4000 Years of Geometry
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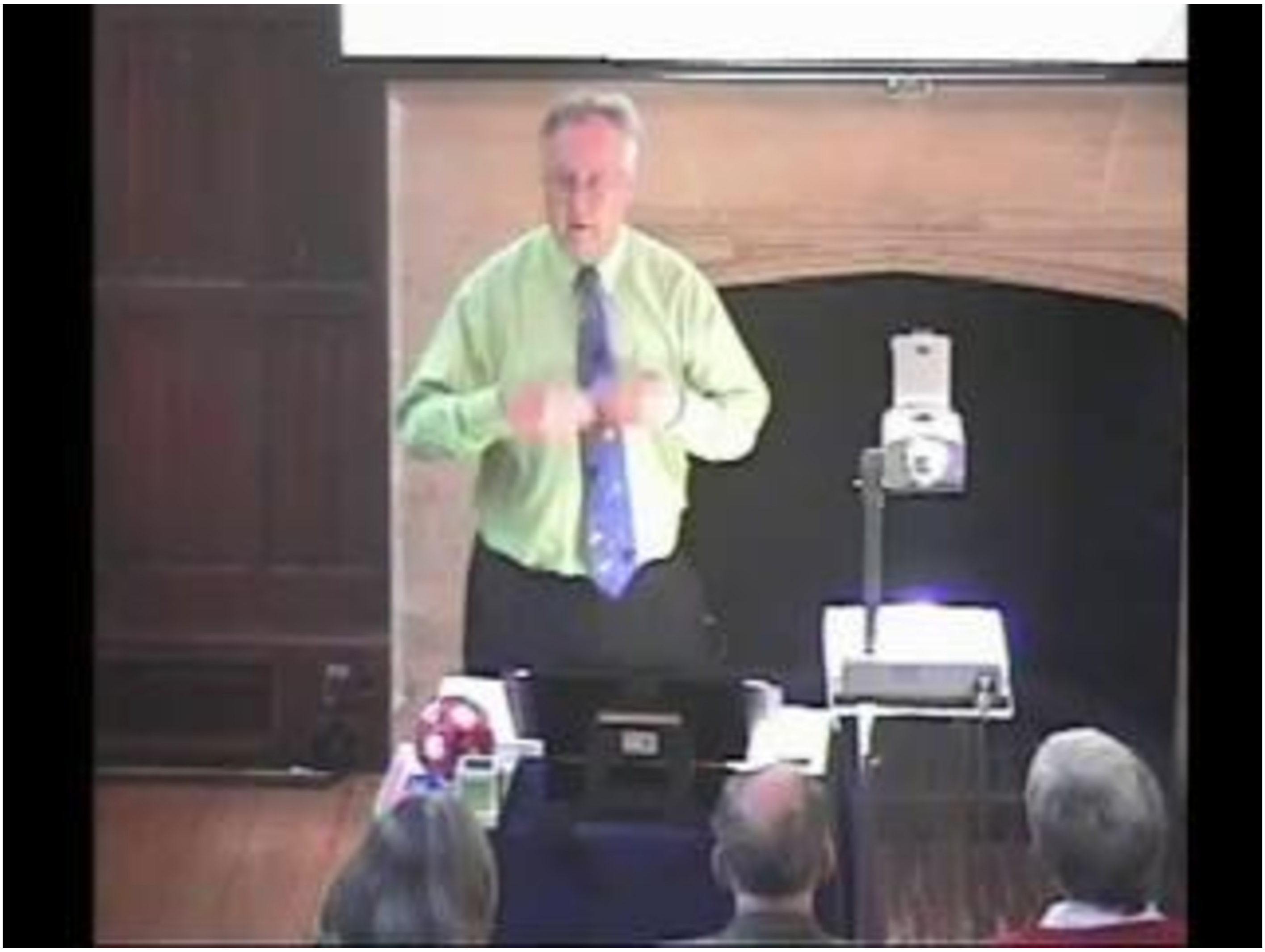

4000 YEARS OF GEOMETRY

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Introduction

History can be presented either chronologically or thematically. Over the past three years I've worked through 4000 years chronologically, starting with the mathematical achievements of Egypt, Mesopotamia, ancient Greece, India, China and the Mayans, then proceeding via the Arabs to Medieval and Renaissance Europe, and concluding by working through the 17th, 18th, 19th and 20th centuries.

My Gresham Chair has now been extended to a fourth and final year, and since I feel unqualified to continue in this vein by summarising the mathematical achievements of the 21st and later centuries, I thought it would be illuminating to take stock by selecting three specific areas and tracing them through 4000 years, - that is, to look at the achievements thematically. Today I'll start with geometry and then proceed to algebra next time, and various aspects of number next month.

So let's start by going back more than 4000 years to the world of Ancient Egypt.

Egypt

From about 2700 BC the pharaohs desired to be buried in massive pyramids. The oldest of these is King Djoser's step pyramid at Saqqara, constructed in horizontal layers and supposedly designed by Imhotep the celebrated court physician, Grand Vizier and architect.

