Part Three - 'Mathematics and Foreign Exchange' and 'When Computing Met Finance'
Transcript

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I have to say that when I set out to prepare this talk, the material that I'm going to actually present today I imagined would be approximately a quarter of what I was going to talk about, but I got into this subject and I found that I was really rather satisfied with the conventional wisdom on some of these things, so what I'm going to talk about is - it says there - arithmetic at the end of the 13th Century, so this is definitely medieval, and I apologise for not going into the later developments in foreign exchange, which have a great deal of mathematical interest, but in a different way.

So the talk's actually going to be in three parts, as usual. I'm going to start by talking a little bit about the financial, commercial background; I then want to talk a little bit about the technology, as it would be said, that is the arithmetical tools that were available; and then I want to see what you can deduce by putting them together.

So we begin with some money. If you were a Viking, at the beginning of the 10th Century, this is what you would have understood by money. This is the Cuerdale Hoard, which was found - Cuerdale is in Lancashire, and all the objects are bits of silver - we now call them hacksilver - and among them, there are some rather special bits of silver which we call coins. They were not of course Viking coins; they were Saxon coins, Anglo-Saxon coins. The Vikings took them over and treated them as part of their money system.

So what do we mean by money? For our purposes today, 19th Century analysis will suffice. A modern economist would give a series of lectures at this point on the functions of money and its uses, but Jevens wrote a book in 1972, which was very popular and very influential, and he found it worthwhile to distinguish from the beginning between a medium of exchange, by which we mean the money objects, the coins the hacksilver, and so forth, and the accounting units which are used to measure value. He believed that money arose mainly because these devices made barter, or avoided the inconvenience of barter. Modern economists would think of other uses for money, and many other kinds of money of course, but let's just think about barter for a moment.

Here we have a picture from the 11th Century, which was printed by Salzman under the title of "A Simple Bargain". So, we think we know what's supposed to be happening here. There's a chap who appears to have a chicken, and a chap who appears to have something else, and they're trying to barter, but it's not as simple as that. First of all, the something else looks a bit like an old sock, which wouldn't actually be much good for barter, and then, if you look a little more closely, you'll see that the chicken man has in his hand a coin, in other words money, but it's not clear whether he's received the money from the sock man or whether he's actually giving the money to the sock man. I've looked at the original of this, or at least what Salzman says about it, and it's still totally unclear what this is supposed to represent. Those of you who get the BHSM bulletin will have read a paper by John, from the Open University, John Mason, who explains how barter came to cover an enormously complex operation in the later Middle Ages, with the intervention of money in all forms, money and credit, whether it was ready money or not. So barter wasn't all that simple.

Coins, however, did simplify the situation, but in order to use coins in trade, you've got to know certain things about them. In medieval times, a coin was worth, that it was accounted for, in the terms of the precious metal that it contained, and you could only measure that by actually doing two things. You first of all had to ascertain the mass, which you could do by weighing, but then you had to work out the fineness, in other words, what proportion of the mass was actually the precious metal. That second thing, the assaying, or finding the fineness, was rather difficult. There are methods known, and have been known since antiquity, for assaying both silver and gold, but they are not something you can do on the spur of the moment. There is a touchstone method, but it's extremely unreliable.

So weighing was normal in trade, and here is a rather depressing picture. These are some poor slaves are being sold, but you can see that the essence of the transaction is that the coins that are being paid for the slaves have to be weighed, and so they are not accounted for by the number of the coins - it is the actual amount of silver or gold that is concerned.
Some improvement came along in Western Europe, Christian Europe, when gold coins were reintroduced. There had been gold coinage in Roman times, and it was reintroduced, gold coinage, in Islam of course, where gold was in supply from Africa, but gold coinage didn't come into Western Europe until, wasn't reintroduced until the 12th, 13th Centuries.

So here we have the four representative coins from that time: starting off with the typical Islamic dinar; and then a coin, Castilian coin, imitating the dinar, the second one, obviously the same design, but with a different kind of lettering; and then we get into the more typical European-type coins, something called an Augustale of Frederick II; and then, playing a large part in the rest of this talk will be the florin, which was the Florentine coin which began to be minted in 1252.

