How Mathematicians Think About Patterns Transcript

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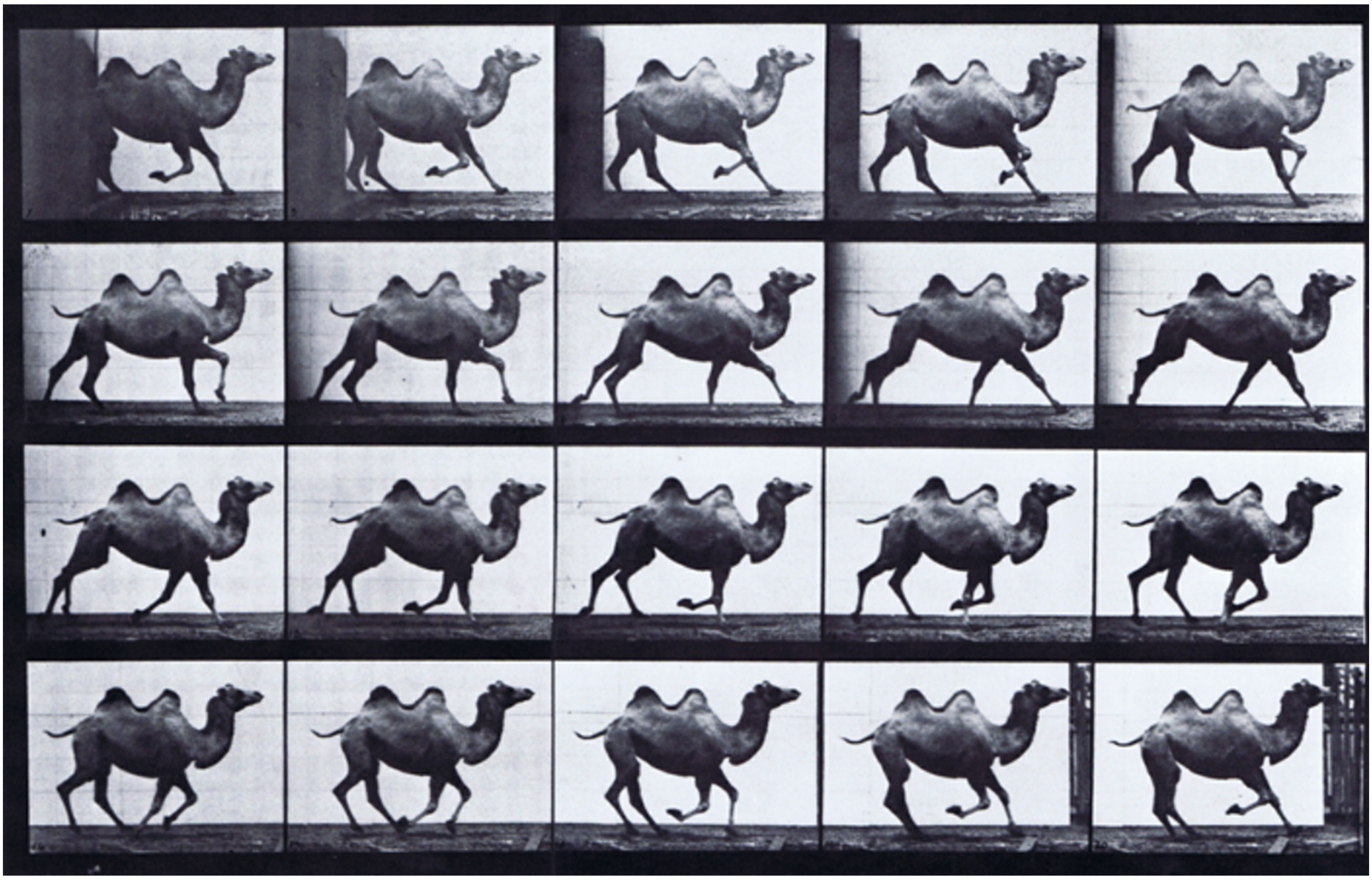

28 February 2013

How Mathematicians Think About Patterns?

Professor Ian Stewart

I want to talk about how mathematicians think about patterns. Well, I want to talk about how I think about patterns, and how quite a lot of mathematicians think about patterns. This is not the only way to do it, but I want to bring out two features of patterns which are particularly important, which is the role of symmetry and the role of what is called symmetry breaking, which is clearly related, and I want to show you how patterns can be recognised, classified, but also understood to some extent in terms of those two basic mathematical ideas. I am going to go through four basic types of pattern: snowflakes; animal markings – now, I gather there was an earlier talk in this little series about zebra stripes. I do not know exactly what was done in that, but there will be a little bit of overlap, and I hope I will not contradict what was already said! I hope it will be a complementary view. There are many ways to think about these things.

In the Christmas Lectures, we brought one of these animals into the lecture room. You can find it on the Royal Institution website – you can actually download and watch the lectures, and I think it was lecture number five. So, we brought a six month old tigress into the lecture room, so people on the front row, you should imagine a tigress being brought in on the end of a chain, with two burly young men, a very well-behaved tigress, but it was the pinnacle of my lecturing career. I have never been able to match that since!