Better known are the magnificent pyramids of Giza, which date from about 2600 BC and attest to the Egyptians' extremely accurate measuring and geometrical skills. In particular, the Great Pyramid of Cheops has a square base whose sides of length 230 metres agree to less than 0.01%. Constructed from over two million blocks, averaging around two tonnes in weight and transported fifty miles by a whole army of workers, this pyramid is an impressive 146 metres high. Moreover, it's not solid, but contains an intricate geometrical arrangement of carefully calculated internal passageways and chambers.

Our knowledge of later Egyptian mathematics is scanty, deriving mainly from a small number of fragile primary sources - notably, the Moscow papyrus (c. 1850 BC) and the Rhind papyrus (c. 1650 BC) which is in the British museum. Part of the latter shows some geometrical problems involving triangles, with one of them enlarged so that you can see the detail. Such problems were used in the training of scribes.

Another problem from the Rhind papyrus is Problem 48: *Compare the areas of a circle and its circumscribing square*. Choosing the diameter of the circle and the side of the square to be 9, they found the area of the square by comparing 1 with 9 setat (the unit of measurement), 2 with 18 setat, 4 with 36 setat, and 8 with 72 setat, and then adding the rows corresponding to 1 and 8 (which add to 9) to yield 81 setat. For the circle, they started by comparing 1 with 8 setat and proceeding as before, giving an answer of 64 setat.

But why 8? The Egyptians seem to have found by experience that the area of the circle with diameter d is the same as that of a square with side $^8/_9 d$ - this is presumably why they started with diameter 9, to simplify the calculations. In terms of the radius the area is $^{256}/_{81} r^2$, which in modern terms corresponds to a value of *pi* of about 3.16, an excellent approximation for almost 4000 years ago.

Mesopotamia

Let's now turn our attention to the Mesopotamians, from around the same time as the Rhind papyrus. Using a wedge-shaped stylus, they imprinted symbols in the moist clay - this is called *cuneiform writing* - and the tablet was then left to dry in the sun. Many thousands of such mathematical tablets have survived, though not many deal with geometrical problems.

However, there's a particularly remarkable tablet that shows a square with its two diagonals, and the numbers 30, 1;24,51,10 and 42;25,35. These numbers are written in their hexagesimal system based on 60, which survives in our measurement of time, and here refer to the side of the square (30), the square root of 2, and the length of the diagonal ($30\sqrt{2}$). The accuracy of the square root 1:24,51,10 is amazing, as becomes apparent when we square it - we get 1;59,59,59,38,1,40, which differs from 2 by an exceedingly minute amount.

Greece

The period of Greek mathematics lasted for about a thousand years, from around 600 BC, and falls into three main periods. The first concerns the semi-legendary figures of Thales and Pythagoras; the second deals mainly with Athens and Plato's Academy; and the third takes us to Alexandria, starting with Euclid and extending over 700 years.

Thales and Pythagoras

Starting with Thales and Pythagoras, we have to admit that we know little about either of them. According to legend, Thales came from Miletus, brought geometry to Greece from Egypt, predicted a solar eclipse in 585 BC, and showed how rubbing feathers with a stone produced electricity. As Proclus commented some 1000 years later:

The famous Thales is said to have been the first to demonstrate that the circle is bisected by the diameter. If you wish to demonstrate this mathematically, imagine the diameter drawn and one part of the circle fitted upon the other. If it is not equal to the other, it will fall either inside or outside it, and in either case it will follow that a shorter line is equal to a longer. For all the lines from the centre to the circumference are equal, and hence the line that extends beyond will be equal to the line that falls short, which is impossible.

You'll notice that this extract is concerned with *mathematical proof*. Starting with some initial assumptions (or *axioms*) we derive some simple results, and then more complicated ones, and so on, eventually creating a great hierarchy of results, each depending on previous ones. An example from *Book I* of Euclid's *Elements* is Proposition 47 (which happens to be Pythagoras's Theorem) where each step depends on the previous results.

The Greeks adopted several methods of proof - the Thales extract used a proof by contradiction (or *reductio ad absurdum*), where we assume that the result we want to prove is incorrect and then deduce a result that contradicts our assumptions. You'll see other examples of this later on.

These ideas were developed around 550 BC by Pythagoras and the Pythagoreans who supposedly gathered around him in the Greek seaport of Crotona, now in Italy.

Pythagoras's name is for no apparent reason associated with *Pythagoras's Theorem*, even though the Mesopotamians used the result a thousand years earlier. But it was in Greek times that the theorem was proved.

The theorem, states that *in any right-angled triangle, the area of the square on the hypotenuse (the longest side) is equal to the sum of the areas of the squares on the other two sides*. So it's a geometrical result about areas - with no mention of any algebraic equation such as $a^2 + b^2 = c^2$.

Here's a dissection proof that would have been typical of the Pythagorean school. These pictures give two dissections of a square of side $a + b$. Removing the four triangles in each case, and comparing the red squares, we see that the largest area must be the sum of the two smaller ones. It's a far cry from the axiomatic proof that would appear later in Book I of Euclid's *Elements*.

A similar dissection proof of Pythagoras's theorem appears in a classic early Chinese text, the *Zhou-bei suanjing* (The arithmetical classic of the gnomon and the circular paths of Heaven): if we draw the square on the hypotenuse and then move the two upper triangles in the figure, we get the sum of the squares on the other two sides.