Because there were a lot of these things coming into circulation, it was necessary to know the fineness of the coins. I've mentioned the weight. We assume that a merchant could ascertain the weight of an individual coin, but in order to ascertain its fineness, he either had to do a very complicated operation, which was beyond the bounds of practicality, or refer to some reference. Here we have the list made by Pegolotti, a Florentine merchant, which was compiled around 1300, and these are just some of the gold coins that were in circulation at that time. I've put little marks against the four that were actually depicted on the previous slide.

So the first one is the florin, and the Florentines claimed that this was 24 carats fine, so you see something there which says the number 24 is the crucial one - 24 meant pure, so everything was divided into 24 parts, and instead of a percentage, which of course they didn't have, it was how many parts out of 24 were gold. The Florentines believed that their florin was pure gold. If you look down, you'll see that the Castilian coin was 23½ carats fine; and the Allessandrian bisant, the Islamic coin, was believed to be 23½; and the poor old Augustale, or Agostantini as it's called here, was only 20 and something, 20 and a bit, carats fine. In a transaction of course, large sums of these things were being used, and it was necessary to know this fineness.

So that's the starting point. Now we move on to the international situation.

So, if you want to send your wool to Florence from London, then, well, how are the Florentines going to pay for it? They could send florins, but that would be a risky business, sending a large amount of money by ship, and so in the 13th Century, the mechanism of the bill of exchange grew up. I'm going to just very briefly explain how that worked. This is based on the account given by the eminent medieval historian, Peter Spufford, who has made a study of all these subjects.

So this is the situation for sending a shipload of wool from London to Florence, and how it's going to be paid for using the bill of exchange mechanism. So there are the four parties, two of them in London and two of them in Florence. On the left-hand side, they're the principals if you like, and on the right-hand side, there are the bankers. What happens is that the agent in London sends the shipload of wool to Florence, and various entries are made in the books to account for that. The remitter, as it's called, in Florence then pays for the wool, but pays a banker, a drawer, in Florence, in Florentine currency, florins, and various accounting things are made. In order to settle up, we then draw up, the banker draws up the bill of exchange, which he sends back to the remitter, and also notifies an agent of his, the payer, in London, and the bill then goes back round the system, is presented by the payee in the London to the payer, and the payer then pays the payee, this time in London money, sterling. So the point is that, in order to carry out this transaction accurately, there is obviously going to be some mathematics involved, and the next step is to ask what technology, that is what mathematical tools, arithmetical tools, were available at this time in order to carry out such transactions.

Now, I'm glad to say that Fenny Smith and I agree on several of the points I'm about to make, so excuse me if I'm repeating things that she said. But we all know that you can't do sums with Roman numerals, and of course the Romans weren't stupid - they managed to manage their empire, large empire, for several hundred years without actually trying to do sums with Roman numerals. For sums, they used the abacus, the calculus, the counters on a grid, and they became very adept at doing that. The word "abacus" can be confusing because many people think of abacus as meaning some eastern object - sliding beads and so forth. Abacus, in this context, means any grid or plan onto which the counters could be used. The other problem with it is that of course the abacus would quite often be just scratched in the sand or drawn on a slate, and so very few of these things have survived, and the counters themselves, although there were undoubtedly lots of the counters, when they're dug up nowadays, the archaeologists tend to say that they were gaming counters or things of that kind, that they're quite often misclassified.

There were other ways of doing arithmetic, and here's a particularly horrid one, moving into the medieval period now. This comes from Bede, and this is finger reckoning. Although it's great fun, I cannot really believe that it was a very practical method of doing arithmetic, and it's really just there for interest.
However, when we come to the turn of the millennium, things begin to improve. This is usually associated with the man who became Pope Sylvester II, who I shall pronounce as Gerbert - if anybody knows better, please tell me - who is credited with various steps in the improvement of arithmetic. It's confusing, because not only is he credited with introducing an improved form of abacus, but he's also associated with the introduction of the Hindu-Arabic numerals. So let's see what actually was available, and for this, I'm going to rely on two accounts, and I rely on these two because they are basically confirmatory - they both say roughly the same thing, although they don't appear to a great extent in the published accounts in the history of mathematics.