I want to talk about sand dunes, and I want to talk about animal movement.

Let us start with snowflakes. Here are a couple of modern pictures, and one rather older picture, of the remarkable structure of some, not all, but a lot, of snowflakes. The detailed structure is very, very different, but you can see that there is a remarkable six-fold symmetry in the snowflake.

I was watching a television programme the other day, and there was somebody with an experiment that said they do not do that. Well, they did not do that in his experiment, but I think that is because it was so carefully controlled that actually, I think you need a little bit of random noise to smooth things out a bit to get this kind of symmetry. I am not quite sure why his experiments did not work, but a lot of other people's experiments do work, so there are still some mysteries about snowflakes in fact.

I just want to emphasise that the six-fold symmetry in this perfect form is not the only thing that can happen. Here is another form of ice crystal, and you can vaguely believe this is some fancy relative of the six-fold symmetric snowflake, but it is clearly not as regular. There are some very interesting questions you could ask about this picture.

This is a real snowflake, chosen I am sure to be pretty close to perfect. If you look, you can just see little bits where it has slightly melted. You can see, if you look closely, that this is not faked up, this is genuine, and what puzzles everyone is this mixture of the symmetries of the thing – and I am going to explain those now – but also the remarkable variety within this range of symmetric structures.

The way mathematicians think about symmetry, since about 1830, is in terms of transformations, in terms of ways of moving an object so that, when you have moved it, if somebody looked away while you were doing it and looked back again, they would not realise you had done anything. So, if I take this snowflake and I rotate it through 60 degrees, you really would not see any difference. It would look exactly the same shape, in exactly the same place. And if I rotate it through 120 degrees, 180 degrees, multiples of 60 degrees, the same thing works.

In addition to these rotational symmetries, the snowflake has six reflectional symmetries. If I put a mirror in the right place and flip it over, it looks exactly the same. So, there are three red lines, showing three mirrors, which in this picture, go between the big lumps of the snowflake, and then there are three more green lines which run down the centre of the protrusions of the snowflake, and that gives us six more.

This snowflake has six symmetries that are rotations, another six that are reflections, twelve in all, and what really matters mathematically is not just how many but the way these different symmetries relate to each other - what happens when you do several of them in turn, what do you get? And of course, one of the features that is automatic from this idea of symmetry as a transformation that leaves things unchanged is, if you do two of them in a row, you leave it unchanged, and then you leave it unchanged again, so what does it look like? It looks exactly the same as it did to begin with. You have moved it twice, but that must be equivalent to moving it once because the twelve I have shown you are the only ways to move a snowflake around by rigid motions and make it fit in the same position. So, mathematically, you can take two of these transformations and combine them together to get another one. So it's like a little multiplication table, if you wish: this transformation, followed by this other transformation, makes this third transformation. This is called a group, and group theory is now a major, major part of mathematics and how we analyse symmetries, but I do not want to teach you any group theory, I just want to alert you to its existence.

Just to make this point that this snowflake really is very symmetric, the top left-hand picture is that picture of the snowflake, just shrunk down; the next five to the right, I have rotated it through 60, 120, 180, 240 and 300 degrees – I actually did this on the computer, just took the picture and twisted it. I did it clockwise, which is the opposite of mathematicians' usual convention, but no matter. Then the pictures on the bottom are the same thing but I reflected it first, so we have twelve – these are the twelve images of that snowflake under its twelve symmetries, and you have to look very hard indeed at these pictures to see any difference. Whereas, if I did it 30 degrees, you would say, no, it is pointing in the wrong direction, okay? So the puzzles are: where does this symmetry come from, what is the physical basis for this, and then, after that, the other puzzle, that is this particular snowflake, but there is all these other fancy-looking snowflakes, lots of different structures, but they have this same group of symmetries.

Way back, a good many hundred years ago, Johannes Kepler was thinking about this stuff, and he wrote a little book called "On the Six-Cornered Snowflake", "De nive sexangula", I think it is in Latin. It was a Christmas present to his sponsor, who was paying his salary basically, and in it, he just did thought experiments about this puzzle of the snowflake and its six-fold symmetry, and he worked his way through and he thought about packing balls together. You know, if you put a lot of coins on the table and push them together, you get a honeycomb pattern. He worked his way towards what we would now recognise as the idea that snow is an ice crystal and it is something in the atomic structure of the ice crystal that is governing this six-fold symmetry. This was back in the time when atoms were something the Ancient Greeks had thought about, and mostly dismissed again, and it was all kind of forgotten, and it was not until about 1900 that physicists actually started to believe in atoms. It is remarkable how recently that happened. So, Kepler worked his way towards the beginnings of the way we would now tell the story, which is snowflakes are formed from ice, ice is a crystal, formed from water.

