Part One - 'The Influence of Amatino Manucci and Luca Pacioli' and 'Louis Bachelier and his Theory of Speculation'

Transcript

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There was a thriving Jewish community which had found sanctuary in the South of France under the Holy Roman Emperors, and I like to think of these people as a group of merchants and traders who were all getting on, trading with each other, and not making any further judgements about each other than whether they were good to trade with.

What did they trade in? Wheat, barley, oats, wine, wool, cloth and yarn; they also dealt with the exchange of money and the lending of money, which was all tied up into it. There was an official ban on usury from the church in those days, but then we see that the Archbishop himself is happily borrowing money from the company at 15%!

I have a sample page of Farolfi’s accounts. They are very neat. It accords to the same paragraph format that we saw before, only there is only one column in the page. There is a wide margin, and they were working in Roman numerals, and doing their sums in the age-old way that people always used to do them from classical times onwards. It was very comfortable and very easy, with counters on a board.

He says, “He gave us, on the day XXVII (that is the 27th) of February in the year novantanove,” which is 1299, and it has been posted where he has given – the complementary entry is that he needs to have it in the ledger, al Quaderno de le spese, on the 8th carte, on the 8th leaf, £26 17s 2d, so we have already got complementary entries, with the positive on one side and the negative on the other.

This document is in Florence; it consists of 56 leaves. There is a lot missing, and by the cross-references one can see there is a whole chunk missing from the end. Some of the pages have been stuck together, so poor old Castellani had to use a mirror and try and sort out what was the original writing and what had been stuck on top.

The first half of the ledger contains the accounts for the debtors, for the expenditure, and the debit side of the head office accounts, so that is the balance account; and the second half contained the accounts for the creditors and the profit and the credit side of the balance, so there is an algebraic opposition.

With all the cross-referencing, he mentions all sorts of other ledgers that other parts of the accounts have been entered in, which we have not got, but we could manage to build up quite a good picture of them.

We have the White Ledger, which is divided into two halves, and we think that the General Ledger was a continuation of it, because there is no closing balance to be seen on this ledger and the General Ledger has no opening one, and it is the earlier references that come from this. So they filled up the White Ledger and they copied the balances over – they continued over into the ledger that we have.

There is also the Red Book, which contained the main merchandise accounts, dealing with cereal, wine, oil and whatever, and that is similarly divided into your two halves. There are the purchases at the front and the sales at the back, and the closing balances were then transferred to the debit of the balance account in the General Ledger.

There are a couple of other ledgers. Cloth had a ledger to itself; and then there was the Expenses Ledger, so there was the current expenses and the things relative to the fixtures, including the furniture and the warehouses.

There is also the Plush Book, and this was for the one partner, Borrino Marsoppi, for his business expenses. They totalled it up from time to time and they debited to the Expenses account and credited it to his account, so you can see again you have got an algebraic opposition, and his account being kept separate from anything else. Then there was another ledger, which seems to be the same thing but for everybody else put together.

Manucci has been quite sophisticated in the matter of rent. He has four accounts which concern pre-paid rent on various premises. You can see £16 paid to someone on 17th May 1299 for four years’ rent in advance for a warehouse, and the sum gets transferred from the White Ledger to the debit of the rent account. Then one year later, exactly, on 17th May 1300, you get one year’s rent, which is £4, and that is credited to the rent account and debited to the current expenses, and you assume that every year from there on, he is going to transfer £4 in the same way.

He does the thing with the other rents. So he has got advance rent of £12 for a shop for four years and, after a year, he transfers £3. He has got £1 10s paid in advance for one year for another warehouse, and that is all written off at the end of the year. Then he has got £12 paid in advance three years ago in 1297, for a second shop, and what we see in these accounts is the last balance of £4 transferred in the same way.
So in some we have got from Manucci, it is very sophisticated. We have extensive cross-referencing; we have different sorts of accounts: we have personal accounts; we have got real accounts, as I say, for goods and for rent; and we have got nominal accounts, for fixtures, for current expenses, and for expenses of eating and drinking.

We have evidence that he balanced the accounts, but we cannot check because we have only got the debit side, and all the references to the credit side are in a chunk that is missing. But Lee did a lot of detective work and chased up various accounts and transactions that the other side was missing because we do not have the manuscript with them on. He managed to make it balance, but it was clear that Manucci’s system had the means of checking his balance, so he has got all these components that Lee put forward as being on the way to double-entry book-keeping, and the only reason that we are not sure whether he is is that we do not have all the evidence.

Now we are going to talk about Luca Pacioli, who included the first printed account of double-entry book-keeping and, in contrast to Manucci, we know quite a lot about him.

He was born around 1445, in a small place called Borgo San Sepolcro near Arezzo in Tuscany, the same place Piero was born. Piero drew it; there is a little sketch in a detail of one of his paintings.

He does give us some biographical information. He began his career as a tutor to the three sons of a rich Venetian merchant in one of the islands in Venice. When he was there, he attended maths classes at the Rialto, and he showed an early start in his career in 1470 by writing a book on arithmetic and algebra for his students. We do not have that any more, unfortunately.

He travelled to Rome in the same year, where he stayed as a guest of Leon Alberti Baptiste. Some time between 1470 and 1477 he was ordained as a Franciscan friar. This is a mendicant order, an itinerant order, so that is quite consistent with him spending all his career as an itinerant teacher of mathematics, and almost every town you can think of in Italy, Pacioli was there at some point, teaching mathematics, and he even spent some time in what is now Zadar on the Croatian coast.

He spent some time at the court of Montefeltro in Molino, and Piero was there as well. In one of Piero’s paintings, you can see the Duke himself kneeling in the foreground at the Madonna and Child, and you can see Pacioli between St Peter and St Francis, and he looks much like the other pictures we have of him, including his woodcut.

He went to Milan, where he was at the court and worked closely with da Vinci on De divina proprietum, or the Divine Proportion. Pacioli wrote the words, da Vinci drew the pictures, and I am sure you are all familiar with the wonderful illustrations that he made of the regular solids. There are a couple of manuscripts of that, and the book itself was published, in black and white, in 1509.