Another classic Chinese problem can be solved using Pythagoras's theorem: the *Bamboo problem*, in which *there is a bamboo 10 feet high, the upper end of which being broken reaches the ground 3 feet from the stem. Find the height of the break.* In modern algebraic notation, which the Chinese didn't have, we can call the answer x and the rest of the bamboo y, so that $x + y = 10$ and (by Pythagoras's theorem) $x^2 + 3^2 = y^2$. Solving these equations gives the result.

Plato

The second great period of Greek mathematics took place in Athens, with the founding of Plato's Academy around 387 BC in a suburb of Athens called 'Academy' - that's where the word *Academy* comes from. Plato's Academy is featured in Raphael's fresco *The School of Athens*, with Plato and Aristotle at the top of the steps.

Plato's Academy soon became the focal point for mathematical study and philosophical research, and it is said that over its entrance were the words *Let no-one ignorant of geometry enter these doors*.

Plato wrote a short dialogue called *Meno* in which Socrates asks a slave boy how to double the area of a square. The boy first suggests doubling the side of the square, but that gives four times the area. Eventually he settles on the square based on the *diagonal* of the original square. It's a wonderful example of teaching by experiment and is far removed from anything of Egypt or Mesopotamia.

Plato believed that the study of mathematics provided the finest training for those who were to hold positions of responsibility in the state, and in his *Republic* he discussed at length the importance of the mathematical arts: arithmetic, geometry, astronomy and music.

Plato's book *Timaeus* is also of mathematical interest and includes a discussion of the five regular, or 'Platonic', solids - the *tetrahedron*, *cube*, *octahedron*, *dodecahedron* and *icosahedron* - in which the faces are all regular polygons of the same type and the arrangement of polygons at each corner is the same: for example, the cube has six square faces, with three meeting at each corner. He also linked four of these polyhedra with the Greek elements of earth, air, fire and water, and assigned the cosmos to the dodecahedron, which had only recently become known.

One hundred years later, Archimedes would find all the semi-regular solids, in which the faces are all regular polygons, but they're not all the same. There are just 13 of these - for example, the truncated cube, obtained from a cube by chopping off the corners, is made up of triangles and octagons. They have delightful names, such as the *Great rhombicosidodecahedron*, and were later discussed by Johannes Kepler.

Several of these polyhedra arise naturally as crystals, and the truncated icosahedron, made of pentagons and hexagons, occurs in real life as a football. Interestingly, it turns out that *any polyhedron made from pentagons and hexagons, with three faces meeting at each point (as in a football), must have exactly twelve pentagons*. Such polyhedra occur in chemistry and architecture, too. In chemistry, they're known as *fullerenes* or *buckyballs*, and are molecules whose structure is that of a truncated icosahedron (C_{60}) or some other polyhedron made from pentagons and hexagons. The names are derived from the American architect Buckminster Fuller, who designed the *geodesic dome*, a structure that pound-for-pound is lighter, stronger and more cost-effective than any other. His best-known design was for the American pavilion at the Montreal 'Expo 67' World Fair.

Euclid

Around 300 BC, with the rise to power of Ptolemy I, mathematical activity moved to the Egyptian part of the Greek empire. In Alexandria Ptolemy founded a university that became the intellectual centre for Greek scholarship for the next 800 years - our third period of Greek mathematics.

A number of important mathematicians were associated with Alexandria. The first was Euclid, who lived and taught there around 300 BC.

Euclid's *Elements* was a compilation of results known at the time, organised in a systematic way. It has been the best-selling mathematics book of all time, over more than 2000 years - possibly the most printed book ever, apart than the Bible.

Euclid's *Elements* consists of thirteen books, traditionally divided into three main parts - on plane geometry, arithmetic and solid geometry. *Books I* and *II* deal with the foundations of plane geometry and the geometry of triangles and rectangles; *Books III* and *IV* then proceed to the geometry of circles. *Book V* is on proportion, which is then applied to the geometry of similar figures in *Book VI*. The final three books are on solid geometry, and conclude with the construction and classification of the five Platonic solids. Let's look at some of these.

Book I starts with definitions of basic terms such as point, line and circle. There are then five geometrical *Postulates*, beginning with some constructions, such as the first [*To draw a straight line from any point to any point*] and the third [*To describe a circle with any centre and distance*]. The fifth postulate is much longer: *If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than two right angles.* This one seems to be of a different style from the others - in fact, as we'll see, it caused no end of problems over the next 2000 years.

Let's now look at the first proposition, which asks us *On a given straight line to construct an equilateral triangle.* To do this, we draw the line *AB* and trace out circles with centre *A* and radius *AB* and with centre *B* and radius *AB*. These meet at a point *C*, and the triangle *ABC* is then the required equilateral triangle. Euclid then *proves* that the construction works - that the resulting triangle *is* equilateral. At each stage he refers to a definition or a postulate, and in later propositions there are frequent references to earlier ones.

Another result from *Book I* is Proposition 5, the famous *Pons Asinorum* or 'bridge of asses', that *in an isosceles triangle, the angles at the base are equal to each other*. This result is credited to Thales, and in medieval universities it was often as far as students of Euclid ever reached: if you could cross the bridge of asses, you could then go on to all the treasures that lay ahead! Other propositions in *Book I* include the results that *the angles of a triangle add up to two right angles*, and that *given any triangle, we can construct a rectangle with the same area*. Since any polygon, such as a pentagon, can be split into triangles, we can construct a rectangle with the same area as any given polygon - and when combined with other results, shows us that we can square any polygon.