This is a picture of the particular form of the ice crystal that is occurring in normal temperatures and pressures, and you can see the two different kinds of atoms, the hydrogen and the oxygen. Hydrogens are black, oxygens are white. There is a honeycomb structure to this. It is in three-dimensions, it is not completely flat, but it is within this structure of the ice crystal as it grows that there is, among other things, there is this group of twelve symmetries, the hexagonal symmetry.

Now, when you actually ask under what conditions the different forms of snowflake occur, this is experimental work, and the two important variables are the temperature, which is in degrees centigrade along the bottom, and the super-saturation, the amount of moisture up in the cloud where the snowflake forms, which goes up the left-hand side, and depending on the combination of those two variables, we get lots of different shapes. At the top, in the middle, under plates, you see these very flowery plant-like things, called dendrites, you see sectored plates, which are like hexagons with more hexagons stuck on, which is what we were looking at just now. Below that, you see just thin hexagonal plates, and further down, it grows a lot more in the third dimension. You can look at this picture and see all of these different kinds of shapes, and some fancy ones that are not particularly hexagonal at all, and if you start playing with other variables, you can get more structure, but the source of the variety of a crystal grown at a uniform temperature, in a uniform pressure, you can sort of predict roughly what kind of shape it will be and it's classified by this.

Why then do we see such variety? Well, let us just take a closer look at dendrites growing. So, here is a little piece of a snowflake, growing this tree-like structure, and in fact, what you see is that, as the side branches get bigger, they put out their own side branches, very like a tree. As a tree puts out branches, the branches put out smaller branches, you get down to twigs, the snowflake does this as well, and it does it because of what physicists would call an instability. If you have a flat edge to the crystal but there is a little bump on it, which is going to happen just because a few bits of ice are going to stick to it, under those conditions of temperature and super-saturation that cause this, that bump grows faster than the flat bits. So once you have got this little bump, it gets bigger, so this instability of the flat edge amplifies. The conditions of temperature and pressure where the snowflake is affect the immediate pattern with which extra bits of ice get added on.

Now imagine a little snowflake going around in a big storm cloud. You have got this tiny little snowflake, ice is forming on it, at any given moment, at all of the points in this honeycomb six-fold symmetric pattern, at all of those points, the conditions are pretty much the same, at a given instant. So you go to your little diagram of what shape crystal you should get and it does whatever it should do at that instant, so it does the same thing at each of the six corners, but a split second later, it is whirled around in the cloud and the conditions are different, so now you re-set and say, oh, what do I do now…oh, I grow this structure at all six corners. So, because the snowflake is small, so on a very small scale, the conditions at the corners are pretty much the same, but on a larger timescale, the conditions at those six corners keep changing. You could actually draw a picture of how the snowflake's temperature and super-saturation move around that diagram, and the rules for growing keep changing as it moves. There is an awful lot of ways to draw wiggly curves through that diagram, so there is a lot of different detailed shapes for your snowflake, but the six-fold symmetry is preserved at all stages, and so the thing grows into this perfectly symmetric shape, barring accidents. This is not absolutely perfect all the time, but when you get one of these symmetric snowflakes, that is pretty much what has happened.

The symmetries of the crystal lattice and the way snowflakes grow as ice crystals combine together in the cloud to give the variety but keep the symmetry.

Let us move on to my favourite symmetric beast. I am a cat fan – the tiger. Tigers are stripy, and the stripes are actually quite elaborate and they are not perfect mathematical stripes. If you look at the ceiling of this building, you see mathematical stripes in the beams on the roof – they are just absolutely like this, they are identical. A tiger is not quite like that, but then a tiger is not just a nice flat mathematical plane. If you look at the rear-end of the animal, technically called a tail, you see very nice rings around the tail, fairly evenly spaced. They shrink a little bit towards the end and then it has got the black tip, but that is the most mathematical part of a tiger I think, the tail. So, can we understand where those patterns are coming from? I want to convince that it has a relation to symmetry.

Tigers are not the only cats with patterns. There is a leopard, a very nice leopard, doing typical leopard things: sits in a tree, dangles its legs down, a lot of the day it spends like that.

A cheetah – runs around a lot…

This is a clouded leopard, such a beautiful pattern that it is an endangered species, much more endangered than most of them, just because some people think this is such a pretty pattern in the fur that they want to make something out of it and wear of it.

Of course Tigers are not the only stripy animals around. Some fish have stripes, some fish have circular stripes, some fish have spots…

When we did the Christmas Lectures in Japan, in the summer, we could not get a tiger, so we got a cat, and this was a cat with a circular stripe on its side, a bit like this fish but it was not blue. It was a beautiful animal.

This gentleman, whose centenary we celebrated last year, Alan Turing, he is the person who founded the mathematical theory of animal markings, and it has gone in and out of favour with biologists ever since. Every time they convince themselves, and the mathematicians, that it does not really work, or the latest incarnation does not really work, a few years' later, some other bunch of mathematicians or biologists, or fairly recently, people in the dental school, come along and say, yes, it does! So, there are places where it works, there are places where it does not – take it too literally and it does not completely match the biology. Think of it slightly metaphorically as a type of model, and it works rather well.