Now I am going to give you a bit of a diversion, because one thing that I have found in the literature is that during the 1300s, they were all using Roman numerals. I’m one of the people who wonder how on earth they did their sums with Roman numerals, how did they manage? They were all waiting for the Arabic numerals to be introduced so that they could do their sums properly! It was not like that at all. They were perfectly happy with it. Italy seems to have been rather further forward than everywhere else. Leonardo Fibonacci wrote his “Liber Abaci” in 1202, and the word seems to be the same as an abacus, which we think of as a bead frame, but it was nothing to do with that. It is how you use the Arabic numerals, how you do commercial sums with them. The Italian merchants adopted it and they embarked on an intensive campaign to educate their children in how to use these Arabic numerals and do calculations with them. They sponsored schools; sometimes they were private schools, sometimes the commune actually paid for the school, where most male children would go and learn how to do their sums with the Arabic numerals. The maestri, the teachers, wrote their own books; the merchants kept their own manuals, and they formed a corpus called the libri d’abaco, all about the Arabic numerals. The first one, apart from Fibonacci, comes from the late 13th Century, there are 15 from the 14th, and there are over 100 from the 15th, and over 200 printed by 1600.

These libri d’abaco are all slightly different, they all cover the same material, so there are lots of overlaps and lots of different emphases: some would mention some things other people would miss out; some would give more detail, some less. They often used the same numerical examples, so you would have the same problems that would all the way through, and they would be common property. Everybody would use them. Sometimes they would be all exactly the same; sometimes their solutions would be different; sometimes people would use arithmetic solutions; sometimes people would show they could use algebra for something – they thought algebra was the great thing of the future, and wanted to show you could do it through algebra; some had more explanation and some had less explanation, and some corrected each other.

But in Northern Europe, we were not the same. We rather liked still using our counters that we had always used since ancient times.
partnership: the objective, or what it is for; how long it is going to last; who is employed; who has invested what;

a bank! Then he shows you how to enter exchange of goods in the ledgers, and bills of exchange.

different accounts; how to pay by draft or through the bank or both, how to enter it in your various books. But

There is more guidance: how to enter all the aspects of your business trips; you might have separate stores with

anything." [33x143] anything, well, you don’t make any mistakes, and if you don’t make any mistakes, you are not going to learn

price, the buyer is paying 102%, that is the 4% difference for the fee, and he shows you how to enter all these

Now, with the Office of Exchange, you had to pay a brokerage fee. This was normally split between the buyer

He writes a section of double-entry book-keeping, in particular the Venetian method, which is what he grew up with, which surpasses all the others, and if you are familiar with this, then you will be able to cope with any other style. It is full of exhortations and prayers and homilies and little bits of advice; the first piece of advice he gives is that you need to be a good accountant and quick at computation and you need to arrange all your affairs systematically.

I am going to give you just a very quick summary of what is in it, because there are loads of translations, and

He tells you how to make an inventory. You have got three ledger books – the Memorandum, the Journal, and

The inventory does have an example, which is dated November 1493. He says the important thing about the

Then he tells you how to maintain your three important books, and again, there is a warning: have your books

Then he shows you how to enter your transactions. There are different ways of purchasing goods, as we saw.

You can pay cash, you can pay with time, you can exchange goods – that’s barter – or you can write out a draft,

Then he tells you how to enter the different combinations using your double-entry in the various ledgers. Then he tells you how to deal with various public offices and, again, keep careful accounts with banks, especially with banks, as the clerks can sometimes mix them up.

He gives you a worked example using 4% as the fee. So what you are doing is getting 98% of the

The one from Adam Riese in Germany was the most influential, and again he offered both sorts. His first edition I think was 1522, and I have been shown an edition from 1576. Obviously there was a demand, and that was the way people liked to do it.

There was a blatant piece of propaganda in 1508, from Germany. It is the Margarita Philosophica, published by

Gregor Reisch, the pearl of wisdom, and she is seen presiding over two chaps doing their sums. It shows poor

old Pythagoras still using his counters. He is depicted looking a bit shabby and care-worn, a bit worried. In

contrast to him, there is a posh young chap, looking very confident and pleased with himself, with a fancy hat

and his frilly shirt and his dainty fashionable boots, and he is going far; he is using the new numbers! It didn’t

work. We were still using them well into the 18th Century.

But let’s get back to Pacioli, in Italy, where education was more forward, and where he is cashing in on the fact

that there is a need for instruction in double-entry book-keeping. He wrote a section of double-entry

bookkeeping, dedicated to the Duke of Urbino, with the intention of giving his subjects all necessary guidance in

the successful conduct of business. He recommends double-entry book-keeping, in particular the Venetian

method, which is what he grew up with, which surpasses all the others, and if you are familiar with this, then you

will be able to cope with any other style. It is full of exhortations and prayers and homilies and little bits of advice; the first piece of advice he gives is that you need to be a good accountant and quick at computation and you need to arrange all your affairs systematically.

I am going to give you just a very quick summary of what is in it, because there are loads of translations, and

many people have written about it, so I am just going to give you a flavour of it.

He tells you how to make an inventory. You have got three ledger books – the Memorandum, the Journal, and

the General Ledger. He goes through all the aspects of commercial life and tells you how to enter various

transactions into the books and the various different accounts you might have, and the all-important closing or

balancing.

The inventory does have an example, which is dated November 1493. He says the important thing about the

inventory is you have got to do it all in one day – everything that you own, all your stock – you need to have a

record of what that is on that one day, otherwise you might find that you are in trouble. You have got to be as
detailed as possible, and it helps you because you have really got to be vigilant in all your business transactions,

and then he says, “There are more bridges to cross to make a good businessman than to make a doctor of

laws,” and the big I like is that, “The law helps those who are awake, not those that sleep.”

Then he tells you how to maintain your three important books, and again, there is a warning: have your books

officially examined, stamped by the commerce officer, and watch out, because there are these people who have

two sets of books and they show one to the buyer and one to the seller!

Then he shows you how to enter your transactions. There are different ways of purchasing goods, as we saw.

You can pay cash, you can pay with time, you can exchange goods – that’s barter – or you can write out a draft,

and various combinations of these. He shows you how to enter the different combinations using your double-

entry in the various ledgers. Then he tells you how to deal with various public offices and, again, keep careful

accounts with banks, especially with banks, as the clerks can sometimes mix them up.