Book III is on circles - I'll mention just a couple of results: *Prop. 20: In a circle the angle at the centre is double the angle at the circumference when the angles have the same arc as base.* A special case is *Prop. 31: In a circle the angle in a semicircle is right*,and a related result is *Prop. 22: The opposite angles of quadrilaterals in circles are equal to two right angles.*

Such cyclic quadrilaterals, where the corners of a quadrilateral lie on a circle, turned up again in 7th-century India, when Brahmagupta showed that if the sides are *a*, *b*, *c*, *d*, we can obtain formulas for the area of the quadrilateral, and for the lengths of its diagonals. I particularly like his method of constructing cyclic quadrilaterals from right-angled triangles. If we have two right-angled triangles, so that $a^2 + b^2 = c^2$ and $A^2 + B^2 = C^2$, we can calculate the four numbers *aC*, *cB*, *bC* and *cA*. Brahmagupta showed that these always give the lengths of a cyclic quadrilateral, and that, moreover, the diagonals must be at right angles to each other. For example, if we start with the right-angled triangles $3^2 + 4^2 = 5^2$ and $7^2 + 24^2 = 25^2$, the formula gives us the sides 75, 120, 100, 35, which we can scale down to 15, 24, 20, 7, which are the lengths of the sides of a cyclic quadrilateral.

Returning to Euclid, we find that the final three books deal with three-dimensional geometry. Of these, *Book XIII* is the most remarkable. It introduces the five regular solids and shows how to construct them. For example, to make a dodecahedron, we take a cube and add to each face a 'roof' whose proportions are such that the faces all become pentagons; these proportions involve the so-called 'golden ratio', whose geometrical properties are worked out in detail earlier in *Book XIII*.

In this book, Euclid also proved that if we calculate the lengths of the sides of a pentagon, a hexagon and a decagon inscribed in a circle, then these turn out to be the lengths of the sides of a right-angled triangle.

Euclid concluded the *Elements* by proving that the only possible regular solids are the five we know about - there are no more. This is the first ever 'classification theorem' in mathematics, and forms a fitting climax to his great work.

Euclid's *Elements* was warmly received, and quickly replaced all its predecessors and competitors. After the invention of printing, an enormous number of printed versions appeared, such as one published in Venice in 1482, and the first English edition, produced by Henry Billingsley in 1570.

Back in Alexandria, Apollonius was writing his celebrated treatise on *conics*. These curves can be traced back to Menaechmus in the fourth century BC, and arise from slicing a cone in various ways. There are three different types - the *ellipse* (with the circle as a special case), the *parabola* and the *hyperbola*.

A 16th-century edition of Apollonius's *Conics*, has many classical figures on the title page, and the frontispiece of Edmond Halley's 1710 edition shows the philosopher Aristippus, shipwrecked on the island of Rhodes, who noticed some conics that had been drawn in the sand and claimed that the inhabitants must thus surely be civilised.

It was also around this time that the *three classical problems* emerged, of doubling the cube, trisecting the angle and squaring the circle. Each asked for a construction that uses only a straight edge and a pair of compasses - no measuring was allowed. Since I'll be talking about these in January, I won't say any more now.

Archimedes

One of the greatest mathematicians of all time, *Archimedes* is mainly remembered for running naked through the street shouting 'Eureka'. He was a native of Syracuse on the island of Sicily, and it is not known whether he was associated with Alexandria.

His impressive geometrical achievements include the Archimedean polyhedra, calculations of the centres of gravity for a triangle, hemisphere and parallelogram, calculations of volumes and surface areas of the cone and sphere, and his celebrated result (which he apparently wanted engraved on his tomb) that the surface area of any horizontal section of a sphere is the same as that of the surrounding cylinder. And although we can't square the circle, Archimedes showed that parabolas can be squared - the area of any parabolic segment is $^4/_3$ times the area of the enclosed triangle, and the corresponding square can then be obtained.

One of Archimedes' best-known results is his estimation of the number *pi* by drawing hexagons inside and outside a circle and comparing their perimeters with the circumference of the circle. This wasn't accurate, so he replaced the hexagon by a 12-sided polygon, then 24, 48 and 96 sides, obtaining successively better estimates. He found *pi* to be a little bit less than $^{22}/_7$, the value we learned at school, and a bit more than $3^{10}/_{71}$ - this gives a value of about 3.14, correct to two decimal places.

This process was carried further by the Chinese mathematician Liu Hui in his *Classic of the Island in the Sun* of around 263 AD. He continued doubling the number of sides up to 3072 and obtained the value *pi* = 3.14159, in our decimal notation. Even more impressive, around 500 AD Zu Changzhi and his son extended this to polygons with 24,576 sides, thereby obtaining *pi* to six decimal places. They also replaced the approximation $^{22}/_7$ by the much better one $^{355}/_{113}$, which gives *pi* to six decimal places, and wasn't rediscovered in Europe until 1000 years later.