Turing was actually interested in a cow, which has these big blotches on the side. They are so iconic that you see them driving around on the side of lorries, all over the world. We were walking past the local grocery store, and there is always one of these lorries parked outside, but last week, it was a different company and it did not have the black blotches on the white background, and we were actually quite disturbed by this. Suddenly, the world had changed and our ten-wheeler cow had disappeared.

So, Alan Turing, in his papers, would discover these plots. They are a little bit of a mystery, but they are clearly his mathematical model, which I am going to talk about, of how these blotches on a cow or that kind of pattern forms, and we are getting some nice blotches. It is based on computer output because, if you look very closely at the little dots, they are actually numbers in, I believe, hexadecimal notation, which is what computers were using. He has clearly copied out all the numbers by hand and drawn contour lines through places where the numbers are all the same, the sort of way you would get on a weather map, you have these lines where the pressure is all the same. Imagine doing that by hand on a grid that just has the numbers for the pressure. Well, this is what he has done. Instead of pressure, it is the concentration of some chemical.

He did also write and publish a research paper on this, and he was interested in stripes, spots and blotches on cows. He was also interested in the shape of creatures like the hydra, which is a little microscopic creature with a lot of tentacles. He worked his way to a theory that what must happen – and this was before modern genetics, and that is one of the reasons the biologists stopped being happy with it, because it was on a level where the influence of the genes was not explicit. The idea was, think of the animal when it is still an embryo, fairly simple. The basis of the eventual pattern of markings is laid down chemically in the cells while it is still in an embryonic stage. You get what is called a pre-pattern. You cannot see it, but if you could detect the relevant chemical, you would be able to. And he said, "I will call this chemical morphogen." This is the typical word scientists use for "I know what it does but I do not know what it is." In this case, Marking-generator – it is a thing that generates the pattern.

The idea is, the pattern itself comes about as a result of two different processes: chemicals reacting with each other, which changes the concentration of those chemicals; and diffusion, which spreads the chemicals slowly across the embryo. So, if you have a lot of chemical at one place and wait, it will slowly spread, and so, these are called reaction diffusion equations. You have complicated local chemical reactions, which are non-linear, in the jargon – they obey quite complex mathematical equations, and then global diffusion across the whole thing, and the diffusion kind of organises what the local non-linear stuff is doing, and the result of this is you get patterns. This was Turing's big insight: you get patterns.

These are experiments at the top and computer-generated pictures at the bottom of these reaction diffusion equations, and we see spots and stripes and more complex-looking spots and actually very complicated, rather irregular structures as well.

Just to drive that point home, here is a fish called a box-fish. It is irregular, but the spacing between those yellow stripes is pretty much the same everywhere. There is a regular spacing to it, but the details are irregular. A computed Turing pattern, which is qualitatively similar. I am not suggesting these match in detail, but they are the same kind of thing.

Now, Turing explicitly in his paper explains this in terms of what he called the breakdown of symmetry. This is my symmetry-breaking that I was talking about. He says, basically – those are his words, but let me summarise: there is a problem. There is, what looks like, quite a difficult problem: the embryo, at the time the patterns are formed, down before the animal grows, is essentially spherical. It is spherically symmetric. Superficially at least, it is like a ball. If I rotate it, you cannot really tell in which direction it is pointing. Now, biologically, there are actually some markers for front/back, left/right and top/bottom already at that stage, but nonetheless, look at it, it just looks round. It is obeying laws of physics and chemistry everywhere to produce the eventual animal, and if it was like a snowflake, which is six-fold symmetric and therefore does the same thing at all six corners, a sphere should do the same thing everywhere. It has not got corners. Every point on the sphere can be moved to any other point by symmetry. So should it not just grow the same pattern everywhere, in which case, your horse that eventually emerges ought to be a sphere, but it cannot result in a horse because a horse is not spherically symmetric.

He has an answer to this objection, and, people would not necessarily phrase this in terms of embryo turning into horse, but there is a period in the history of science where people did not quite understand this point, and the idea was, if you have a symmetric system, it's behaviour will be equally symmetric. And Turing says, no, this is not true, this does not happen, and it does not happen because the symmetric state can become unstable. Very tiny perturbations, little bumps on the sphere in the case of the horse, changes in the chemistry at one point or some region, can destroy the entire symmetry of the object because those small changes can grow. Instead of damping down and everything staying symmetric, they can grow, they can take over the structure, and the whole thing changes and it loses some of symmetry that you thought was there. He explains this – Turing obviously had a good sense of humour. I mean, this is in a journal paper.