Now, with the Office of Exchange, you had to pay a brokerage fee. This was normally split between the buyer

and the seller. He gives you a worked example using 4% as the fee. So what you are doing is getting 98% of the

price, the buyer is paying 102%, that is the 4% difference for the fee, and he shows you how to enter all these

various things in the ledgers. It is a bit complicated, so he is quite comforting and he says, “If you don’t do

anything, well, you don’t make any mistakes, and if you don’t make any mistakes, you are not going to learn

anything.”

There is more guidance: how to enter all the aspects of your business trips; you might have separate stores with
different accounts; how to pay by draft or through the bank or both, how to enter it in your various books. But
you can detect a certain theme all through: don’t trust anybody any further than you can throw them, especially
a bank! Then he shows you how to enter exchange of goods in the ledgers, and bills of exchange.

You record all the details of everything in the Memorandum. For example, you have got all the details of your
partnership: the objective, or what it is for; how long it is going to last; who is employed; who has invested what;
what the assets and liabilities are; and you keep the partnership account separate in the ledger from all other dealings.

Then you have different expense accounts: the ordinary, the household; the business; the wages. It is quite interesting because the Extraordinary Expenses were things that were lost. So you might have lost things through sea voyages, or lost money, or it might be stolen, or you might have just lost it at the gambling tables. Actually, the example he gives is archery, so it is a bit like people betting on horses – you might bet on the outcome of an archery competition. That is what occupied them, in those days. But okay, so you’ve lost it, well, put it down in the accounts.

Then he shows you how to enter in the ledgers your income and expenses; profit and loss; how to carry the books forward; how to correct mistakes, because the mistake is there and you have got to leave it there, so you have to make complementary entries to put it right; and the all-important closing and balancing of the accounts, and getting an assistant to go through one ledger while you do all the totals in the other, and you have your trial balance just to make sure everything is okay, and you are still buying and selling while you are doing this, so what to do with all the transactions that are happening as you do it.

Then there are almost the last essentials. The ledger has got to be closed every year – that’s the way we do it, and that means you always know where you are, and “frequent accounting makes for lasting friendship!” And then there are the little addenda: how to keep your correspondence in order, and a summary of the rules for keeping a ledger, a summary of what has all gone before; and a summary of all the important transactions – where you entered it in the Memorandum, you record it in the Journal, and you post it to the Ledger.

It is full of inconsistencies and it is certainly not an ideal first text for a learner. There is no sample set of accounts. There are individual examples, but no sample set. It is a bit like a technical manual for a video – it is perfectly clear if you know what you are doing, but if you do not know what you are doing, you don’t know where you are! But it was popular, and essentially it was for merchants who knew what they were doing – they wanted something to refer to for the details. There was a second edition in 1523, which was very influential, particularly among 16th Century mathematicians, and the two great figures, Cardano and Tartaglia, both based their big arithmetical, algebraic works on Pacioli’s Summa, and they both included a small section on double-entry book-keeping.

So what was his influence? Basil Yamey says that the best books on accountancy that came up in the 15th Century were all based, to a large extent, on Pacioli’s Summa. There were some others, but they never seemed to be so popular.

Manzoni, in 1540, is based, to a large extent, on De Scripturis. He tidied it up – he probably got rid of all those homilies. He showed how to correct a badly kept set of account books and, most importantly, he has a sample set of accounts that you can look at.

Jan Ympyn in Antwerp kept closely to it, but he improved it; he had an illustrative set of account books, and this was very, very popular. There was a French edition in 1543, and there was an English edition by Grafton in 1547.

Then there was Hugh Oldcastle, in England, a Profitable Treatise, and he borrowed very closely from Pacioli, even up to not containing a model set of accounts.

Schweicker in Germany produced a derivative of Manzoni based indirectly on the Scripturis. Interestingly enough, for all the charges that have been levelled against Pacioli for plagiarism, neither Manzoni nor Oldcastle actually mention Pacioli. They don’t say we have pinched this from him; they just copy it, so everybody was doing it!

Ympyn, there were many editions of his books, and some of them mention him and some of them don’t.

What happened later? Well, things just took off and burgeoned, so after that enormous numbers of treatises appeared on double-entry book-keeping, with a great variety of subject matter and treatment, but there is one thing that runs through all of them: Pacioli’s name is continued to be remembered and he is still called the “Father of Double-Entry Book-keeping”. The first time I ever came across Pacioli was in the 1990s, in the great run-up to the year 2000, and there was ‘Pacioli 2000’, a wonderful new software package for keeping accounts, with a really grotty picture of him on the front!

So here we have two important figures in the history of double-entry bookkeeping. Neither of them invented it, but they both had their part to play. Manucci was a steady practitioner – he just did it – and he was one of a body of people that they were all using whatever tools and skills they had available, probably swapping notes. “Oh, I found this a good way to do this,” “Oh, I didn’t write this down and I wish I had, so let’s make sure we write this aspect down in the future,” and they were all going forward, and all crystallising their way towards double-entry book-keeping. It was already there and established by the time Pacioli came along, but he was a flamboyant teacher. His main aim in life was to get it disseminated, get it promoted, and he promoted double-entry bookkeeping, along with all the other good things, in the Summa. So they both deserve their place in history.
Mark Davis
Louis Bachelier and his Theory of Speculation

I want to talk about something which definitely relates to the 20th Century and really has no precursor, and that is the work which followed on from Louis Bachelier.

At one point, Louis Bachelier was a PhD student in Paris, in the late-19th Century. He presented his thesis in the year 1900 on The Theory of Speculation. He was a PhD student with Henri Poincaré, who was certainly a most distinguished figure in Paris at the time. In traditional French style, he presented the thesis, and then there was a handwritten evaluation by the jury of the thesis, and there was the comment that the material studied in the thesis was “far away from the topics usually considered by our students,” but nonetheless they, quite rightly, gave him a lot of credit for originality, which you could hardly deny that he had.

The point about Bachelier’s thesis is two-fold. On the mathematical side, he independently came up with a whole lot of things which were precursors of a much of probability theory in the 20th Century, so he was a manifesto writer in a sense, although I do not think he would have thought of himself that way.

