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MARILYN AND THE GOATS: PROBABILITY PARADOXES

A Lecture by

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Gresham Lecture

Marilyn and the Goats

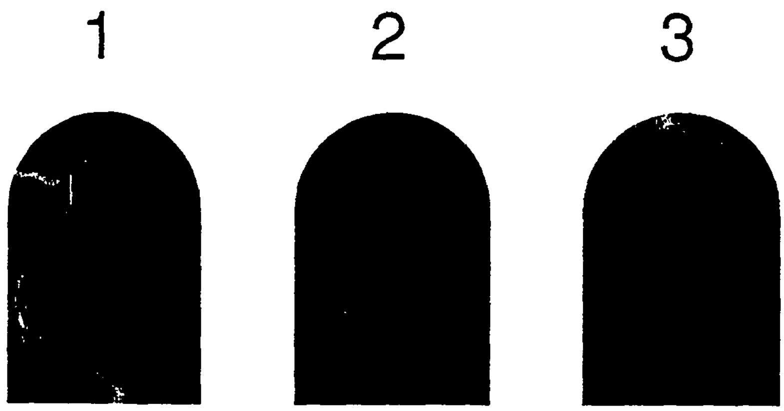
Ian Stewart

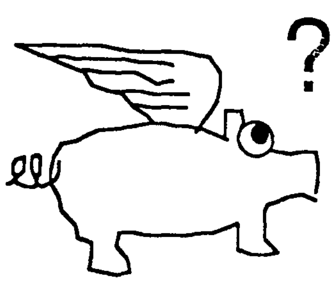
I'll get to the goats bit, and Marilyn, a bit later on, OK? But first:

1. The Myth of Pigasus and Andromeda

Pigasus, the great flying pig, crash-landed awkwardly on the sand: a perfect three-point landing, two knees and a nose. Perseus extracted himself from the dune into which the great beast had flung him, and looked around for the fair Andromeda. *It would be easier*, he thought, *if only the ground would stay still.* "Funny," he said, when his head finally cleared. "I expected her to be chained to a rock somewhere."

"I bet she is," said Pigasus. "In one of those three caves over there."





"The tide gets up pretty high in these parts," Pigasus mused. "It would certainly submerge those caves. We'd better get her out quickly."

"Great!" said Perseus sarcastically, as they approached the caves. "The entrances are all sealed by rocks!"

"If you tie Hyppolyta's magic girdle to my tail and the other end to a rock, we can easily shift the beggar," said Pigasus.

"True," said the Hero. "Let's get started on the first cave, then."

"Um," said Pigasus.

"Um?"

"I mean, aren't you forgetting something?"

"Like what, O great Tub of Lard?"

"Like, Andromeda's in one cave, and the other two contain the gorgon Medusa and her sister Stheno, either of which will turn us both to stone merely by gazing upon us."

"I have my trusty golden shield, to reflect the stony gorgon gaze back upon its owner," said Perseus triumphantly.

"It's an *iron* shield, and the word you are seeking is *rusty*, Perseus."

"I suppose we'll just have to guess."

"*You* may have to guess," said Pigasus, "but we flying pigs have hidden talents, given by the goddess Demeter. I *know* which cave Andromeda is in."

"Terrific! Then tell me!"

"Unfortunately, we flying pigs are sworn to secrecy, and if we reveal our hidden knowledge the goddess Demeter causes us to be encapsulated in a bubble twenty kilometres below the surface of the Earth, never to escape throughout all eternity."

Perseus thought about this. "I'll give you a carrot if you tell me," he said.

Pigasus found himself seriously considering the offer. *Oh, the power of the deadly sin of greed!*, he thought, and shook his head. "However, I *can* help you."

"How?"

"First, I can reveal that the chances of Andromeda being in any particular cave are equal: one chance in three. Now you guess a cave, and then we'll see what else I can safely reveal."