One of my favourite Greek mathematicians is Pappus of Alexandria, from the early 4th century AD. I'd like to mention two contrasting results of his, and later we'll see a third. In his treatise *On the Sagacity of Bees*, he credited bees with geometrical forethought in planning their honeycombs. After showing that there can only be three regular arrangements (with triangles, squares and hexagons), he noted that the bees in their wisdom chose the hexagon, perceiving that it would hold more honey than the other two. As with polyhedra, we can relax the condition that all the polygons should be the same, and we get a whole variety of bathroom-floor tilings which are *periodic*: their patterns repeat for ever.

A few years ago, Roger Penrose (a former Gresham Professor of Geometry) produced his *Penrose tilings*, that are not periodic. They are built from two or more shapes (such as a 'kite' and a 'dart', or a 'chicken' and a 'duck') which appear all over the plane, but not in any regular way, and arise in the study of certain types of crystal.

Pappus's other result is one of the great theorems of mathematics, and you might like to try it for yourself. Draw two lines, and mark points *A*, *B*, *C* on one and *a*, *b*, *c* on the other. Then join *A* with *b* and *c*, *B* with *a* and *c*, *C* with *a* and *b* - this provides three new points of intersection. Amazingly, whatever points you originally chose on the two lines, these new points always lie on a straight line.

Islamic mathematics

The period from 750 to 1400 was an important time for the development of mathematics, since it led to an awakening of interest in Greek and Indian culture in Baghdad, which lay on the trade routes between Europe and the East. Islamic scholars seized on the ancient geometrical texts, translating them into Arabic and extending and commenting on them. Fortunately, we have several primary sources for Islamic mathematics, including the earliest surviving source of Euclid's *Elements*, dating from 888 AD and now in the Bodleian Library in Oxford.

Influential among the Islamic scholars, especially when his works were translated into Latin during the Renaissance, was ibn al-Haitham. Known in the west as Alhazen, he was a geometer whose main contributions were to the study of optics. A celebrated problem, 'Alhazen's problem', asks: *at which point on a spherical mirror is light from a given point source reflected into the eye of a given observer?* An equivalent formulation is: *at which point on the cushion of a circular billiard table must a cue ball be aimed so as to hit a given target ball?*

Alhazen also tried to prove the 'parallel postulate'. We recall that Euclid's *Elements* is built on five 'postulates', or self-evident truths. Four of these are straightforward, but, the fifth postulate is different in style, resembling a theorem that ought to be provable from the others: if two lines include angles x and y whose sum is less than 180°, then these lines must meet if extended indefinitely.

Over the centuries many mathematicians tried to prove this fifth postulate, often by recognising other results and proving them equivalent to it; it is then sufficient to prove any of the other results. The most familiar of these is the 'parallel postulate': *Given any line l and any point p not lying on l, there is a line parallel to l that passes through p*. Alhazen gave a 'proof' that involved moving a line perpendicular to the given one, so that its other end traces out a parallel one. This attempted proof was criticised by Omar Khayyam, a mathematician and poet who wrote on geometry and who is known in the west for his collection of poems known as the *Rubaiyat*.

The Islamic world soon spread along the northern coast of Africa and up through southern Spain, and Córdoba became the scientific capital of Europe. Islamic decorative art and architecture spread throughout the region, where celebrated examples include the magnificent arches in the mosque at Córdoba and the variety of geometrical tiling patterns in the Alcazar in Seville and the Alhambra in Granada.

Renaissance art

We now turn our attention to the development of perspective. One notable feature of Renaissance painting was that, seemingly for the first time, painters became interested in depicting three-dimensional objects realistically, giving visual depth to their works, as contrasted with earlier works such as the Bayeux tapestry where such depth is not to be found. This soon led to the formal study of geometrical perspective.

One of the first to investigate perspective seriously was Filipo Brunelleschi, who designed the self-supporting octagonal cupola of the cathedral in Florence. His ideas were developed by his friend Leon Battista Alberti, who presented mathematical rules for correct perspective painting and stated in his *Della pittura* [On painting] that 'the first duty of a painter is to know geometry'.

Albrecht Dürer was a celebrated German artist and engraver who learned perspective from the Italians and introduced it to Germany. He produced a number of drawings showing how to realise perspective, and his famous engravings, such as *St Jerome in his study*, show his effective use of it.

The Universities

Around this time, European universities started to appear - in Bologna, Paris and Oxford. By the beginning of the 13th century, Oxford University had a recognised head who in 1214 was given the official title of 'Chancellor'. This was Bishop Robert Grosseteste, who taught at Oxford in the early 1400s and founded the tradition of scientific thought in medieval Oxford. He was particularly interested in mathematics, and wrote in praise of geometry as follows: *The usefulness of considering lines, angles and figures is the greatest, because it is impossible to understand natural philosophy without them. By the power of geometry,* *the careful observer of natural things can give the causes of all natural effects.*

Until about 1600, the curriculum was in two parts. The first part, studied by those aspiring to a Bachelor's degree, was based on the ancient *trivium* of grammar, rhetoric and logic. The second part, leading to a Master's degree, was based on the *quadrivium*, the Greek mathematical arts of arithmetic, geometry, astronomy and music; the geometry would have been mainly extracts from the first few books of Euclid. Together, these made up the *seven liberal arts*.