You find in there things which are generally attributed to other people, because other people did it better than he did, frankly, but nonetheless, the ideas are there, and among them are Brownian motion, which is an object of study for a lot of the 20th Century right up to the present day. That is generally thought to have been systematised by Einstein in his paper on Brownian motion in 1906, but this was studied by Bachelier. He used the idea of Markov process. None of these things was defined, so Markov defined the Markov process in 1909, but lots of properties of it are used, implicitly, by Bachelier in his thesis in 1900, and specific equations related to Markov processes, like the Chapman-Kolmogorov equation, which has acquired its name from works dating from the 1930s. He got the connection between Brownian motion and analytic objects like partial differential equations and the heat equation, which became a theme in modern analysis for decades ahead, and he obtained technical results in Brownian motion which appear in modern textbooks, like the so-called Reflection Principle, for which he got the right formula. There a rigorous proof of that thing, covering it in detail, that dates from about 1956. So he was way ahead of his time in terms of getting results.

That was the mathematical side of what he did. Then there was the economic side. The purpose of his thesis was not directly to introduce all these objects in probability theory, but he used them to study pricing of financial options. That is something where I think the history is a bit fuzzy. So, a financial option is the right to buy something. So if I do a trade with, say, William, and I say, “Okay, William, you can buy from me one share of IBM stock at $100 three months from now,” that is a right, but not an obligation. So then, if William pays me some money for that right, then if the price is above $100, he can buy from me at essentially below market price, and he can walk away with a profit; or if the current price is below the strike price, then he can just tear up this contract and walk away. So he is paying me something now in order to acquire a right later on, which has either zero or positive value, but never negative value, and the question is how much should he pay me now for this. That is something which nobody had really considered in any scientific way before.

Actually, these things were traded centuries before that. There is little book, published by Princeton University Press, called “The First Crash,” by Richard Dale, and he delves into the history and points out that, in the 17th Century, you could trade options in London, and there were all kinds of quite sophisticated financial contracts. Of course, this is just gambling. I mean, no one had the slightest idea of what these things were really worth, and they probably did not really care very much either. So there was a big history of trading, but no real history of analysis, and I suppose that Bachelier was more or less the first person to have a go at this. It turns out that the formula he came up with is remarkably close to the Black Scholes, which is the gold standard of option valuation, which was produced in 1973. So, from the formulaic point of view, he was very close to the right answer, but he did not have the right idea. It was a concept, but he did not really have a theory that worked. However, at the same time, the theory he did have was not improved on by anybody for 65 years beyond that. There is a ‘twin track’ kind of history with this. Bachelier in 1900 independently introduced a whole lot of things which turned out to be important later on, and they were, in fairly linear fashion, carried out by lots of other people, some of whom knew about Bachelier and some of whom did not, but lots of them did. He certainly was not forgotten. What he did was too sloppy to be really much use in the end, but he had the kernel of the right idea.

So the mathematics side was a fairly linear progression. The economic side was a step function, where he was completely ignored. He had no impact whatsoever on economics for 60 years. Hardly anyone in economics had ever heard of him, and then they only heard of him from statisticians. Then, in the 1960s, he ignited a spark somehow, and a lot of development in financial economics then took place in a very rapid fashion, and within 10 or 15 years we had a huge industry of option trading, essentially using formulas which are not so different from Bachelier’s 1900 formulas.

The other curious part of the story is the probability side. No one on the probability side. No one had the slightest interest in financial economics. If they had any applications in mind at all, they were probably in physics. So, there was certainly no intention of developing some kind of calculus which would be useful for people in financial
markets. No one knew anything about it, they had no interest, it was completely unknown, and yet, when somehow this all became relevant, when people in financial economics were alerted to this circle of ideas, it turned out that the mathematics that had been produced could not have been better tailor-made for the job than what had been produced. So somehow, there was this completely independent activity which, when reconnected with the other half of Bachelier, was exactly the right tool for the job. That is remarkable, and that is what this story is all about.

The key thing in Bachelier’s thesis was Brownian motion. The Brownian motion is a model for random – you can think of it as the model for a stock price. That is the way Bachelier thought of it. It is a random motion and the idea is that it has a continuous path, so if you think of this in continuous time, so you’d have a continuous function, and if you look at the increment at any time, gain or loss over some fixed period, then those gains and losses are independent over independent periods, and they have a normal distribution. The gain over one time has a normal distribution with mean zero and variants equal to the length of the interval. That is the standard definition for Brownian motion.

It has all these properties that the ‘probability-ists’ love to study. It is a Martingale, it is a mark of process. Martingale is the model for a fair game. If you simply repeatedly toss coins, and you win or lose a pound on heads or tails respectively, and you look at the evolution of your fortune, then that is a Martingale. You expect to have tomorrow what you have today, so the probability of gain or loss is equal at every time, and you are sitting with this pile of coins and your expected fortune at any future time is just equal to the size of the pile of coins you have today. That is a stochastic process, but that property is called Martingale. That had not been introduced in any formal way by the time Bachelier came around, but he used it because the idea in the thesis was that markets display some kind of balance, so that every trade in the market has a buyer and a seller, and there cannot be any kind of consistent bias in favour of one or the other, otherwise, the market would not be in equilibrium and everybody would be piling in on the same side. So the only way to have an equilibrium in the market, according to Bachelier, is to have a situation where the expected profits on both sides are zero, otherwise somebody is getting consistent advantage. The expected gain of the speculator is nothing. What that is saying is that if you hold a contract, then it is a Martingale. Whatever its value is now, you do not expect, on average, to win or lose anything in the future, otherwise the market could not be in any state of balance.

That was the basic philosophy behind what he did. What he then did was to go ahead with this assumption – and this is never really explicitly stated in any very clear way, but this is what is going on – and he assumed that this price process would be Markovian, so its values in the future only depend on its value now, not on how it got there. He introduced transition density, which says that if you start at some point, then at a given time, your probability of being at some point, at time, is given by some density function. He derived what is now known as the Chapman-Kolmogorov equation, which says that if you think of two, three times like this, and you start here and you end at the third time, then you must go through some point, an intermediate time, and so you can break down the probability of getting from a point here to a point over there by just looking at wherever you happen to be at the intermediate time and integrating out that dependence. That gives you a relation which has to be satisfied by the transition functions. That is called the Chapman-Kolmogorov equation. You can find this in every textbook on Markov processes. He noted that the Brownian motion, the way I just described it – saying that if you start at a point, then the value at some future time is normally distributed with a mean equal to the level you have started at and the variant is equal to the length of the time interval. So if you take that as your transition function, then it does satisfy the Chapman-Kolmogorov equation. He did not think to point out that all kinds of other things might satisfy this as well, which they certainly do, so there is an element of rough and readiness about all this, but that was one of two routes he took to getting the analytic properties of Brownian motion organised. One of them was get the Chapman-Kolmogorov equation, do some analysis – look at what happens over short times, and you find that the transition function satisfies the partial differential equation, which is the standard heat equation, the heat equation describing the flow of heat from some fixed initial temperature distribution. It is the evolution of temperature distribution, say, on a plate or something like that. If you start at a given distribution, then time flows on, so that the heat kind of diffuses around, and it obeys the second order partial differential equation, and then the same equation is satisfied by the transition function for Brownian motion. That is a key result in probability, because you start with some sort of description of random motion, you want to work out something about what is the probability of things, and it turns out the way to do that is to solve the partial differential equation, and doing that is a well-studied problem in analysis and numerical analysis, so that turns the problem from something which is potentially difficult to something which is just a standard piece of applied mathematics.