If a rod is hanging from a point a little above its centre of gravity, so you have got something like a ruler which is pivoted just above the middle, and it swings, it will hang vertically downwards in stable equilibrium. Now, when it does this, it is essentially symmetric, from left to right, for example – it is hanging in the middle. He says, imagine a mouse climbing up the rod. If the mouse keeps climbing up the rod and goes past the middle and up towards the top, at some point, the whole thing's top-heavy. What does it do? It flips right over. So this equilibrium becomes unstable, and instead of just hanging steadily, it flips over and it starts swinging from side to side, and when it swings from side to side, in fact it is the time symmetry – it is in the same place at all times – that gets lost. It is not in the same place at all times – it is moving, but it is moving periodically. It is repeating the same motion over and over again, assuming there is no air resistance or friction, so, after a certain period of time, it does the same thing – does it again, does it again, does it again. Now, if you actually draw a picture of the position of the pendulum, with this mouse hanging on for dear life, swinging from side to side, and you draw it against time, you get a curve that repeats the same structure over and over and over and over again. It is like stripes in time…yeah? So, we are seeing a pattern, but in this case, we are seeing it in time.

In fact, I only realised Turing had written this after I had prepared my version of much the same thing. So, instead of his rod, I have a rod as well, but it is pivoted at the bottom and held by a spring, and the spring is such that it likes to sit vertically, and if I move it to one side or the other, I stretch or compress the spring, and the forces that the spring exerts to push it back again are the same on both sides. Although I have only drawn the spring on one side, this whole thing as a system is left/right symmetric. When it is vertically upright, it is in a left/right symmetric state. If I flip the whole thing over, the rod still appears to be in the same place.

Now, I do not use a mouse. If I had known Turing had written about a mouse, I might have put a mouse in, but it is more dramatic if you take an elephant and put him on top of the rod, and you do not have to be an engineer to look at this and say that is not going to work. If that is a really heavy elephant, strong rod, but the spring is not going to be able to support the elephant, so it is going to do something like that. Well, now the whole thing has moved off to one side and the state of that rod is no longer left-right symmetric. If I flip that picture, the rod goes over to the other side, yeah? So although the mathematical equations, the whole system has this left/right symmetry, the state that you observe does not. Why not? Because that vertical upright state, although mathematically possible, is not stable – the slightest breath of wind and it will start to move and then the weight of the elephant takes over and down it goes, until, eventually, you compress this amazingly powerful spring that will resist the downward force created by the elephant. So where has the symmetry gone?

There is another solution that looks like that. So, you can have a symmetric system with a symmetric state, but if that loses stability, the system goes to an asymmetric state, but there are traces of the symmetry still remaining, there is a symmetrically related state that is still there.

Now, let us look at the same idea but with a richer range of symmetries - symmetries of the whole plane. Again, you can look around this room and see some examples. I talked about stripes in the roof. People online will not be able to see these stripes, but imagine a whole pile of stripes, okay? So, you have the symmetries of that stripe pattern are, if you move it one stripe along, it still fits, move it two stripes along, it still fits, but move the stripes into the gap between other stripes and you will notice it has changed. So, it has symmetries, but they are not slide it to any position, they move it very specific distances at right angles to its stripes, or slide it along the stripes…you will not see any change. So symmetries of the plane are a much richer source of patterns, and there are basically four kinds, three major ones and another one I might as well mention.

So, take some shape in the plane. Here it is. It happens to be a schoolteacher dragging a donkey, for reasons that have nothing to do with this talk. It is a shape. You can rotate that shape, you can reflect it in some mirror, not necessarily left to right but it can be, or you can also do a thing called a glide reflection, which pushes it along and flips it over.

These are the basis of symmetries of patterns in the plane, so let us go back to Turing and the tiger and animal markings. Jim Murray, a good many years ago, put Turing's equations onto a domain that is shaped a bit like a skin of an animal. Now, it is not entirely the case that, when the embryo grows, that is not exactly the right shape of domain, but there are a number of reasons in the way the patterns form that unwrap the animal into a blob with four bits sticking out and a tail-end is not such a bad thing to do. What he discovered was you can get interesting patterns forming and you can also get these. There is one at the bottom that has one stripe – it is completely black. There is one just above it that has half a stripe, if you wish. One above it has black at one end and white at the other end, a complete, single unit of stripes. There is one that has black at two ends and white in the middle, and you can actually find sheep, for example, and goats, which have that sort of pattern. Then, as you change the mathematical numbers, you get all sorts of different patterns.

He also looked at the rather symmetric bit that is on the back end of big cats, and he found you can get stripy patterns. It is a tapering cone. So you solve Turing's equations on a tapering cone and you find stripes, and you find spots, and you find a wonderful set of patterns that are stripy at the thin end and spotty at the thick end. In a sense, the mathematical reason is quite straightforward. If you have got stripes going round, they will be stable until there is enough room for them to do something different, and the main thing they can do that is different is that the stripe breaks up into a series of equally spaced spots. So, as the amount of room increases, you can get stripes breaking up, by symmetry breaking, into spots. So, he proposed a theorem.

Here is a computer calculation showing this kind of thing but with a perfect cylinder. You get stripes, but it is possible for them to break up into a pattern of spots. It is actually a hexagonal pattern. That is very interesting about it. The successive stripes break in alternate places: one breaks like this, the next one it is down here, and then it is back like that, and then it is back like that. If it is tapered, you can see similar things occurring. So, Jim Murray has a theorem. Here is the theorem.