"If you insist. *I* guess... that.... Andromeda is in the middle cave."

"Excellent," said the flying pig. "I can now reveal that there is a gorgon in the left hand cave."

Perseus flung his sword to the ground in anger. "Stupid pig! How does *that* help?"

"I thought the information might induce you to change your mind."

"Oh, come on! That's idiotic! After all, there's got to be a gorgon in at least one of the caves I didn't choose! How can it help me to choose the right cave if all you do is tell me *which one* the gorgon is in? What I need to know is which cave *Andromeda* is in!"

"So you're sticking to your choice, then?"

"Why not? All you've done is cut down the possibilities. I now know that either Andromeda is in the middle cave or she's in the right hand cave. The chance she's in the cave I chose, the left hand one, is fifty-fifty. So there's no advantage in changing."

"Have it your own way,"said Pigasus. "Only..."

"Only *what*?" cried Perseus in exasperation, setting off towards the middle cave.

"Only you'd have double the chance of being right if you changed your mind," said the flying pig.

"Eh? Piggy-wig, you're off your head."

The pig shook its enormous snout, and the twisty tail at its far end jiggled in sympathy. "Look, the chance Andromeda is in the middle cave is one in three, right?"

"So you say."

"Fine. So the probability that you are wrong is twice the probability that you're right. To be specific, the probability you're right is 1/3 and the probability you're wrong is 2/3. OK?"

"Of course!"

"I agree. But let's follow it through. If you're right, and you swap to the right hand cave, then you'll get us both stoned by a gorgon. If you're *wrong*, however, and there's a gorgon in the middle cave, then because I've kindly eliminated the other gorgon, Andromeda must be in the right hand cave. That is, if you're *right* and swap, we're remarkably lifelike statues; but if you're *wrong* and swap, we snatch the fair Andromeda."

"Yes, yes! What difference does *that* make?"

"Thing is," said the pig, "we've just agreed that you're twice as likely to be wrong as you are to be right. So swapping to the other cave will make you right twice as often as you are wrong. In other words, your probability of locating Andromeda is 2/3 if you swap, but only 1/3 if you don't."

"But – "Perseus sagged to the sand, his head in his hands. "O Massive Barrel of Dripping, you know that Heroes have no taste for higher mathematics! How can I decide which argument is correct?"

"You could try taking well-meant advice," suggested the pig.

"No! The chances *must* be even! You've eliminated one cave, so I've got two to choose from! Each is equally likely to be right!"

"That wasn't the order we did it in," muttered the pig under its breath. More loudly it said: "OK, you know best. We flying pigs are going to become extinct soon anyway, no more than a sarcastic metaphor, so why should I care whether you get your mathematics right or not? I'd rather be a rock than stuck in an underground capsule for the rest of time. You go right ahead."

"I intend to," huffed Perseus. *Really, these flying pigs are quite insufferable.* But nothing could be done about that. *If pigs had no wings, they'd walk.* Fat chance! He roped Pigasus to the middle rock with Hyppolyta's magic girdle, and slapped the beast firmly on its substantial rump. The rock rolled away from the opening and...

Place your bets, ladies and gentlemen! I'm offering odds of 3:2 that Perseus is wrong. If he is correct that the odds of success are even, then on average you'll win. Of couse, if Pigasus is correct that the odds against success are 2:1, then on average you'll lose. I'm happy to carry out as many trials as you want. Remember the procedure:

- Andromeda is definitely in one of the caves. The chance of her being in any particular cave is 1/3.
- *First* Perseus chooses a cave.
- *Then* Pigasus points to one of the other two caves, and tells him (truthfully) that there's a gorgon in it.
- *After that*, Perseus is offered the opportunity to change.

He prefers to stick with his first choice, and I am offering odds of 3:2 that he's wrong. That is, if Andromeda is in the cave that Perseus chooses then I pay you 3 francs; but if there's a gorgon in the cave, you pay me 2 francs.