There is a little result about the distribution of the maximum of Brownian motion, which you find any modern textbook on the subject would include, and it was given by Bachelier in his thesis – but, as I pointed out, it depends on something called the Strong Markov Property of Brownian motion, and that was not introduced or properly studied until Hunt in 1956, and it was not really until then that you could actually say that that result had been proved. So it was a long, long way ahead of its time.

If we turn to the economics side of this, his objective in the thesis was to use these stochastic methods to study the valuation of options, and so, as I explained before, the standard call option is just the right to buy something at a fixed price at a specified time, some time in the future, and you have the right to buy a price, and if you do that, then either it will be worth nothing – so if the spot price of your asset at the exercised time is less than the
strike price, there is no value in exercising the option, so it has value zero. If the spot price is higher, then you can buy for the lower price and sell for the higher price, and you get a profit which is equal to the difference, so you get this ‘hockey stick’ function as the exercised value for a call option as a function of the spot price of the asset that it is written on at the time when you exercise it.

Of course, this thing has non-negative value. The exercise value is going to be either zero or something positive; it is never anything negative. So if you want to acquire that right, you should certainly pay some premium for it, and the standard way in which that is done nowadays is you pay the premium at time zero. So if I want that contract, I pay now a premium and then I exercise later if the conditions are such that I should exercise. But that was not the way that things worked out in Bachelier’s day. The way that the French markets worked at the time was that you did not pay anything – you entered an option contract with no payment on either side, and then at exercise, you paid the fee, or the premium, but only if you did not exercise it. So either you exercised it and you just got the exercise value, or if you did not exercise, then you had to pay a premium. So the ‘hockey stick’ diagram then gets kind of modified to take account of that.

But in the old French system, it turns out that if there is a gain, then the best thing to do is exercise when your spot price is above the strike price minus the premium. That is the optimal exercise and then you creep into profit, so the value is zero at that time, but when you move off to the left of the strike price, then of course you are losing something, and if you do not exercise, then you just lose the value of the premium. So the whole ‘hockey stick’ has been shifted downwards, and you get a loss on one side and a profit on the other side, and that is the right thing because you have paid nothing for this contract at time zero, so there must be some profit and loss, otherwise you just get something for nothing.

This goes very well with his idea that the speculator’s net expected process is zero, so his idea for pricing this thing was the expected value of that, the pay-off from that function should be equal to zero. His model for the evolution of the price is Brownian motion. It is an easy calculation. You can set your students this problem, if they are not historians, which would be to figure out what the value of that contract would be, and all you have to do is just look at the expected exercise value, minus the premium, and it is Brownian motion, so the expectation is just the integration of the usual normal density function, so you can write down what that is, and then you just have to solve it for the one unknown thing, which is the premium, and you come up with a formula, and that is Bachelier’s option pricing formula.

So that is not really a theory, because why should this have the expectation zero – what exactly is the reasoning here? That was not really clear. It was clear in some general sense. It is obvious that there must be some kind of balance in the market, but does that really translate into something about probability when you and I may not agree on what this probability is? I may have some view of what the future of the market is, you may have some different view, so how can we come up with one formula for this thing? We have to agree on one price if we are going to trade, but we may not have the same view on the market, so how can this possibly be the correct thing? There are all kinds of objections to this. It is not a theory; it is a formula, but it is a pretty good one, it turns out.

As I pointed out, if you look at the economics, absolutely nothing happened in this area for 65 years. Why not? There are obvious reasons for this. The main reason is simply that the time was not right for it. There was no really serious market in options. Options was a gamble, there wasn’t a whole lot of trading, it was a very peripheral part of financial economics, no one was really interested, so there was not really any perceived need to get into this area at all.

Another point, which should never be forgotten, is that it would be completely impossible to handle option trading without computers. It is too complicated. A big part of this is hedging, which is doing some offsetting trades to manage the risk, and you just cannot do that. Certainly you could not do it with Roman numerals, but I do not think you could do it even with these wonderful newfangled gadgets of Arabic numerals. I do not think you could do that without a computer. So it would be impossible, even with the best system of quintuple entry bookkeeping, to manage an option book in any sensible way; it is just not feasible, and you have to bear that in mind. I think it is very important to bear in mind that the whole thing goes along with the technology, and unless you have the technology it is simply impossible, however good your theory happens to be.

And then, people’s minds were just elsewhere. There were a couple of things going on in the 20th Century – World War One and World War Two – and then was there was the Great Depression, which certainly was not any great incentive to indulge in financial engineering on any great scale, so all of this was a reason why financial markets were just directed in a different area. After the Second World War, there were various agreements like the Bretton Woods Agreement in the late-1940s, which regulated exchange rates, and that really removed the opportunity for kind of the serious financial engineering in the foreign exchange markets, because the rates were, in theory, fixed. So it was not until the 1970s, when the whole exchange rate mechanism broke down, that there was suddenly a huge opportunity to invent new ways of managing foreign exchange risk.

