cents (it's the ratio between 12 pure fifths and 7 octaves) has to be clawed back from the existing fifths to resolve the comma. We could simply narrow one of the 5ths to absorb the surplus (creating an example of a *wolf* fifth which 'howls') and hide it like a guilty secret in a foreign key area (as is commonly done in Pythagorean tuning). Unfortunately, this makes some keys and intervals particularly sour and unusable (although did provide composers with exotic soundworlds when required). However, by 'tempering' the tuning we can close the loop more elegantly. Such temperaments aim to resolve these commas by tampering pure intervals.

There are literally 100s of documented temperaments over the centuries. One of Valotti's 1728 *sixth-comma* systems – as the name implies – splits up the comma into six parts. These slices are distributed into a string of 6 fifths from F to B. One of Werckmeister's 1691 tunings, on the other hand, is a 1/4 comma system distributing the comma between four of the fifths to close the loop (Figure 10).

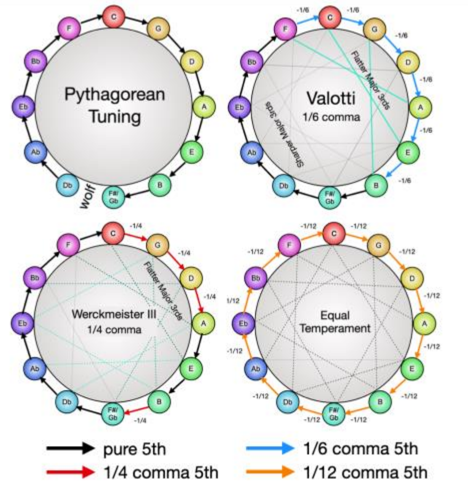

Figure 10: A survey of temperaments with four types of 5th, resulting in various key characters and major 3rd flavours. Mermikides (2023).

You might wonder why – if these altered fifths are no longer 'pure' – they are placed around the common keys such as C, F and G. Why not hide their impurities away in the dark forest of foreign keys? The answer lies in the thirds. A pure 5th is a little too wide to complete the circle, but the pure major 3rd is far too narrow. So narrowing the fifths in a temperament narrows the major 3rds closer to the ‘pure’ major 3rd – which would otherwise be very wide Pythagorean major 3rds. This narrowing of the nearby fifths, creates a more stable sweetness to commonly played chords.

In this way ‘well’ temperaments were crafted to make all keys usable, but not the same: some sweet and harmonious others a little more brittle. 12-tone equal temperament (which we might call 1/12th comma) distributes the comma perfectly evenly over a 5th, none of them are pure, and neither are any of the major thirds, but they – and all the keys – have the same relative tuning, a highly convenient system allowing countless musical devices and masterpiece works, despite its theoretical impurity.

A control of temperament allows access to pieces in any key (and endless modulations within a piece). A seminal work exploring such key fluidity is J.S.Bach’s *The Well-Tempered Clavier.* Composed in 1722 (and a 2nd book in 1742), it consists of a prelude and fugue for each of the 24 major and minor keys. This would require a temperament that works from every key, but it is not – as is often misunderstood – written for equal temperament. It is – as the title plainly suggests – for a *well*-tempered system, where every key is usable and has its own character. For centuries it was unclear which temperament Bach intended for these pieces (although we know of his favourites). But in 2005, the music theorist Dr. Bradley Lehman put forward an intriguing hypothesis. The original handwritten cover of the piece contains a decorative swirl. When turned upside down, all the information needed to construct a temperament is there, a marking of C, then a series of loops with various knots which are consistent with a distribution of the Pythagorean comma. Other scholars have contested Lehman’s hypothesis, but knowing Bach’s mischievous love of puzzles, the specificity of those loops, and the beguiling idea of an answer hidden in plain sight for two and a half centuries, it is too irresistible not to share.

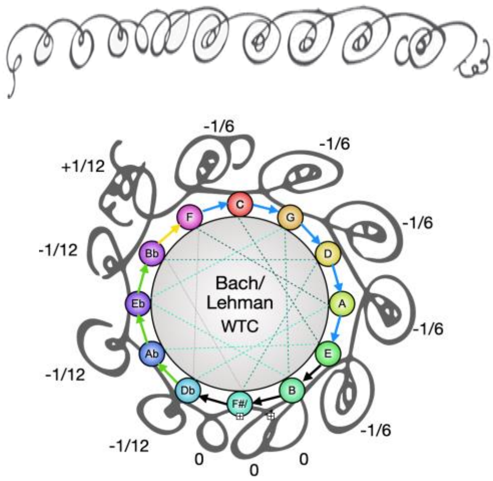

Figure 11: The decorative swirl on the title page of the *Well-Tempered Clavier* (top), its eleven loops and one break (encoding the comma deviations in each 5th) has been suggested (Lehman 2005) to describe Bach’s intended temperament (bottom). Mermikides (2023).