The first one, this is a book by Florence Yeldham, published in 1926, called "The Story of Reckoning in the Middle Ages", which I would recommend. She gives this manuscript from Ramsey Abbey, and she describes the accompanying instructions from the manuscript. This is, if you like, an improved form of abacus. The first thing to notice is you've got the Roman numerals heading the columns, so you've got...working from the left, you've got units, tens and hundreds, and then repeated again, units, tens and hundreds in the thousands, if you like, and so on up to...I think there are nine lots of the columns. So that's Roman numerals. But then, hidden amongst the arches at the top, you will see Hindu-Arabic numerals, and also, while we're talking about the writing, what's written along the bottom are in fact the Roman names for fractions, which indicates that division sums were being done here as well. So, the other important thing, which is not apparent from this diagram, but which is confirmed by the descriptions, is that the counters that were used in Gerbert's abacus did in fact have the Hindu-Arabic numerals written upon them, and the calculations were done by moving these numbered counters around.

A similar account is given by Turchill, who was believed to be a clerk in the royal household, often said to be a clerk in the Exchequer. That is slightly confusing. The book which I use, rely on for this, is the account given in Poole's book on the Exchequer in the 12th Century. Poole was of course mainly interested in the Exchequer, but the Exchequer is rather misleading, because although an abacus of this kind was undoubtedly used in the Exchequer, and indeed gave its name to the Exchequer - the chess board - it was a simpler kind of abacus for a quite different purpose. It was for counting the sums of money that the sheriffs brought in from the counties, and it was only really...the sums that were done were only really additions and subtractions. But Turchill I think must have been a clerk in a more senior position, if you like, and he wrote about the abacus in general, and his account tallies very much with the one that was given in the manuscript described by Yeldham.

So, what were the two distinctive features? The grid was arranged in the columns, and the counters were labelled with Hindu-Arabic numerals. This is - Gerbert was around the turn of the millennium. In fact, I think he was Pope at the turn of the millennium. These accounts are 1100, beginning of the 1100s, the 12th Century, so this is all quite some time before even Fibonacci, and my conclusion is that, in the higher realms of government, and perhaps in the large monastic estates, Gerber's abacus, with all the frills, was well-known in the 12th Century. Now, that shouldn't deflect us from noticing that of course older forms of the abacus, with plain counters, useful for doing additions and subtractions and accounting and so forth, were used by merchants and, as Fenny has said, we know that that carried on into I say the 17th Century - you may say the 18th Century. People still used the jetons and the counters for doing simple arithmetic. However, from a mathematical point of view, the real interest is that Gerbert's abacus evolved into a system where you weren't just moving the Hindu-Arabic numerals on the counters around, but actually using the same procedures, the same algorithms, for doing the calculations with the numerals themselves. In other words, what we would often call pen-reckoning, but doesn't have to be with a pen. It could be traced in the sand with a stick, or it could be they were written on a slate, and for both of those reasons, it's unlikely that we'll find many extant examples of them, but nevertheless, my belief is that the Gerbert abacus and the pen reckoning algorithms, using Hindu-Arabic numerals, were very similar, and in certain quarters, they were well-known by this time.

Here is one of the few examples of pen-reckoning, from 1320. These are just long addition sums, but it's clear that that's using the sort of standard methods. There are the carrying symbols and so forth, and the numerals are almost recognisable to a modern eye there, so you can check that out.

So, one reason why I think there's some confusion about what was going on, about the technology that was available, is that there are misleading clues. One of the misleading clues is the oft-quoted fact that in 1299, the Florentine bankers' guild is said to have banned the use of the Hindu-Arabic numerals, and you will read this in a number of standard texts without really any further comment, but of course they weren't actually banning the use of the Hindu-Arabic numerals in the banks. All they were saying was that you must carry on publishing your accounts in the traditional method, using the Roman numerals to record these things, so that everybody, or at least everybody who was interested, could actually understand them. In other words, the general population who could understand Roman numerals could see what was going on. They were not banning the use of Hindu-Arabic numerals in the banks, and in fact, as I say there, almost certainly, for about 100 years at that time, the Hindu-
Arabic numerals had been in use. And just to echo again something that Fenny said, we know that this led to the establishment of the schools in Florence and other Italian city states, where the children were told - confusingly it says abaco, abaco and algorismo, but one shouldn't think of those as being distinct things. The algorithms for doing calculation is really what's being concerned there.