What I want to do is go through the mathematical development, which I said was linear, from Bachelier onwards, and I will just mention the key people and the key events in this, and then we will go back to the economics and ask how this thing got back into currency, if you like, and then we will see the whole picture.
Brownian motion is called Brownian motion because it was an observation by a botanist called Robert Brown in 1826, that if you look at particles of pollen or something suspended in fluid, through a microscope, you see that they undergo very irregular motion, and that was an observation by Brown. Either Brown or some later scientist looked at these motions and they observed how far these little pollen particles moved in a certain time, and they got something like a square root law, so the average displacement was proportional to the square root of the time. The root mean square displacement was a proportion of the square root of the time. That was an empirical fact which had to be explained somehow, and that is what Einstein did. So Einstein formulated this whole thing in the context of the molecular theory of gases, and he worked out what the mutual bombardment of a collection of particles would do, and he got the same transition function. He got a transition function, and he got the heat equation, satisfied by it. That was a major achievement. In fact, there is a whole experimental side going along with this, which I will not go into, but after this theory, experimental physicists were trying to establish whether this theory was correct, and it gave an estimate for Avogadro's Number, which was subsequently verified, so this story is a big part of early 20th Century physics.

However, it was not – as it stood. It was not really a big part of 20th Century mathematics, because you need to have a theory, a mathematical theory, of Brownian motion, and that was an analytic theory because it tells you what the transition function has to be, but is there a real mathematical object that corresponds to this, and that is not obvious at all. What Einstein pointed out in his paper was that this Brownian motion model could not be valid down to arbitrary small timescales because it is just a fact of the theory that the root mean square displacement is proportional to the square root of the time. For the average velocity of the particles, it is like one over the square root of the time. So if you look at a very short time, the average velocity is blowing up, so you have this very weird thing. So this must be something which is valid down to some minimum timescale, and below that, we would have to do something else, because it is simply not a physical model if you allow infinite velocity.

But, from the point of view of mathematics, do you have to re-scale everything as for small timescales or don't you? Norbert Wiener, in his extraordinary paper in 1923, showed that you didn't. This was the first example of a distribution of probability, of a random variable, which was not finite dimensional. If you think of an ordinary one-dimensional random variable, you just have a density function on the real line; if you have a vector, something in n-dimensional space, you have a multi-dimensional, multi-variant distribution function; but Brownian motion is a path, it is a continuous function, and that is not a finite dimensional set, and so can you have a probability distribution on something that is not a finite dimensional set. That was an obvious question that was left over from integration theory, as developed by Lebesgue and others in the early part of the first decade of the century. I think this was the first concrete example of something that was a probability distribution on an infinite dimensional sense, in this case, the space continuous functions. So the property was you have a probability of distribution that describes the motion of a particle, and that motion is continuous, and the distributions are exactly the ones of the Brownian transition function as derived by Bachelier and Einstein. That was a major achievement in mathematics, and it put Brownian motion firmly on the map as a well-defined mathematical object; it really was not as a result of what either of the previous contributors had done.

Many distinguished contributors came after this. I was arguing with somebody yesterday about whether you would have to put Kolmogorov among the 20th Century's best five mathematicians. You could argue that either way, but he is certainly up there somewhere. He was an extraordinary character, who worked in many areas, but a major one was probability. The standard framework for probability as we see it today, based on measured theory, is the result of Kolmogorov's work in the early '30s.

Paul Levy is an interesting guy because he, in a way, was slightly more in the Bachelier tradition. He was a French mathematician working in the '30s and '40s, and I think he was responsible for a shift of perspective, which is maybe not so easy to describe, but is a key thing is probability: there are always two sides to it. One side is analytics. You can look at a probability distribution as a distribution function; it satisfies some equation like the Chapman-Kolmogorov equation, or like the heat equation, so you just look at the analytic objects which describe the probability distribution of something. Probability, essentially, did that right up until the 1930s, and did not really do very much else.

There is a book by Leo Bryman on probability - it is one of the standard textbooks for first year graduate courses on probability. He says, in the preface to that book, that there are two sides to probability, it is rooted in two different places. One of them is analysis, that is the study of distribution functions; and the other is gambling. Of course this is true. No one can deny the connection between probability and gambling. Bryman credits Michel Loeve, who wrote a big tome on probability and David Blackwell, who is a statistician at Berkeley.

There have always been those two ways of looking at things, so somehow, the mathematicians had settled on this analytical side because it was more comfortable to them, but they somewhat neglected the other side, from which you get a whole lot of intuition and, in the end, a whole lot of mathematical technique. I think probably the contribution of Paul Levy was to shift the attention to the other side, so you use methods which are definitely based on looking at the sample paths of some process as opposed to just looking at the probability distribution of that process.
You get things like stopping times, where stopping time is, the first time some process hits the level or something like that, and that is something which is essentially stochastic. You have to think in terms of the sample paths to do that, and the reflection principle which I mentioned, which Bachelier produced, if you want a proof because you look at some level, you look at the first time your process hits that level, and you think about what happens after that. It will never just pop out of an analytic treatment of the problem; you have to think in terms of how the process actually behaves, so you have to think in pictorial terms to do this. Bachelier did this, and his big contribution was to re-cast probability in that kind of spirit. Paul Levy, who was a much more professional guy, proceeded with the same kind of agenda, so he cleaned up a lot of this and introduced much more formal methods and produced a lot of results, but in a way it was a development of Bachelier’s thing. Actually, they were antagonists, but that’s another story.

In the post-'30s period, we already mentioned the idea of Martingales, and from Bachelier’s perspective, the Martingale was the key thing, although he did not have that name; the name Martingale was introduced in the late-'30s. But he was using what essentially was a Martingale property in his expectation zero hypothesis. The big book is by Joseph Doob, who published his book in 1953. Everyone loves this book. It gives a very, very nice treatment of Martingale theory, much of it originally due to Doob himself.

The other thing that turned out to be a key factor in this whole story is the theory of stochastic differential equations, or stochastic processes, and this is due to Kiyoshi Ito, who amazingly enough was doing it in the middle of the Second World War in Japan. The first paper was published in 1944 under the proceedings of the Imperial Academy of Sciences in Japan – truly amazing. We already saw, right at the start of this talk, that there is a connection between Brownian motion and the heat equation, so what is that connection in some more general context? If you just move away from just Brownian motion but you look at other sorts of random, some kind of transformation of Brownian motion, you get other sorts of equations. What is the connection between partial differential equations and stochastic processes? In order to solve that, Kiyoshi Ito invented this thing called Ito stochastic calculus, which has been tremendously influential over the last 50 years and is an essential component of the analysis that we are using in finance.