Mozart’s Semitones

Many Western contemporary listeners and musicians take the existence of 12 notes in an octave for granted, any talk of a world between them (other than some expressive bends and slides) is the stuff of history, other cultures, eccentric experimenters, or not considered at all. From child learners to PhD candidates, the question of “why there are twelve notes” is among the rarest questions – if ever asked. Music is beautiful and challenging enough without such complexities. There are twelve divisions, with ‘enharmonic’ equivalents: G# is ‘the same’ as A-flat for example, even if notated differently. Guitar tablature, MIDI notes and where musicians place their fingers or pitch their notes is equivalent. So, it can come as a shock for some to learn how relatively recent this 12-note vision is, even in the most mainstream of Western practice.

The British composer Thomas Attwood (1765–1838) studied with Mozart in Vienna from 1785–1787 and was reportedly one of the maestro’s favourite pupils. Still this might not have been the most comfortable experience for Attwood (himself a fine composer, organist at St.Paul’s Cathedral and a founding member of the Royal Philharmonic Society), particularly given Mozart’s *A Musical Joke* – a mischievous parody of compositional ineptitude – is likely based on Attwood’s studies.

Among Attwood's study notes, is an intriguing page in Mozart's hand (Figure 12, top left) explaining the distinction between "major and minor semitones" (*mezzi tuoni grandi* and *mezzi tuoni piccoli*). What could this mean? Surely semitones are semitones? Mozart is most likely referring to a form of 'meantone temperament'[1], well established by this time. These temperaments can deal with the "problem of major 3rds" directly. If you recall, pure major 3rds are too narrow to complete the octave circle, so any system where stacked major 3rds do so (like equal temperament) sharpens them considerably. There is a solution, and that is demonstrated in Figure 12 (upper right). Narrow major thirds (from C to E to G# to C) fall well short, but by creating a new note 'Ab' sharper than G# we can ensure that there is a sweet major 3rd both to and from C. By splitting more accidentals (D# and Eb, etc), we can accommodate sweet 3rds for other notes also.

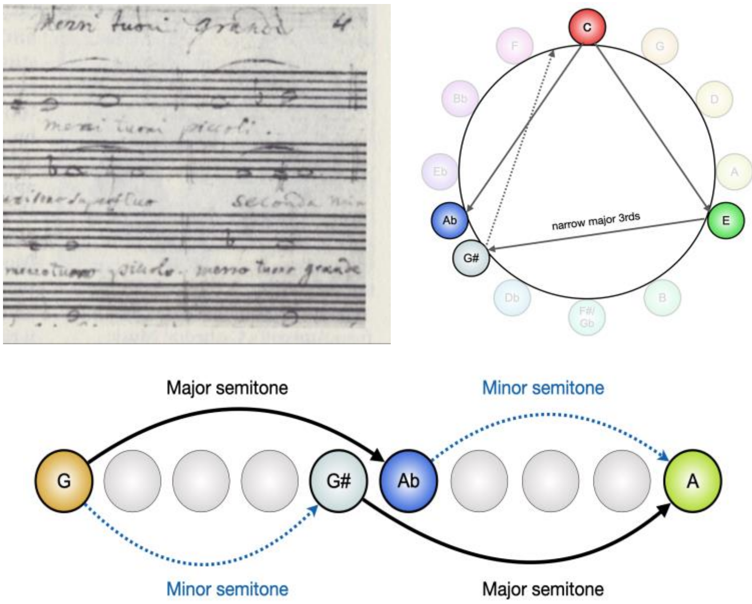

Figure 12: Clockwise from top left: Mozart's notes to student Thomas Attwood on 'Major and Minor Semitones' (c.1786), an illustration of splitting accidentals to ensure narrow major 3rds, and a description of major and minor semitones in 55-division temperament.

So, depending on harmonic context, we are given options. One such temperament (articulated by Tosi in 1743) splits each tone (say G to A) into 9 equal parts, with minor semitones from G to G# and Ab to A; and major semitones from G to Ab, and G# to A. This splits the octave into 55 equal divisions.[2]

The splitting of accidentals and their tuning placement was not just theoretical: Pedagogical material from the 17th and 18th century, and instrument design (such as 'split-key keyboards'), used such tunings directly and practically (see Figure 13). The surprise musicians exhibit when discovering such documentation, revealed how – through equal temperament's adoption - extensively and rapidly such distinctions were abandoned in general music pedagogy in the following two centuries. Advocates of historical tuning have set about performing Mozart's repertoire in such temperaments, attempting to capture its original soundworld.

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[1] 'Meantone' because all whole tones are identical, while the notes between them – unlike equal temperament – are asymmetrical.

[2] In the chromatic scale, there are 5 whole tones (C-D, D-E, F-G, G-A, A-B) each with 9 divisions (so that's 5 x 9 = 45 divisions). There are also 2 major semitones (E-F, B-C) with 5 divisions each (5 x 2 = 10). This makes 55 equal divisions of the octave andresults in note placements very close to a temperament known as sixth-comma meantone, to which Mozart may have been referring.