A spotted animal can have a striped tail, but a striped animal cannot have a spotted tail. Proof: in order to get stripes breaking up into spots, you have to have a larger region of the domain. On the whole, the body of the cat is a lot bigger than the tail. Indeed, if you have stripes and spots mixed up at the rear end of the tail where it is nice and thin, you will see the stripes and then they will turn into spots.

To demonstrate this theorem, here we have a spotted cat with a spotted tail, there we have a striped cat with a striped tail, and here we have a spotted cat with a striped tail, a cheetah, and I don't have a picture to show you of a striped cat with a spotted tail. So Jimmy is right!

Okay, I want to tell you about sand dunes, where all of this and the relation to symmetry is, in some ways, easier to understand because we understand what a sand dune looks like. A pattern of chemical concentrations is something that we are not quite so familiar with. So, what I want to show you is sand dunes actually have very similar patterns to animal markings, and the reason is they have got basically the same symmetries.

One of the commonest patterns for sand dunes are called linear dunes. They are basically stripes. They are stripes across the desert. Instead of colours, it is whether the sand is high or low. They are formed by strong winds that blow in a constant direction. These are in Algeria, photographed from a satellite.

This is the biggest set of linear sand dunes, or linear dunes – they are not sand – in the solar system, and it is on Titan. It is a photograph from the Cassini satellite, and there is just this vast set of linear dunes on Titan. So these things do not just occur on the Earth.

Now, there are also these wonderful crescent-shaped or barchans dunes. These are in Egypt… Those are on Mars… Here is a more complex field of these dunes, and this is in the Namib Desert. The way they form is you essentially get a pile of sand builds up as the wind blows, and the wind blows over the top, and it forms a big vortex that sweeps out the bit in the middle of the crescent. If you have got a crescent, the region in the middle is where this vortex is sweeping stuff out and funnelling it out to the edges. If the crescent is shaped this way, the wind is blowing that way, so the arms of the crescent are being pushed along, and in fact the entire dune moves. So, as the wind blows over the top – this is the standard way all sand dune patterns form – sand gets moved around. The shape of the dune affects what happens, and it also affects how the wind blows. It affects the aerodynamics. So there is this interaction between the wind and the sand, and even if it is a steady wind, once the sands change shape from flat, the effects of the wind are different in different places. Sand erodes from the windward face, gets pushed over the top, dumped on the other side, the leeside, and the sand moves, the dune moves.

People who study this kind of thing, they classify dunes into six main patterns. There are others but these are the main ones. So, there is: the barchans or crescent dunes; linear, longitudinal – basically, they point in the same direction as the average wind speed; transverse dunes, which are like stripes as well but they are at right angles to the wind speed; parabolic dunes, which form usually at the edge of where the desert meets water, and they are kind of pinned in place by the vegetation; barchanoid dunes, which are wobbly; and star dunes, which are dotted all over rather randomly.

We can model this mathematically by saying, well, we will take a model desert with flat sand, wind blows at constant speed, fixed direction, and the desert is infinite. The symmetries are all rigid motions, but we have got a wind. So it is rigid motions that preserve that wind direction. So I can slide the desert, but not rotate it. I can reflect it, but only in a mirror that points the same way as the wind.

Then I can look at the group of symmetries and say, if symmetries break, then that means some of the symmetries no longer apply to the state of the desert, but some smaller set of them may. Remember I was talking about the mouse on the pendulum and saying if you draw a picture…? If that pendulum was always in the same place, the picture of where it was would just be a horizontal straight line, and you could move it to any position, through any distance, any time, and it would look exactly the same. But if it is oscillating periodically, I can move it by one period, two periods, three periods, minus one period, go the other way… So, a continuous set of symmetries in some direction, when it breaks, what it tends to do is it breaks up into discrete translations in that direction, through multiples of some fixed distance or time, and this is what we see in the dunes.

Think about transverse dunes. The wind is blowing vertically. The symmetries could slide those dunes up and down to any position you like. They could slide the deserts sideways to any position, and they can flip it left-right. What symmetries does this pattern of dunes have? Any translation at right angles, any distance whatsoever, fixed distances along the wind direction, so the symmetry along the direction of the wind has broken, and you can predict this pattern of stripes just from the symmetries. You need to know when that pattern is stable, but you can predict that is a possible pattern.

Now with the barchanoid dunes, where it is like that except they have become wavy, we have broken another set of symmetries, the left/right translations. They are not through any distance anymore. They are through specific multiples of the wavelength as well. So, a second group of symmetries has broken to get the barchanoid dunes. In fact, if you imagine taking the sharp pointy bits and just pinching them off, you would get a pattern of spots which would again have specific translations in both directions.

Linear dunes are very similar to the transverse ones in terms of the symmetry, except it is now the translations at right angles to the wind that have broken to give stripes. The ones in the wind direction remain.

Barchan dunes, you can imagine some kind of lattice of copies. This is a very idealised picture. Or you might imagine that they are perhaps staggered with respect to each other, so there is one, then one, then one, then one, like that. So again, there, symmetries in both directions have broken.

