Euler
Transcript

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Euler

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"Read Euler, read Euler, he is the master of us all…." So said Pierre-Simon Laplace, the great French applied mathematician.

Leonhard Euler, the most prolific mathematician of all time, wrote more than 500 books and papers during his lifetime about 800 pages per year with another 400 publications appearing posthumously; his collected works already fill 73 large volumes tens of thousands of pages with more volumes still to appear.

Euler worked in an astonishing variety of areas, ranging from the very pure the theory of numbers, the geometry of a circle and musical harmony via such areas as infinite series, logarithms, the calculus and mechanics, to the practical optics, astronomy, the motion of the Moon, the sailing of ships, and much else besides. Euler originated so many ideas that his successors have been kept busy trying to follow them up ever since.

Many concepts are named after him, some of which Ill be talking about today: Eulers constant, Eulers polyhedron formula, the Euler line of a triangle, Eulers equations of motion, Eulerian graphs, Eulers pentagonal formula for partitions, and many others.

So with all these achievements, why is it that his musical counterpart, the prolific composer Joseph Haydn, creator of symphonies, concertos, string quartets, oratorios and operas, is known to all, while he is almost completely unknown to everyone but mathematicians?

Today, Id like to play a small part in restoring the balance by telling you about Euler, his life, and his diverse contributions to mathematics and the sciences.

A potted history of his life divides into four main periods. He was born is Basel, Switzerland, on 15 April 1707, where he grew up and went to university. At the age of twenty he went to Russia, to the St Petersburg Academy, where he became head of the mathematics division. Things got so bad there that he went to Berlin in 1741, where he stayed for twenty-five years. Things got so bad there, for different reasons, that he returned to St Petersburg in 1766, where he spent the rest of his life, dying in 1783.

Leonhard Euler was born on 15 April 1707 in Basel, in Switzerland, where his father was a Calvinist pastor of modest means, who wished his son to follow him into the ministry. On entering the University of Basle at the age of 14, not unusual in those days, the young Euler duly studied theology and Hebrew, while also writing on law and philosophy.

While there, he encountered Johann Bernoulli, possibly the finest mathematician of his day, who was impressed with Eulers mathematical abilities and agreed to give him private teaching every Saturday, quickly realising that his pupil was something out of the ordinary. Euler also became close friends with Johanns sons, Daniel and Nicholas, although Nicholas died soon after.

Euler took his Masters degree at the age of 17, and entered divinity school to train for the ministry:

I had to register in the faculty of theology, and I was to apply myself to the Greek and Hebrew languages, but not much progress was made, for I turned most of my time to mathematical studies, and by my happy fortune the Saturday visits to Johann Bernoulii continued.

Eventually, Bernoulli persuaded Eulers reluctant father that his talented son was destined to become a great mathematician, and Euler left the ministry.

His first mathematical achievement occurred when he was just twenty. The Paris Academy had proposed a prize problem involving the placing of masts on a sailing ship, and Eulers memoir, while not gaining the prize, received an honourable mention. (Later, he was to win the prize twelve times!)

He next applied for the Chair of Mathematics at the University of Basle. He didnt get it, but meanwhile Daniel Bernoulli had taken up a position at the St Petersburg Academy in Russia, and invited Euler to join him there. The only available position was in medicine and physiology, but jobs were scarce so Euler learned medicine and physiology in the process of which his study of the ear led him to investigate the mathematics of sound and the propagation of waves.

In the event, Fate dealt a cruel blow. On the very day that he arrived in Russia, the liberal Empress Catherine I, who had set up the Academy, died. The heir was still a boy, and the faction that ruled in his place regarded the Academy as something of a luxury. Euler found himself in physics, rather than medicine, which was probably a relief for all those future patients who might not have appreciated being operated on with straight-edge and compass. Euler decided to keep his head down and get on with his work, while living at the home of Daniel Bernoulli and working closely with him.

In 1733, Daniel Bernoulli had had enough of the problems of the Academy and returned to an academic post in Switzerland, while Euler, still aged only 26, replaced him in the Chair of Mathematics. He determined to make the best of a difficult situation and settle down, so he got married, and had thirteen children, of whom only five survived to adolescence. Euler always enjoyed having the children around him he even managed to carry out his mathematical researches with a baby on his lap!

The 1730s were productive years for him, and Id like to show you some of the areas in which he became involved. At the same time, he was also acting as a scientific consultant to the government preparing maps, advising the Russian navy, testing designs for fire engines, and writing textbooks for the Russian schools although he drew the line when he was asked to cast a horoscope for the young Czar.