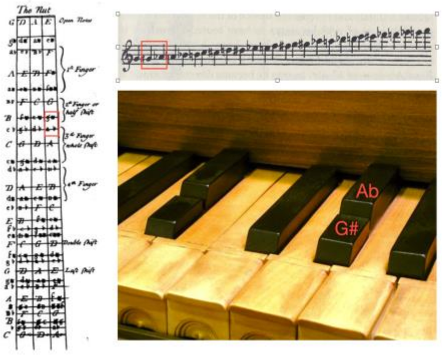

Figure 13: Clockwise from left: Preuller's fingerboard diagram (1797), Woldermar's Violin Method (1803) and an image of a split-key harpsichord (1715). In each an example of the G#/Ab distinction is annotated.

Blues Microtonality and Impure Thoughts

The discussion of microtonality may be scant in contemporary music studies, but it exists in the music all around us, particularly when instruments (like vocals and strings) are not bound by even-tempered tuned strings, resonators and digital resources. Perhaps the most prevalent, extensive and accepted use of microtonality to the Western ear is found in the language of the blues. Many scholars suggest its tuning origins are based on just intonation, but it has since coalesced with equal tempered harmony in a complex and beautiful manner. Quite contrary to the simple description of a 5 or 6 note blues scale, a serious study of seminal blues recordings reveals a beautiful gradient across the octave spectrum, with 'clusters' and 'complexes' along its continuum. The peaks and ridges of this pitch gradient are totally established, understood and *felt*, by skilled practitioners and listeners. These idiosyncratic blue notes relate to just intonation ratios, but also the harmonic ambiguity which occurs around these notes in a typical blues progression. For example a blues progression usually involves dominant 7 chords on the root, fourth and fifth of the key, this results in a minor and major 3rd, and a minor and major 7th, and we see this bittersweet ambiguity in the microtones (occasionally termed 'blue notes' or 'blues curls') *between* these minor and major landmarks. If they are notated at all, they are simplified with ornamental symbols, but they are as important to blues expression as any 'normal' note. Figure 14 captures the blues note continuum.

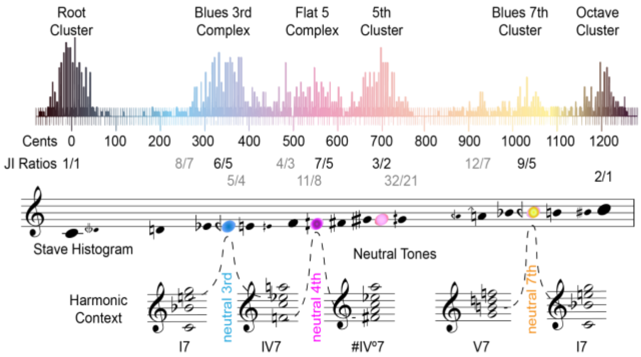

Figure 14: An illustration of the blues note continuum (using data sourced from seminal blues recordings. An octave is shown, with clusters and complexes, rather than exact points. Three idiomatic peaks occur between the major and minor third, major and minor seventh, and the perfect and augmented 4th. These all relate to harmonic adjustments that occur with common blues progressions. (Mermikides (2023), with empirical data fromCutting (2018)).

The blues and many other music forms reveal that there is in fact no easy dividing line between equal temperament, 'free' tuning and just intonation, and no easy objective measure of quality. In practice, each tuning system offers unique opportunities. Hindustani music allows us to really hear the beautiful low-integer shruti and subtle ornamentation because of the fixed drone; the 'compromised' well and equal temperament allow otherwise impossible stunning music to emerge. Furthermore, there is a limit to our ability to discriminate pitch, and *any* note is within perceptual distance of some (albeit perhaps complex) harmonic ratio. It is also impossible to produce and transmit (even with electronics) an *absolutely* perfectly intoned note.[3] In short, any note can be argued to be both ‘pure’ or ‘impure’ depending on our perspective. As humans we also acclimatise to tuning, we are not passive listeners to some (im)pure musical ideals but actively build our musical experiences. And perhaps most importantly of all: dissonance is good. Vibrato, glides and human ‘deviations’ of pitch – if ‘imperfect’ – can be incredibly expressive, Music – as well as striving for a theoretical perfection –embraces the perfectly imperfect.

Endnotes: From Light to the Stars

Pythagorus’s vision imagines a cosmos of beautiful order, symmetry and harmonic resonances, creating a universal music of balance and order guiding the movements of the stars, music and our inner lives. This idea is beguiling, and there is in fact fascinating evidence that under some conditions planetary systems do indeed favour harmonic ratios. However, sonifications of light frequencies and our own solar system reveal challenging and spicily dissonant frequency relationships, cosmic blues notes and phasing beats, as well as simple harmonic ratios. As we continue to explore the universe, it reveals among its laws an inherent – perhaps fundamental º chaos, with perhaps no deterministic centre. Black holes, cold deaths, entangled particles, dark energies, quantum foam and overwhelming swathes of empty time and space, are themes in the music of our spheres. And it is this mix of order and chaos, consonance and dissonance, purity and grit – reflected in the cosmos and our perfectly imperfect lives – that is the essence, and in fact true beauty, of music.

[3] Javanese Gamelan and a range of traditional Indonesian music routinely use ‘stretched octaves’ and imply equal temperament (with 5, 7 and 9 divisions of the octave) creating beautiful shimmering timbres, no less beautiful than low integer just intonation.

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