Parabolic dunes, they have lost the symmetry of translation altogether in one direction. There is nothing there – you cannot move them. But you can still move them specific amounts, left to right.

And finally, the star dunes, which probably are best thought of as not really having much symmetry at all. Everything is broken. They do have local, vaguely rotational symmetries. Star dunes form when the wind, on average, does not go in any particular direction. One day it is coming from the north, another day it is coming from the east, another day from the south, south-west, whatever – no prevailing direction, so the pattern has no prevailing direction either.

So now we see that the same symmetry principles are applying in two different mathematical systems or two different physical systems: the markings on animals, where it is a chemical pattern, which we now know is related to the genetics but that is another story; and the pattern of sand dunes, where it is a pattern of sand. Roughly speaking, any physical system in the plane that starts out being symmetric under lots and lots of rigid motions in the plane, you will get the same kind of range of symmetry types of pattern, so the same mathematics will apply to lots of different systems, and that is what makes this very powerful.

Okay, I would like to end up with something that I'm pretty sure I talked about at Gresham College about eighteen years ago, possibly using the same slides, to some extent, which is animal movement.

Here is a lovely sow, trotting along, so what is mathematical about that? I was actually giving a lecture to some schoolchildren in Malta and this is the point at which they woke up, and once we got the trotting pig, then I had their attention! So, you probably know the poem, yes?

“A centipede was happy – quite!
Until a toad in fun
Enquired, which leg goes after which?
This raised its mind to such a pitch,
It lay distracted in a ditch,
Considering how to run.”

I am not going to talk about centipedes. They are a little bit too complicated, although the mathematics I am going to tell you about does seem to apply to them, with a few interesting predictions. The main point here is that animals move in lots of different ways.

We have a hare. The hare is bounding. The horse there is trotting. The elephant is walking, and I can be pretty certain about that because, essentially, there is only one pattern that elephants move with. That is not quite right, but it is close. The insect has six legs, and that one I think is doing what they call a tripod-gait, which is essentially that the legs move in groups of three, one at the front, one at the back, and the opposite one in the middle. If it is not doing that, it is doing a thing where essentially it is like a walk, it is this leg, that leg, this leg, that leg, this leg, that leg, and then all the way up - metachronal gait, that is called. I am not quite sure what the bull is doing. The cheetah is doing what is called a rotary gallop, for the cheetah. It is going so fast that actually the back legs have got in front of the front legs, as you can see – it is catching up with itself basically.

These things are called gaits, and it is actually useful. It is a mixture of mathematics and physiology, basically. So, there are medical reasons for wanting to understand human gaits, because people have problems wrong with their knees and hips.

It is good for sports science, Mo Farah did not just win his medals by being very good at running. There is a lot of analysis and training is aimed at specific things that the sports scientists have understood about how to run fast.

Or you can use it to analyse dinosaur tracks and understand how the dinosaurs move. Or you can use it to look at human evolution. Or you can look at robots with legs – how can you make them work? Very useful to the US military, who want to defuse bombs on firing ranges and things like that.

Or you can use them to explore planets. We have just landed another wheeled robot on Mars. That is because the engineers really understand wheels. Legs will be the coming thing, but not quite yet because the wheel is a little more reliable, but they have gone to some lengths with the latest Mars rover to make sure that its wheels do not get stuck and so forth.

And future applications: send out robots all over the solar system, if not beyond, and they may not have wheels and they may not have legs either – they may have all sorts of structures.

The subject of gait analysis was started by this chap, born Edward Muggeridge, rapidly became Eadweard Muybridge, sounded posher, and he took photographs of animals moving, and let me run you through one or two of these.

When the elephant walks, and indeed when any other four-legged animal walks, the pattern is this. Look at when the left rear leg hits the ground. It will be followed by the left front, then the right rear, then the right front. If you look closely at that picture, this is what you see: that foot, that foot, that foot, that foot. If you ever see them, and I have sometimes seen animations on television – they have got better now, but in the early days of animations of dinosaurs and things, sometimes it was that foot, that foot, that foot, that foot, back leg, back leg, front leg, front leg. No, they do not do that. Back, front, back, front – that is how it works.

Our pig, like the horse, is trotting, and you can recognise a trot because the legs are locked together in pairs. A diagonal pair hits the ground, and then the other diagonal pair hits the ground – those two, then those two. All of these have patterns.

If we go to the canter, now that is rather complicated: left rear, then a diagonal pair together, and then the missing leg hits the ground after that, and that is why you get this rather odd-looking structure in the legs.

Transverse gallop, an elk here is doing it: left rear and then right rear, but with a slight delay, and then left front and right front, with a slight delay. Those two, then those two, clippity-cloppity, clippity-cloppity, right?

A camel pacing. I still do not believe this: both legs on the same side move together, and then the other two do. What I really do not believe about it – I can just about live with the camel – giraffes do it. It is an enormously tall animal, and it picks up both legs on the same side... It is going to fall over. We were in Botswana a few months ago, and no, they do not move – they can move fast, but they can also move quite slowly, and they do it the same way. It is quite a puzzle.