The invention of printing

Johann Gutenberg's invention of the printing press (around 1440) revolutionised mathematics, enabling classic works to be widely available for the first time. Previously, scholarly works, such as the classical texts of Euclid, Archimedes and Apollonius had been available only in manuscript form, but the printed versions made these works much more accessible.

At first the new books were printed in Latin or Greek for the scholar, and many scholarly editions appeared. We've already seen printed editions of Euclid and Apollonius, and another attractive work was Nicolas Oresme's *Latitude of Forms*, a geometrical approach to the study of motion.

Increasingly there were works in the vernacular, such as Billingsley's *Euclid*, which we saw earlier. Possibly the most famous vernacular writer in England was Robert Record, whose books include the *Grounde of Artes* (on arithmetic) and the *Pathway to knowledge* (on geometry).

The production of books rapidly led to a standardisation in terminology and notation. Record introduced the term *straight line*, which is still used, and also several entertaining examples that unfortunately didn't catch on, such as *sharp angle* and *blunt corner* for acute and obtuse angles, *touch line* for a tangent, and *threelike* for an equilateral triangle.

Gresham College

An interesting institution founded in the late 16th century for the teaching of geometry and other subjects was Gresham College in London. The Gresham Chairs arose from the will of Sir Thomas Gresham, which provided £50 per year for each of seven professors to read lectures in Divinity, Astronomy, Music, Geometry, Law, Physic and Rhetoric. In particular, the geometry lectures were to be read twice each week, in Latin (for visiting scholars) and English (for mariners and any others interested): *The geometrician is to read as followeth, every Trinity term arithmetique, in Michaelmas and Hilary terms theoretical geometry, in Easter term practical geometry.*

The first Gresham Professor of Geometry, in early 1597, was Henry Briggs, who invented the method of long division that we all learnt at school, and was also the co-inventor (with John Napier) of logarithms. After leaving Gresham College in 1620 he took up the newly-founded Savilian Chair of Geometry in Oxford, until his death in 1631.

Various geometrical results

I'd now like to turn to a few geometrical problems discussed in the early 17th century.

Around 1637 Roberval solved a famous problem involving a popular curve of the time. The problem was to find the area under one arch of a *cycloid*, the curve traced out by a spot of mud on a bicycle tyre as the cycle is wheeled along. Roberval found it to be three times the area of the rolling circle. His ingenious solution involved relating the area under the cycloid to that under another curve whose area could easily be calculated.

An important type of geometry, which developed from the study of perspective around this time, was projective geometry. We've already seen Pappus's theorem, concerning six points joined up in a certain way, and around 1640 the 16-year-old Blaise Pascal obtained his 'hexagon theorem' about six points on a conic joined up in a similar way. Another result from around this time was Desargues theorem about two triangles in perspective.

A major development in geometry occurred in 1637 with Descartes's celebrated treatise *Discourse on the method*. This included a lengthy appendix on geometry, containing his fundamental contributions to analytic geometry.

One of the problems tackled by Descartes in the *Discourse* was an old problem of Pappus (c. 300 AD), which asked for the path traced out by a point that moved in a specified way relative to a given number of fixed lines. The statement of the problem is complicated, but Descartes' approach was revolutionary - he named two particular lengths arising in the problem x and y and then calculated all the other lengths in terms of them, producing a quadratic equation. This showed that the desired path was a conic (parabola, ellipse or hyperbola).

Although Descartes did not define the rectangular *Cartesian coordinates* (x, y) that are named after him, his use of algebraic methods to solve Pappus's problem set in train a gradual movement from geometry to algebra that continued for about 100 years, culminating in the work of Euler, probably the most prolific mathematician of all time.

In his *Introductio* of 1748 Euler actually *defined* the *conic sections*, not as sections of a cone, but in terms of their algebraic equations. Starting with the equation $y^2 = \alpha + \beta x + \gamma x^2$, he showed that we get an ellipse if γ is negative, a parabola if γ is 0, and a hyperbola if γ is positive. He then yanked the whole argument up to three dimensions, to *quadrics*, which come in seven types, and studied them algebraically, discovering the *hyperbolic paraboloid* in the process.

Another preoccupation of Euler's was mentioned in a letter to his friend Christian Goldbach, in 1750. For centuries, as we have seen, mathematicians had been studying examples of *polyhedra*, but Euler was the first to look at their edges, and to ask how many there are. In his letter Euler made the remarkable observation that the numbers of vertices (corners), edges and faces are always related by a simple formula:

(*no. of faces*)+(*no. of vertices*)=(*no. of edges*)+ 2;

for example, a cube has 6 faces, 8 vertices and 12 edges and 6 + 8 = 12 + 2. This formula is sometimes incorrectly credited to Descartes, but Descartes didn't have the terminology or motivation to derive it. Euler's explanation of the result, however, was deficient, and a correct proof was not given until forty years later, by the algebraist and number-theorist Legendre.

In his later life, Euler produced a delightful result on pure geometry. If we take any triangle, three particular points of interest are the *orthocentre* (the meeting point of the perpendiculars from the vertices to the opposite sides), the *centroid* (the meeting point of the three lines joining a vertex to the midpoint of the opposite site), and the *circumcentre* (the centre of the circle surrounding the triangle). Euler proved the pretty result that these three points always lie in a straight line, now called the *Euler line* of the triangle, and that the centroid always lies exactly one third of the distance between the other two.