The first topic Id like to mention is the theory of numbers, an area to which he contributed throughout his life. In December 1729, he received a letter from his colleague Christian Goldbach, who is best remembered for the Goldbach conjecture involving prime numbers numbers with no proper factors, such 11 and 13, but not 15 which has the factors 3 and 5. Goldbachs conjecture is that every even number can be written as the sum of two prime numbers for example, 10 = 5 + 5, 20 = 13 + 7, 30 = 19 + 11, and so on.

Goldbach wrote to Euler about the so-called Fermat primes of the form (2 to the power 2n) + 1: for example, for n = 0, 1, 2, 3 and 4 we get the prime numbers 3, 5, 17, 257 and 65537. The seventeenth-century French mathematician Pierre de Fermat had conjectured that these numbers are always prime numbers, but Euler managed to show that the very next one, a ten-digit number, is actually divisible by 641. Since then, no other Fermat number has been shown to be prime, so it was rather an unfortunate conjecture.

(Demonstration of Eulers proof)

Eulers way with numbers, and his calculating abilities, became legendary. One day, two students were trying to sum a complicated progression and disagreed over the fiftieth decimal place. Euler simply calculated the correct value in his head. This led the French physicist Francois Arago to exclaim:

He calculated without any apparent effort, just as men breathe, as eagles sustain themselves in the air.

Another challenge he was given was to find four different numbers, the sum of any two of which is a perfect square. Euler managed to produce the quartet:

18530, 38114, 45986 and 65570.

His major work on number theory was much later, so well return to this topic later.

A very different preoccupation in the 1730s was his work on infinite series finding the sum of infinitely many terms. For example, we have:

1 + 1/2 + 1/4 + 1/8 + 1/16 + . . . = 2,

but can we sum the following series?

1 + 1/2 + 1/3 + 1/4 + 1/5 + . . . , where the denominators increase by 1;
1 + 1/2 + 1/3 + 1/4 + 1/5 + . . . , where the denominators are all prime numbers;

1 + 1/2 + 1/3 + 1/4 + 1/5 + . . . , where the denominators are all perfect squares.

Both the second and third series have no sum they are divergent.

(Demonstration that the second series diverges)

However, if we look at the first n terms, then their sum is very close to log n in fact, as Euler proved in the 1730s, when n becomes large, their difference gets closer and closer to a strange number now called Eulers constant. Its about 0.5772…, but no-one knows anything much about it we dont even know whether its a fraction, a/b.

One other puzzle that certainly exercised many minds at the time was to find the exact sum of the fourth series, and one of Eulers earliest achievements was to prove that the sum of this series is the highly non-intuitive number p2/6. This number occurs throughout statistics for example, if everyone in this room were to write down ten pairs of numbers, then the proportion of all those pairs that have no common factor would be very close to its reciprocal, 6/p2.

Eulers proof of this is hardly rigorous by todays standards he was not above writing such equations as 1 1 + 1 1 + 1 1 + . . . = 1/2 if it arose from his calculations.

(Demonstration of Eulers proof)

As a bit of light relief, lets look at a recreational puzzle that Euler solved in 1735: its the problem of the seven bridges of Königsberg. The problem is to cross over each of the bridges just once, and return to your starting point. Euler employed a counting argument, counting the number of bridges out of each land area, and proved that this cannot be done.

(Demonstration of Eulers solution)

Eulers considered this problem in the context of a desire that Leibniz expressed for a type of geometry that doesnt involve metrical ideas such as length or distance. Its what we now call topology, or rubber-sheet geometry the problem is the same if we draw it on rubber and stretch it.

Heres a letter that Euler wrote to Giovanni Marinoni, Court Astronomer in Vienna, in 1736, with his own drawing of the bridges, and describing what he thought of the problem:

This question is so banal, but seemed to me worthy of attention in that neither geometry, nor algebra, nor even the art of counting was sufficient to solve it. In view of this, it occurred to me to wonder whether it belonged to the geometry of position, which Leibniz had once so much longed for. And so, after some deliberation, I obtained a simple, yet completely established, rule with whose help one can immediately decide for all examples of this kind whether such a round trip is possible.

These days, Eulers solution of the Königsberg bridges problem is considered as the first paper in graph theory its now solved by looking at a graph, or network, with points representing the land areas and lines representing the bridges. But Euler never did this the graph that represents this puzzle was not drawn for another 150 years.