So, mathematically, we represent these as a pattern. Each leg moves periodically. The phase is the timing of when it hits the ground. So, for the wall, you have this pair hit the ground at time zero, a quarter of a period later, the front leg on that size, half a period later, then three-quarters of a period, the legs the other side. Yes? You can see what these pictures show. You can look at all of the different gaits in terms of these patterns of phases. So, for the trot, it is nought on one pair, a half on the other pair. For the pace, nought on one side, a half on the other side. For the bound, the hare, which I have not shown you pictures of, back, then front, yeah? Think of a dog moving really, really fast…

Then it gets more complicated for things like the transverse gallop, the rotary gallop, which is what a cheetah uses, which is very similar but one end is switched, left/right, compared to the others, the canter, which is really quite bizarre. One diagonal pair is going out of phase nought, a half, nought, a half, and the other pair are hitting the ground together, which are the two basic two-legged gaits. Human beings do this. When we walk, it is left, right, left, right, left, right, and when it is children hopping along, it is both feet together. If you try and do that when you get older, it does not work very well. A horse cantering does both at the same time. It is remarkable.

So, we have at least eight – there are actually more – patterns, and five of these are very symmetric. The pronk is the most symmetric of all. In fact, the most symmetric, even more symmetric than the pronk is simply stand there and do nothing. All the legs are doing the same thing and they are doing it at all possible times. The time symmetry has not broken. If the animal wants to move around, it breaks that symmetry, and the simplest thing is to move periodically, and if it is not got anything else to do, it kind of wanders around in a fairly rhythmic movement.

What is it that is creating these patterns? Thos is reverse engineering the physiology from the behaviour, and to cut a long story short, they are produced by something called a central pattern generator. This is a network of nerve cells that creates the basic rhythms. There is a whole pile of other stuff that goes on to control whether you stop and have a look round, or whether you see the lion coming and run away at high speed. But the default rhythms come from a network of nerve cells somewhere in the spine. It sends signals to the muscles in the legs. It is not in the brain, it is not in the legs, it is somewhere in between, and you can deduce what it probably looks like – it is actually very hard to dissect out – by studying the mathematics.

Let us take the simplest one. If I have two mathematical units coupled together, there are two common patterns that you will see: they both do the same thing at the same time, or they take turns. I am showing it here with two pendulums. You have got pendulums that are swinging like this, or pendulums doing that. It is possible to do other things, but it is much rarer. This is to do with the symmetries interacting with the phase shifts on the different oscillators.

You can do the same thing with four oscillators in a ring, and then you find patterns like all four of them doing the same thing, or they alternate: left-right, left-right, in terms of nought, a half, nought, a half, and so on. You get different patterns. So, if you draw pictures of the phases, in fact, these are the four patterns you expect in this ring: all zero; nought and a half, alternating; a quarter shift as you go anti-clockwise; a quarter shift as you go clockwise. The first one is like the pronk. The second one is a bit like a pace or a trot, particularly. These two look like a walk, and one of them is walking backwards and one of them is walking forwards. Now, if you try and match that up to the animal, it does not actually quite work. What about the others? What we think is going on, at least in the simplest possible network that can do this, is you do not use four oscillators, you use eight, and you use two for each leg. Physiologically, we think one of them essentially pulls the leg forwards and the other one pushes it back again. They control muscle groups that have that effect, because muscles pull, they do not push, yeah? But you can think of the patterns by just looking at the bottom four. These are perhaps the muscles that push the leg, and then above them are the ones that pull.

If you write down all the patterns that are possible in this kind of network, for example, you can get a pattern where the phases go nought, a quarter, half, three-quarters, up the left-hand side, and then feed back and keep doing it, but on the other side, it is delayed by half the period. This is one of the patterns. There are eight patterns for this network.

If you just look at the bottom four, that is the walk. That is exactly the walk. Whereas, if we have this pattern, which can also occur, and just look at the bottom four, you recognise the trot. That is the bound – nought, half, nought, half. That is the pace and that is the pronk. You can expect to do it with four, but there are some subtle mathematical reasons why that does not seem to be correct, and then these things can occur as well, but it is a more complicated story.

Now, when we did the maths, we realised there is one pattern we had not noticed anywhere, which looks like this: nought, a quarter, half, three-quarters, like a walk, but the same on both sides. Now, it is not the bound. The bound is back, front, back, front, at equal intervals of time, nought, half, nought, a half… This is nought, a quarter, what has happened, what has happened, nought, a quarter, something's…did not know.

Then we went to the rodeo and we saw the bucking broncos, and then of course we realised that good old Muybridge had seen all this before. The back legs hit the ground, then the front, and then actually the whole animal hangs in the air for quite a long time – look at that picture…from top to…top to the second row, it is done one and it is still not come down completely, and when we looked at the video of the thing we saw in the rodeo, it is almost exactly nought, a quarter, for the legs, and then the animal sort of getting itself back on the ground again. It is not 100% confirmation of the theory, but it is at least consistent.