Where it comes from is just looking at the sample path – again, it is a sample path thing. You look at Brownian motion, and you compute its so-called quadratic variation. You look at these increments and you sum up squares of increments, and you do that over a partition of some finite time interval. Then you refine that partition, so you get it smaller and smaller, larger number of points and smaller and smaller distances between the points, and as you take that limit, it turns out the limit is simply equal with probability one to the length of the interval. So you get a random motion, but for every part of that motion you have a deterministic quantity, and that is its quadratic variation. It is a totally amazing property of Brownian motion, but this is the property that distinguishes Ito calculus from ordinary calculus. What Ito was pointing out, or the idea behind this, is that when you do a series of expansions involving Brownian motion, you get the usual two terms, but then you get a third term which involves the square of the increment and that is not a second order term anymore. That is a first order term, because of the property at the top. That is an increment – that square is of order DT, not of order DT squared, so you have to keep it in the list of terms. You cannot throw that last term away; you have to include it. That gives you the Ito formula.

Ito developed this. His calculus is not the same thing as ordinary calculus because you are dealing with functions which are rougher than ordinary functions. Any function which has a bounded variation, the correction term at the end would disappear, but Brownian motion has exactly the right properties, and you can replace the DB squared by DT, and that is the Ito formula. So that’s Ito’s great result, and you find this everywhere. Part of the Black Scholes formula is the Ito formula.

The 1960s saw a big consolidation period for all of these sorts of ideas. The next character in our cast list is Paul-Andre Meyer, who ran a school of probability in Strasbourg which became very, very famous, and he proved the so-called super-martingale decomposition theorem, which opened the way to extending Ito’s stochastic intervals, to general martingales. So instead of just integrating Brownian motion, you could integrate with respect to any kind of martingale, so that, from the practical point of view, enlarges the modelling framework to encompass a whole lot more things.

The connection between the heat equation and stochastic processes was really fixed once and for all by Strook and Varadhan in 1969, the so-called martingale problem formulation diffusion, that somehow, martingales came in, in a very nice way, and even in the analytic side of probability.

Then there is the famous result by Bichteler and Dellacherie in 1979 which was a termination point for the whole theory of stochastic integration. Meyer ‘and Co’ invented something called a semi-martingale, which is a sum of a martingale plus some bounded variation term, and the whole theory is based on this. The question is why? It seemed rather artificial at first. But then, Bichteler and Dellacherie had a very beautiful theorem in 1979 which showed that if you wanted any kind of reasonable theory of integration for any kind of process, it would have to be a semi-martingale, otherwise there is no reasonable theory. So they were saying, yes, the other guys got it right, that is the correct class of process to look at; you cannot really look at anything else, and so it just drew a line over the whole thing. The theory was there for semi-martingales, and they showed that there is no other theory, and that set the whole thing very nicely in context.
So the bottom line here is, this whole sort of area started out as an area for pure mathematical specialists. I should think if you looked at anything up to 1963/4, and you said how many people in the world really understood Ito’s stochastic calculus, then the answer was probably in the hundreds. There was no textbooks, no one had any applications in mind; it was a very, very narrow speciality, and if you look at the number of people who understand it now, then I supposed it must be virtually everybody within two miles from here and one mile from here down there understands it! So now, you get textbooks – this is a piece of pure applied mathematics essentially. There are well-developed rules, it is widely understood, there are courses – taught by people like myself — on this area, and it is just simply part of the picture.

If we then move on to the economic side of this, there is a story behind this. The key figure here is Paul Samuelson. Samuelson is an economist. I actually talked to him in 2003 in connection with the Bachelier book, when he was a young man of 89 – and he is a young man of 89, just an extraordinary guy, so he is now a young man of 93 presumably. Samuelson is best known I suppose as the author of a textbook on economics. The first edition came out in 1947, and there have been successive editions.

Every economics student in America learns economics from Paul Samuelson and his team of updaters. He got interested in option pricing in reasons of which I am not entirely sure, but somehow, in the 1950s, it started to be a germane question again – how to manage financial risk – and so options were things to consider in a more serious way. Samuelson started doing this in the 1950s, and then he received a post-card from somebody called Jimmy Savage. Jimmy Savage was a leading mathematical statistician of the late-40s/early-50s period, a very famous guy in mathematical statistics. He knew about Bachelier, because mathematicians did, and he thought it was about time the economists woke up, so he wrote post-cards to his economist friends saying if you haven’t read Bachelier, do. As far as we are aware, the only person that actually did was Samuelson and, when he did, he realised that there was a whole cornucopia of technique there which he needed to know. He was really sort of inspired by this. When I went to talk to him, I asked, “What was it like 65 years before you were doing this; what on earth had Bachelier got to teach you?” He replied that it was the methods, so the whole idea of treating things as stochastic processes, with probability distributions and the stochastic analysis side: it seemed to be just the right way to handle this sort of problem, and that is why they wanted to do it. It turns out that there was a big debate.

It is very hard nowadays to appreciate what this debate was all about at the time, but I think the idea was that if you think of what economists thought they were there for, in the 1950s, or maybe at any time, if you are a financial economist trying. They are trying to explain the process of price formation, so you have a whole lot of interacting agents who are trading things with each other, and decide how to arrive at prices. That is what financial economists thought they were there to do: to look at the structure of a market, and ask how prices arise from the activities of agents.

There were beautiful theories about this in the 1950s by Kenneth Arrow and Gerard Debreu in particular, where they show that you have a group of people who have some goods but they can trade goods with other people, and they do that, and they do that in a way which is in their own best self-interest, then the whole system can be put together in such a way that you get a unique set of prices for trading commodities between the agents. This was a beautiful piece of work.

Then, along comes Paul Samuelson and says, okay, we are going to throw away all that and we are just going to say our price is Brownian motion. So the economists say, well, you’re crazy, this isn’t economics, this is just playing around, because it has no connection with how these prices are formed. That was the sort of debate that was going on, and this is why Samuel ran into some opposition and why there was some debate. But in fact he was on the right track, and the reason he was on the right track was because, when you start looking at option pricing, you are not really interested in why the price of your IBM stock is what it is. What you are interested in is why is it that the price of an option on IBM stock is related to IBM stock the way it is, and that is a completely different question, and one that is best answered by a stochastic process approach, and this is what Samuelson realised. That was why he was interested.