In the same year, Euler published his first treatise, Mechanica, on the dynamics of a particle. However, his most important work in this area came later, in 1750, with his work on the motion of rigid bodies both free, and rotating about a point. By choosing the point to be the origin of coordinates, and axes aligned along the principal axes of inertia of the body, he obtained what are now known as Eulers equations of motion. The important concept of moment of inertia is also due to him. Even later, in 1776, he proved a very basic theorem, that the rotation of a rigid body about a point is always equivalent to a rotation about a line through that point.

Much of this work used differential equations equations that involved the latest developments in the differential calculus. These ideas were to continue to be developed during the rest of the seventeenth century, culminating in the five-volume treatise on celestial mechanics by Laplace around 1800.

It was around this time, in the late 1730s, that Euler went blind in his right eye. Although he attributed it to overwork, particularly for some close work that he had been doing on cartography, it was more probably due to an eye infection. This didnt diminish his productivity, however. He continued to write on acoustics, musical harmony, ship-building, prime numbers, and much more besides.

In 1741, with his fame preceding him, Euler received an invitation from Prussias Frederick the Great to join the newly vitalised Berlin Academy. With the political situation in Russia still uncertain, he accepted it, and stayed there for 25 years.

At first, Euler got on well with Frederick, bringing him strawberries from his garden. But later, especially after the seven years war between Germany and Russia things began to cool, as Frederick started to take more and more interest in the workings of the Academy. Being very sophisticated, cultured and witty at least, he thought so he found Euler to be lacking in sophistication, rather a country bumpkin, in fact. In return, Euler found Frederick pretentious, not to say petty and rude Frederick even referred to him as my cyclops. The story is also told that he was reproached by the Queen of Prussia, Fredericks mother, for not conversing. Madame, he replied, I come from a country where, if you speak, you are hanged.

Even so, he still managed to work on a dazzling range of topics, writing works in the 1740s and 1750s on the theory of tides, the motion of the moon, hydrodynamics (the flow of a river), and the wave motion of vibrating strings.

His most important work was probably his text on functions, the Introductio in Analysin Infinitorum, published in 1748, during his Berlin years. This work presented the calculus in terms of the basic idea of a function indeed, Euler introduced the notation f(x) for a function of x. Other notations that he introduced were S (for summation), i (the square root of 1) and e (the exponential number). He also popularised the notation for p, although that was actually due to William Jones in 1706.

In the 1748 Introductio, Euler expressed certain well known functions as infinite series or power series. He believed that every function, such as sin x, can be expanded in powers of x. Indeed, Newton, Leibniz and others were familiar with such expansions as

These formulas had also been discovered some centuries earlier by Indian mathematicians.

He then introduced one of the greatest masterstrokes in the whole of mathematics. At first, the trigonometrical functions sine and cosine seem to have nothing in common with the exponential function ex, but if we introduce the complex number i, and play around with the power series, we can easily deduce the fundamental formula linking them, eix = cos x + i sin x, from which we deduce, on putting x = p, that eip + 1 = 0, an equation that include the five great constants of mathematics.

There were many other interesting things in the Introductio. Over the past one hundred years, since Descartes, there had been a gradual swing from geometry towards algebra, and this reached its climax when Euler actually defined the conic sections, the ellipse, parabola and hyperbola, not as sections of a cone (as their name suggests), but in terms of their algebraic equations. Starting with the equation y2 = a + ßx + ?x2, he showed that we get an ellipse if ? is negative, a parabola if ? is zero, and a hyperbola if ? is positive. He then yanked the whole argument up to three dimensions, to quadrics, which come in seven types, and studied them algebraically, discovering the hyperbolic paraboloid in the process.

Yet another interesting topic in the 1748 Introductio is partitions, or divulsions of integers, as Leibniz called them when he introduced them in a letter to Bernoulli. In how many ways can we split up a number into smaller numbers?

Let p(n) be the number of partitions of n for example, p(4) = 5, corresponding to the five partitions 4, 3+1, 2+2, 2+1+1 and 1+1+1+1 the order doesnt matter. So we can draw up a table of values but how would you show that p(200) has the value 3,972,999,029,388?

To find this number we use Eulers pentagonal number formula, which he obtained in his 1748 Introductio in analysin infinitorum and which yields p(N) by iteration. Even now, its still the most efficient way of finding p(N).

(Demonstration of Eulers pentagonal number theorem)

A particularly nice result on partitions, due to Euler, concerned odd and distinct partitions. An odd partition is one where all the separate terms are odd: for example, the number 9 has eight odd partitions 9, 7+1+1, 5+3+1, 5+1+1+1+1, 3+3+3, 3+3+1+1+1, 3+1+1+1+1+1+1 and 1+1+1+1+1+1+1+1+1.