I recommend this book by Alexander Murray, which I don't see quoted very much in the history of maths literature. He approaches it from a medieval historian point of view, but in fact, he has several chapters which I find compelling and convincing on this subject.

Ah, there we go! You've seen this one before, and like you, I think it's a source of confusion. We have this smiling Boethius it's supposed to be, which of course is complete nonsense historically, doing his Hindu-Arabic sums there, algorithms; and we have poor old Pythagoras, looking very glum, trying to use an abacus, but he's using the plain counter sort of abacus and not enjoying it very much, by the look of it. So this is to be thought of as a sort of modern Bostock and Chandler sort of book, which sort of tells you how to do things at a certain level but doesn't actually lead you on to the higher levels of thought.

Okay, well let's see what happens when we try to put these two things together?

Let's now sort of see when we try to put these things together. Now, there's a problem here, that in order to describe this succinctly, to a modern audience, I'm more or less forced to use modern notations, symbols and so forth. We must not think that this was the way the medieval arithmeticians thought about it. They didn't even have an equals sign - as we know, that was Robert Recorde 200 years later. They certainly didn't have the idea of let x equal such-and-such, and then doing elementary algebra. So it was all done in a more...verbal kind of way, and the arithmetics of the time would talk about the rule of three, for example, which to a modern eye, is just a simple proportion sum, but they would have a number of mechanisms for working out, if it was a is to b as c is to d, depending on which one was the unknown that you wanted to calculate, you had different rules and methods for doing it. Okay, but I'm going to talk in these terms and hope that we can thereby elucidate what was going on, without overriding the mode of thinking that lay behind it.

So, by 1300, of course there were many places, centres of population, where trade was flourishing. Let's first of all think about a single place, x, and in this place - we'll go back to Jevens' distinction - there would be a money object, which I call Mx and there would be accounting units, which merchants would use to keep their books. The relation between the two would be determined by the authorities in that place. If it were London, even in earliest times, they were subject to the king for the coinage and so forth, so the king would say how much a money object that the king had issued ought to be accounted for, what it was worth, if you like, in accounting units.

Well, let's look at London. So the accounting unit was the sterling penny, which is just an abstract object from this point of view. It's an accounting unit. Now, there was something called a penny, a coin, a money object, but it wouldn't necessarily be the same as an accounting unit. It might have been clipped, and if you wanted to make sure you were getting your money's worth in any particular transaction, you might have to weigh out the silver coins and make sure you had the right weight of silver, and even then, you would be trusting that the coins were of the correct fineness, that is contained the correct amount of silver. So for larger transactions actually, in London at this time, things were calculated in marks, and a mark was said to be worth 160 sterling pence, or if you prefer to use the next accounting unit, 12 of these sterling pence was an accounting unit called a shilling. There was no coin called a shilling - that's something that's quite clear - no coin called a shilling until the 16th Century in this country, but the accounting unit of a shilling goes back much further. So the mark is not a pound or a certain number of shillings; it's a certain number of pence - it's actually 13 shillings and 4 pence, a very odd amount - and when, in England, they started to mint gold coins, they tended to be actually in these proportions, nobels of 6 shillings and 8 pence, and things like that.

In Florence, on the other hand, they had of course a different kind of penny - denari piccoli - small pennies, and they were small, and they were often not made of silver - well, the objects were not. The accounting unit, however, which was the abstract thing, was - the accounting unit that they used was called the denari piccoli. By this time, we're talking about 1300. The gold florins were in use in Florence, and it was decreed that a gold florin was worth 348 of these accounting units - that's actually 29 twelves, 29 twelve times...yes, 29 times 12, yes, that's right. So the Florentine shilling was called a soldo, and it was 29 of those.