Constructing polygons

Using only straight-edge and a pair of compasses, Euclid had shown how to construct an equilateral triangle, a square and a regular pentagon, as well as polygons with 6, 10, 12 and 15 sides. Which regular polygons can be constructed in this way? In the 1790s Gauss described how to construct a regular 17-sided polygon, and then solved the general problem completely: *a regular polygon with n sides can be constructed if and only if n is obtained by multiplying any number of different Fermat primes* (primes of the form 2-to-the-power-2^r + 1 (such as 3, 5, 17, 257 and 65537) *and then doubling as often as you please*. So we can construct regular polygons with 80 sides (since $80 = 2^4 \times 5$) and 102 sides (since $102 = 2 \times 3 \times 17$).

19th-Century France and Germany

Moving to France, out of the turbulent years of the French Revolution and the rise to power of Napoleon Bonaparte came several important developments in mathematics. Napoleon was an enthusiast for mathematics, and one of his greatest supporters was the geometer *Gaspard Monge*, who accompanied him on the Egyptian expedition of 1798. Monge taught at a military school, where he studied the properties of lines and planes in 3-dimensions. While investigating positionings for gun emplacements in a fortress, he greatly improved on the known methods for projecting 3-dimensional objects on to a 2-dimensional plane; this subject became known as 'descriptive geometry'.

Monge's skill as a teacher at the newly founded École Polytechnique helped to establish descriptive geometry and to inspire his talented students. The best of these was Poncelet, the 'father' of modern projective geometry. While languishing in jail during Napoleon's invasion of Russia, Poncelet developed the idea of a 'projective transformation' and studied the properties of figures that remain unchanged under them. This work was influenced by that of Monge and Desargues, but much of it was intuitive and unrigorous and was not well received by the mathematical powers-that-be in Paris, such as Cauchy.

But one idea that was truly revolutionary was that of *duality*, a real break with the past. Poncelet took a conic, such as an ellipse, and selected a point outside it (called a *pole*), drew tangents to the conic, and joined the points where the tangents meet the conic; he obtained a line called a *polar*. As he moved the pole along a line, he obtained different lines all passing through the same point. Using this idea he was able to take general theorems about points lying on lines and recast them as theorems about lines passing through points - prove one theorem and get another one for free!

Other French mathematicians, such as Gergonne, tried to carry this idea to its logical conclusion, claiming that points and lines should be regarded as interchangeable concepts with the same logical status:

any two points determine a line; *any two lines determine a point*.

(This is not our usual Euclidean geometry, where two lines determine a point only when they are not parallel.) This idea was as profound as it was controversial, but caused difficulties which the French mathematicians were unable to resolve.

The scene next moved to Germany, where Möbius and others attacked the problem using algebra, much as Descartes had done two centuries earlier when he introduced coordinates x and y in order to solve the geometrical problem of Pappus. Möbius's idea of 1827, which he called his 'barycentric coordinates', was to represent each point, not by two coordinates $[a, b]$, but by three $[a, b, c]$, determined only up to its multiples, so that [1, 2, 3] and [2, 4, 6] are the same point, and to represent lines by equations of the form $ax + by + cz = 0$, instead of the usual $ax + by = c$. The duality between points and lines is then very simple: $[a, b, c] \leftrightarrow ax + by + cz$. These algebraic ideas proved to be highly successful and were developed by several mathematicians, mainly in Germany.

Non-Euclidean geometry

As we have seen, Euclid's *Elements* are built on five 'postulates': four of these are straightforward, but the fifth is different in style, resembling a theorem that ought to be provable from the others.

Over the centuries many mathematicians tried to prove the fifth postulate, often by recognising other results and proving them equivalent to it; it is then sufficient to prove any one of the other results. The first really significant advance appeared in the book *Euclid vindicated*, written by the Italian Gerolamo Saccheri in 1733. He considered a geometry in which the parallel postulate is *not* assumed, and he considered the sum of the angles in a triangle.

If the angle-sum is 180°, we get Euclid's geometry. Saccheri proved that if the angle-sum is more than 180° then we get a contradiction: the parallel postulate can be proved both true and false, so there is *no* geometry with this property. Finally, he tried to repeat the process for triangles with angle-sum less than 180°, but he was unable to do so. If he *had* been successful, the parallel postulate would then have been deducible from the other axioms, showing that Euclid's geometry is the only one.

Saccheri was proved wrong in spectacular fashion. Around 1830, working independently in Russia and Hungary, Nikolai Lobachevsky and János Bolyai constructed a geometry in which the angle sum of every triangle is less than 180°. This geometry satisfies the first four postulates, but not the fifth one, and is a very strange geometry. For a start, given any line l and any point p not lying on l, there are *infinitely many* lines through p which are parallel to l. Also, if two triangles are similar (that is, they have the same angles), then they are congruent - and that's certainly not true in Euclidean geometry.

Sadly, Bolyai and Lobachevsky never enjoyed the credit due to them for their spectacular discovery, for it was not until after their deaths that their geometry was understood and generalised.