A distinct partition is one where all the terms are different: for example, the number 9 has eight distinct partitions 9, 8+1, 7+2, 6+3, 6+2+1, 5+4, 5+3+1 and 4+3+2.

Euler proved, using his generating functions, that for any number, the number of odd partitions is always equal to the number of distinct partitions an intriguing and unexpected result.

Another preoccupation was mentioned in a letter to Goldbach, in 1750. Euler had been looking at polyhedra, such as a cube, and observed that the numbers of vertices (corners), edges and faces are always related by the formula:

(no. of faces) + (no. of vertices) = (no. of edges) + 2.

For example, the cube has 6 faces, 8 vertices and 12 edges and 6 + 8 = 12 + 2. This formula is sometimes credited to Descartes, but Descartes didnt have the terminology or motivation to derive it: it was Euler who introduced the concept of an edge. Euler was unable to prove the result, however the proof came forty years later, by the algebraist and number theorist Adrien-Marie Legendre.

Eulers most popular book was his Letters to a German princess. This was a multi-volume masterpiece of exposition (Euler always was a very clear writer) that he produced because he was asked to give elementary science lessons to the princess of Anhalt Dessau. The result was a collection of over two hundred letters that Euler wrote on a range of scientific topics, including gravity, astronomy, light, sound, magnetism, logic, and much else besides. He wrote about why the sky is blue, why the moon looks larger when it rises, why the tops of mountains are cold, even in the tropics, and even the electrisation of men and animals, whatever that was. It was one of the best books ever written on popular science.

The last of his Berlin works that I want to mention was his 1755 massive tome on the differential calculus, containing all the latest result, which he followed in 1768-70 by a three-volume treatise on the integral calculus. Regrettably, I have no time to talk about these.

With all his difficulties with Frederick the Great, Euler must have felt very relieved when in 1766, when he was 59, he received an invitation from Catherine the Great to return to St Petersburg. Things had improved greatly, thanks to the enlightened Empress, and Euler was received royally.

He continued to work enthusiastically, soon producing a delightful result on pure geometry. If we have a triangle, then there are three particular points of interest. The first is the centroid the meeting point of the three lines joining a vertex to the midpoint of the opposite site. The second is the orthocentre the meeting point of the perpendiculars from the vertices to the opposite sides. The third is the circumcentre the centre of the circle surrounding the triangle. Euler proved the pretty result that these three points all lie in a straight line now called the Euler line of the triangle and that the centroid lies exactly one third of the distance between the other two.

I mentioned earlier that Euler had a life-long interest in number theory. In his later life, Euler was to work on results associated with Fermat. Fermats little theorem states that, if p is a prime number, and a is any number that is not divisible by p, then ap1 1 must be divisible by p. For example, if we take the prime p to be 29 and a to be 48, then we can deduce that 4828 1 is divisible by 29. In 1760, Euler extended this result to numbers other than primes, introducing the Euler f-function and proving that, for any numbers a and n, af(n) 1 is divisible by n.

Yet another connection with Fermat was provided by Fermats last theorem. We all know that there are numbers a, b, c that satisfy a2 + b2 = c2 for example, 32 + 42 = 52, or 52 + 122 = 132, but Fermat conjectured, in the margin of his copy of Diophantuss Algebra, that for any power n higher than 2, an + bn could never equal cn. In his number theory book of 1770, Euler proved this for n = 3 (the sum of two cubes cannot equal another cube), and for n = 4 (the sum of two fourth powers cannot equal another fourth power).

The last few years of Eulers life, though more peaceful than his earlier ones, were full of personal tragedies. In 1771 his house burned down, with the loss of his library, and almost his life, and fortunately his manuscripts were saved. Shortly after, his beloved wife died. And finally he lost the sight of his other eye but again, his productivity remained undiminished, as he wrote on slates with his two sons as amenuenses.

He worked up to the very end. In his Eulogy by the Marquis de Condorcet, we read about his final afternoon:

He had retained all his facility of thought, and apparently, all his mental vigour: no decay seemed to threaten the sciences with the sudden loss of their great ornament. On the 7th of September 1783, after amusing himself with calculating on a slate the laws of the ascending motion of air balloons, the recent discovery of which was then making a noise all over Europe, he dined with Mr Lexell and his family, talked of Herschels planet (Uranus), and of the calculations which determine its orbit. A little after, he called his grandchild, and fell a playing with him as he drank tea, when suddenly the pipe, which he held in his hand, dropped from it, and he ceased to calculate and to breathe.