So suppose we're trying to send our shipload of wool from London to Florence. Obviously, the transaction is going to be governed by what we would call an exchange rate, and I'm going to tell you how I see this from a modern viewpoint, and then
we'll look at what the Florentine bankers wrote down about this.

So I would define the exchange rate as the number of florins that equals one mark; in other words, it's the relationship between money objects, actually coin - well, actually there wasn't a coin called a mark. There might have been 160 pennies called a mark, but in fact, there might have been a weight of silver called a mark, or there might even be an ingot of silver called a mark. The use of ingots is quite well documented. So there's that exchange rate, which is the modern way of defining it, and of course there's the reciprocal one if you're going the number of marks that equals one florin and so forth. Now, this is not a constant number, fixed for all time, because it depends on economic factors, which is a sort of catchall term for saying that if there was a shortage of gold or a shortage of silver, gold and silver are used for other things rather than making money objects, and therefore they have a value which is affected by shortages and so forth. Other things as well can affect the exchange rate. So this is variable, and if we're going to do any sums involving the exchange rate, we've got to have a range of exchange rates available so that we can work out which one to use on a particular occasion.

This is what Pegolotti did. This is his table for the exchange between London and Florence, and it gives a range of values of a parameter alpha, and the corresponding values of a parameter beta.

So let's just look at one of the lines, focus on the one which has 34 in it, okay. What that says is that when 34 sterling pennies go for a florin, then the mark, the sterling mark, goes for 6 lire, 16 soldi, and 5-and-11-seventeenths denari. So each line is a statement of the form that I've written at the bottom. You'll notice there's the confusion...not confusion, but at least the double use. Some of the things here are the actual money objects that are going to be handed over, and some of them are the accounting units which are going to appear in the bankers' books. So that's why it has to be written in that form. So the exchange rate, e, that I had on the previous slide, has to be interpreted in terms of the Pegolotti table. So Pegolotti is making a table of values of a parameter beta in terms of parameter alpha.

Well, I think I've said one of these things already. The first one is that the number of denari in order to avoid having large numbers of denari, you have these higher units, super units, of the soldi and the lire, which incidentally, are the same multiples, 12 and 20, as would have been used in London, except that in London, it was a sterling penny, a shilling, and a pound. That's actually just a matter of convenience. We've also noted that the two types of money, or two uses of money, are signified in the table, because when payments were made, money objects had to be involved, and when the books were kept, the accounting units were involved.

Now, of course, we're already getting into the stage where we see new types of money being created, because the bill of exchange that I talked about is now becoming an object of money in itself, because it doesn't - if you'd got a bill of exchange which said somebody was going to pay you something, you didn't actually have to present it for payment yourself, you could try trading it on to somewhere else. So we already begin to see this layer upon layer of financial operations beginning to take place.

Here are the sums, and once again the caveat about this is not the way a Florentine banker would have done it, but it tells us in modern terms what the sums were, and there's some point in doing that. So, first of all, if the alpha in Pegolotti's table was the number of sterling pence that made a florin, and we can translate that into the number of marks that make a florin, and in terms of the e that I had before, that was the reciprocal one over e. Similarly, the beta, the answer that comes out - so the beta thing, to remind you, is the number of Florentine units, florins, that are equivalent to a mark. That comes out to be e times v_f, v being the 384 number, the number of denari piccoli that make a florin, and eliminating, doing the algebra, which of course medieval arithmeticians didn't do it this way, you see the relationship between beta and alpha straightaway. So in order to do this, make up his table, Pegolotti had done this sum, and that's why those strange fractions came into it. He had divided v_f times v, and v_f - I've done this in general because the other thing perhaps to keep at the back of our minds is the fact that exchange was not just between two places. There was a whole network of places between which exchange could take place, and so we would make suitable substitutions for London and Florence, and we'd get the exchange between Florence and Bruges and places like that, for example. So, that's what the table is. It's a table basically of division sums.