He looked at Bachelier and he said, one thing that was no good about Bachelier is the fact that the price is Brownian motion, so that has a normal distribution, so it can be positive – in fact it will be positive – it will take negative values to the positive probability, whereas prices must be positive things. So he suggested using geometric Brownian motion. This was called geometric Brownian motion, and that is basically Samuelson’s price model. As soon as you do that, you realise you have taken a non-linear function of Brownian motion, and if you want to do anything, you have to use Ito calculus. That is why Ito calculus became a definite part of this particular picture.

The equation I have written down is the standard Black Scholes, Merton, Samuelson model for prices, where you have a stochastic differential equation, which just looks like a geometric increase, like constant interest rate is some sort of noise perturbation, and if you work out the explicit solution for that, it involves Ito calculus to get from five to the next equation. You get an exquisite solution for price in terms of a Brownian motion, so the price turns out to be a non-linear function of Brownian motion and time. That is good.
Samuelson himself did not get the Black Scholes formula. There is a lovely paper called Rational Theory of Warrant Pricing by Samuelson, which has all kinds of wonderful ideas, but it does not have the idea. The idea was produced by Fischer Black and Myron Scholes, with Robert Merton sitting on the sidelines, but the actual paper is by Black and Scholes. There is a quote from the abstract of their paper which says: “If options are correctly priced in the market, it should not be possible to make sure profits by creating positions of long and short – or taking long or short positions and their underlying stocks. Using this principle, theoretical variation formula is derived.” That was the key idea: that you should look at the trading, so there should be no inconsistency in holding an option and trading the asset. If you could have a situation where you could trade the underlying asset and produce something which has the same value as an option, then the amount of money you would need to start trading with must be the value of the option, otherwise there is riskless profit – you could just buy one and sell the other, and you would walk away with money for nothing. The idea is no arbitrage. The price is something that should be fixed by fixing the initial capital you need to replicate its payoff by trading in the market.

It turns out that works. So there is a value, there is a uniquely specified value, based on that principle, for a call option, and that value is the Black Scholes formula, for which two out of three of these people won the Nobel Prize in Economics, if you think the Economics Nobel Prize is a Nobel Prize, which not everybody does! The one who did not was Fischer Black, who unfortunately died before receiving the Prize, but I think everyone agrees that he was the smartest. Well – maybe not everybody agrees on that!

After that, there was a big cleaning up operation. Black and Scholes had a paper. It was a little bit rough, they missed a few tricks, and there is still the underlying principle was clear up to a point, but not completely clear, but there was a cleaning-up operation over the next six or seven years, which put the whole thing in good shape.

Just to mention a few things. There is work by Cox, Ross and Rubenstein in the late 1970s, and they introduced the Binomial Tree, which is just like a discrete, geometric, random walk, an approximation to Brownian motion, and if you have that structure, it is very easy to do all the calculations, much easier than the Black Scholes formula itself. It starts out with something that has its price S0 today, and then the assumption is that it just moves to one of two prices tomorrow, and then tomorrow, it will just move to one of two prices the day after that, and so on. So you get a tree structure for prices, and then everything is finite probability, you can work out everything, extremely easily. What you find is that you do not have to specify – the really nice thing about this is when you have a tree structure like that, in order to find out what the value of an option is, you do not need to know what the probabilities are for moving up and down. There is a set of implied probabilities which give you the pricing formula, so it turns out that the pricing formula for options in that binomial tree are just given by taking expectation, or average value of the option pay-off, with respect to some implicitly defined set of probabilities, and that is called the risk neutral measure or martingale measure. Those prices, those probabilities, are the probabilities that make the underlying asset a martingale. The martingales pop up in a very natural way, and you get a list of five properties that it says that you can only have – a little simple model – there will be no arbitrage, so there is no opportunity for riskless profit, if and only if you can find some set of probabilities which make the prices martingales. Then, any contingent claims or any option has a unique value, which is consistent with absence of arbitrage, and the unique value is the replication value that I already pointed out, and you can express it in terms of just an expectation. That is saying that Bachelier’s idea is correct, if you use the equivalent martingale measure thing. So it is not the real probabilities; it is some probabilities that are implied by the structure of the market. That works for the binomial tree, and then the question is how far can you push this list of properties, how far does it extend? It certainly extends to the Black Scholes theory because that works in exactly the same way. It is not obvious that it does, but it does, and you get a list of properties. The question is how much further can you go.

This was gradually answered over the next little while. There are two famous papers by Harrison & Kreps and Harrison & Pliska in the late-1970s, where they brought in general martingale theory, modern theories of stochastic integration, and they formulated those requirements in a general context. By that time, you had fundamentally got the whole picture, so that turned financial economics into mathematical finance, in the sense that everything you see is a well-defined mathematical object. There are no economic principles which do not have a clear mathematical formulation, so the whole thing has been mathematical.

The definitive answer to this question about how far the relationship between no arbitrage and martingale measures would go was still not answered. It was answered much later in 1994 by Delbaen and Schachermayer.

If you look at the way that that impinged on the outside world, everything happened very quickly. 1973 was the Black Scholes formula, and that was the same year that a traded market in options started in Chicago: the first traded options market, as opposed to over-the-counter deals in options. I do not think the two things are related directly. I think that it takes longer than a few months to plan an options market, so they must have got going before Black Scholes was invented, but the two things did happen in the same year.

1979 is a key year, because this process of mathematisation was really done by Harrison, Kreps and Pliska in those three years. The basis of Bichteler and Dellacherie, as I mentioned, shows that, semi-martingale is the only way to go, so the theory was correct because it was impossible to have any other theory.
The trading in the fixed income world, in interest rates, really started taking off in 1980, around the same time. There was a huge expansion of the range of assets which were treated by option-like methods.

Returning to the point about computation, this was the beginning of the cheap memory era in terms of computation. You cannot run an options business without cheap memory and good computers. 1979 was the first IBM computer, and it opened up the whole era – the technology was in place, as well as the mathematics being in place, and that combination won the day.

Then I should just mention that if we look at what is happening now, we are all in a state of slight uncertainty because there are certain little problems in the markets, which you may be aware of, right now, and this is relating more to credit risk trading. Trading in credit as an option-like endeavour began much later, in the late-1990s, between 1996 and 1998. That, like the swaps market earlier, was a huge part of the market, but it is one which was still far less well understood than the ones before. You have a feeling that maybe the original economists got it right, and that you had to look a little back, a little further into the process at price formation and so on to really understand the credit risk market, and so maybe this whole thing has been just a big detour in the wrong direction – who knows?

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