In particular, the German mathematician Bernhard Riemann (of Riemann hypothesis fame) studied some generalised ideas of distance and curvature in the 1850s, and then used these to obtain *infinitely many* different geometries, each one equally valid and each one a candidate for the physical space we live in; indeed, it was one of these geometries, rather than Euclidean geometry, that proved to be the natural setting for Einstein's theory of relativity many years after Riemann's death. This was truly a revolution of deepest significance in the way we think about the world around us.

By 1870 the world of geometry had become very confused. In addition to the wealth of non-Euclidean geometries, there were the Euclidean and projective geometries and many others besides. The remaining years of the century saw many attempts to sort out the mess and to put the various geometries on a firm axiomatic foundation.

The most famous of these attempts was the so-called *Erlanger Programm* of the German geometer Felix Klein, proposed at his inaugural lecture at the University of Erlangen in 1872. Klein described a geometry in general terms as a set of points (such as the plane) with a collection of transformations on them - for example, Euclidean geometry consists of the transformations that do not change an object's size or shape, such as rotations, reflections and translations. By such considerations Klein was able to classify *all* geometries and to prove the surprising result that, in some sense, *every* geometry is contained inside projective geometry.

Much time was spent in the second half of the 19th century sorting out such difficulties by developing mathematics as a hierarchical structure based on sets and elements, just as Euclid had built up geometry from the undefined notions of point and line. By the end of the century Moritz Pasch and David Hilbert had formalised the Euclidean, projective and non-Euclidean geometries.

Surfaces

I'd like to conclude with a couple of topics from the 20th century.

We are all familiar with surfaces such as the sphere and the bagel-shaped torus. Slightly more complicated are 'pretzels', which are like toruses with several holes; an example is the double-torus with two holes. These surfaces are *orientable* - they have an inside and an outside, and each can be obtained from a sphere by attaching a number of 'handles' to it. Other surfaces are *non-orientable* - these include the *Möbius strip*, which has only one side and is obtained by twisting one end of a long strip of paper through 180 degrees and glueing the ends, and objects called the *projective plane* and the *Klein bottle* which cannot be constructed in three-dimensional space. To obtain a non-orientable surface, take an orientable one, cut some circular holes in it, and stick on Möbius strips along their common boundary (you can't do this in three dimensions!) - this is called adding a *cross-cap*.

One project in the 20th century was to classify all surfaces, and this turned out to have a surprisingly simple answer. For the orientable case, every surface is essentially a sphere with a number of handles attached. For the non-orientable case, every surface is essentially a sphere with a number of cross-caps attached.

Fractals

I'll ask the question: *how long is the coastline of Britain?* If you try to measure it with a ruler, or if you look at the country from far above, you can estimate the length of the coastline. But as you measure more accurately, or get closer to Earth, you become aware of more and more inlets and bays, and the length increases accordingly. In fact, the closer you get, the longer the coastline seems to become - imagine walking around it with a pedometer - and then with a six-inch ruler, and so on. In fact, the coastline of Britain has infinite length, even though it encloses a finite area.

This property is a standard feature of fractal patterns, a topic of great interest in the 20th century. In 1906 the Swedish mathematician Helge von Koch described his *snowflake curve*. To construct it you take an equilateral triangle, and then replace the middle third of each side (which we can think of as the base of a smaller equilateral triangle) by the other two sides of the triangle, giving a 'peak' on each side of the original triangle. Now repeat this process with each of the lines in the resulting picture. If we carry this on for ever, we obtain the snowflake curve, which has infinite length, encloses a finite area.

A new way of obtaining geometrical fractal patterns was discovered by Benoit Mandelbrot. Consider a transformation in which we take a complex number (one of the form $a + b$i, where i is $\sqrt{-1}$), square it to give a new number, and continually repeat the process. For example, if $c = 0$ and we start with the point 2, we get successively 4, 16, 256, ... going off to infinity, whereas if we start with the point $\frac{1}{2}$, we get successively $\frac{1}{4}$, $\frac{1}{16}$, $\frac{1}{256}$, ... tending to 0. Any point inside the circle with centre 0 and radius 1 stays inside the circle, while any point outside the circle goes to infinity; those points on the circle stay on the circle. We call this boundary circle the *Julia set* corresponding to $c = 0$ (after the French mathematician Gaston Julia), and its inside is the *keep set* (because we keep its points in sight).

Different values for c give a wide range of different boundary curves (Julia sets): for example, $c = 0.25$ gives us a 'cauliflower', while $c = -0.123 + 0.745i$ gives us a shape rather like a rabbit. The Julia sets for some values of c are in one piece, while others are in several pieces. If we draw a picture of all the complex numbers c for which the Julia set is in one piece, we obtain a bizarre picture called the *Mandelbrot set*. This fascinating set arises in the study of chaos theory and has given rise to a whole range of beautiful geometrical designs under the heading of *fractal art*.

Conclusion

This brings to an end 4000 years of geometry, and I hope you found the journey worthwhile. If not, may I recall a student in Alexandria who had begun to read geometry with Euclid and asked him 'What advantage shall I get by learning these things?' Euclid called his slave and said 'Give him threepence, for he must needs make profit out of what he learns.' I do hope that you don't *all* want to claim your threepences from me.