But now, we can begin to see a little bit more what's going on - hopefully this has thrown some light on that. So, if you're a clerk in the bank in Florence, what you want to know is how you're going to account for the shipload of wool, which is supposed to be worth so-many marks, and what you're going to put in your books, and the answer to that... Well, no, first of all, the answer to that depends upon what is the exchange on [London], what is today's exchange rate, the parameter alpha. So Pegolotti's table told you that if alpha was 34, then beta was whatever it was, okay, and in order to get the accounting answer that the clerk required, he would have to multiply the number of marks by the value of beta, the value of beta that corresponded to the given
alpha. So the clerks were doing multiplication sums - relatively simple, probably could be done with the abacus, the unnumbered abacus, the abacus with plain counters. Certainly, the later books tell us how we could have done that kind of thing. But in order to draw up the table, either Pegolotti, or the person who calculated it for him, had to do a division sum, and quite a complicated division sum. He had to divide this number by alpha in order to obtain the correct value of beta. So, this seems to me to make it very clear that even by this time, there were two different levels - perhaps more than two, but anyway, at least two different levels of expertise in arithmetic. You had the level of Bostock and Chandler calculus, where people can do the sums - in other words, when you know that the derivative of x squared is 2x or whatever it is - and you had some other people who really understood a bit more and who could do more complicated calculations. The second group of people, who would be, if you like, in the back room at the bank, they would be either, by this time, probably not using Gerbert's abacus, but actually using the Hindu-Arabic numerals pen-reckoning and so for, but their expertise, their methodology, had developed from the Gerber version.

So, to round off... There are a large number of conclusions, and if you go to that book by Alexander Murray, which I mentioned, you will find a very useful attempt to put all this in the context of the growth of numerate thinking in the later Middle Ages, asking such questions as to what extent the growth in mathematics - Cardano, Tartalli and so forth, the solution of algebraic equations - to what extent that depended upon the base that had been established earlier on for commercial reasons. Of course we shouldn't think that commercial arithmetic was the only stimulus to the development of arithmetic. Astronomy was equally important, and the calculations that had to be done by the astronomers would have used, I believe, similar sophisticated methods and certainly weren't done with the plain counters that the merchants used.

So, well, let's just, as I say, summarise what's here. Different levels of expertise, and it's not always apparent from the evidence that's in front of us. The evidence can be unbalanced because we have the printed books from the 16th Century, which are telling us all about how merchants can do their calculations and so forth, and there's a lot of that. Evidence for the pen-reckoning method is harder to come by because it was essentially ephemeral, and whether it was done by scratching the figures in the sand or by writing them on slates, it tended to disappear.

And then finally, the generalisation about the numerate thinking, that there is in fact evidence that the commercial considerations were important in developing numerate thinking, possibly only indirectly, however, in that the abacus schools and so forth led to generations of Italians in particular, but other countries as well, where numerate thinking and so forth was part of the training. There were some things that didn't happen, and it's always...the dog in the night-time is always an interesting speculation. It doesn't seem that there was any serious progress in the concept of number at this stage. You might think that, given the fact that the exchange rate, for example, small changes in the exchange rate could lead to quite significant differences in the way the sums came out, you might think that that would lead to the idea of the number as a continuum, but in fact, the standard method throughout the Middle Ages of dealing with smaller quantities was to invent smaller units, so you had farthings, and then, at one point, they invented something called a mite, which was a fictitious thing equal to one twenty-fourth of a penny, but there was no idea of what would lead to the decimal notation, that is, having tenths, and then tenths of tenths, and tenths of tenths of tenths and so on, having the same multiplier over and over again. Although that was, in the Hindu-Arabic system, used for multiples, it didn't seem to come in for fractions, sub-multiples, until the 16th Century, as we know, and without that, it's hard to lead on to the idea of the number continuum and the infinite decimals and so forth which you require to deal with that, and of course the calculus itself, which requires the notion of small change to be codified in some way.

So, I hope that that's shed a little light on the topic. I'm, as I say, I was heartened by the fact that your talk had come to similar conclusions in some of these aspects, and I'd be interested to hear any comments that people have. Thank you.