I agree, it's a sucker bet. Are you going to accept it? Write your decision on a piece of paper. Are Perseus's chances 1/2, whether or not he swaps (as he thinks); or are they 1/3 if he stays and 2/3 if he swaps (as Pigasus argues)? If Perseus is right, you win; if Pigasus is right, I do.

2 An Experiment

In order not to give you a clue too soon, I'm going to describe an experiment. It's not hard to write a computer program to simulate the problem, and to run it a large number of times, counting how often Perseus succeeds and how often he fails. You might like to try this before reading any further. I'm going to describe something very similar, but instead of using a computer I'll use a standard table of random numbers, namely the *Cambridge Elementary Statistical Tables*. That should avoid any chance of bias, and in principle you'll be able to check my calculations.

Here's the method. Only the digits 1, 2, and 3 will be used, one for each cave. I'll call these 'acceptable'. I run through the table of random numbers, looking at acceptable digits in the order in which they are printed.

1 The first acceptable digit A determines which cave Andromeda is actually in.
2 The next acceptable digit G determines Perseus's guess.
3. The next acceptable digit P that is *different* from A and G determines what Pigasus

tells Perseus. (Pigasus must choose a cave with a gorgon, so he can't use A; and he has to choose a different cave from the one Perseus chose, so he can't choose G. Note that A and G may be equal: in this case Pigasus is choosing at random from the two caves with gorgons in them.)

4 Assuming Perseus doesn't swap, count whether or not his guess is right.

5 For comparison, assume he takes Pigasus's advice, and swaps to the unique cave that differs from G and P. Count how often his amended guess is right.

6 Move on to the next acceptable digit and repeat the whole procedure.

The next table shows what happens on the first 20 attempts.

The verdict of the Cambridge Statistical Tables

Andromeda's cave A	Perseus's guess G	Pigasus's selection P	Never swap	Always swap
2	1	3	wrong	**right**
1	3	2	wrong	**right**
1	1	2	**right**	wrong
2	2	3	**right**	wrong
1	3	2	wrong	**right**
3	2	1	wrong	**right**
3	1	2	wrong	**right**
2	1	3	wrong	**right**
2	2	3	**right**	wrong
2	1	3	wrong	**right**
3	3	1	**right**	wrong
2	3	1	wrong	**right**
2	1	3	wrong	**right**
3	1	2	wrong	**right**
3	3	1	**right**	wrong
2	3	1	wrong	**right**
2	2	3	**right**	wrong
1	3	2	wrong	**right**
2	1	3	wrong	**right**
1	2	3	wrong	**right**

If Perseus's strategy is never to swap, he is right 6 times and wrong 14 times. Using Pigasus's strategy, *always* to swap, he is right 14 times and wrong 6 times.

Pigasus must be right. *It's best to swap!*

3 So What's with Marilyn and the Goats? All is Revealed.

This result is so counter-intuitive that many people find it hard to accept it. Recently there's been quite a fuss in American newspapers about precisely this problem. A lady called Marilyn vos Savant, who is listed in the *Guinness Book of World Records* as having the highest IQ ever recorded, writes a regular column called *Ask Marilyn.* It is syndicated in several hundred American newspapers. Marilyn posed the same question, but instead of Perseus seeking Andromeda she described a game show contestant who has to choose one of three doors. Behind one is the major prize — a car — and behind the other two are goats. After the contestant has chosen, the host point to a door, different from the one chosen, and says "there's a goat behind that one". Should the contestant then switch to the third door? Marilyn explained, just as Pigasus has done, that the chances of getting the car are twice as good if you swap.

Here are some of the responses from her readers — all at universities, colleges, or research institutes:

• "You blew it! As a professional mathematician, I'm very concerned with the general public's lack of mathematical skills. Please help by confessing your error and, in the future, being more careful."
• "There is enough mathematical illiteracy in the world, and we don't need the world's highest IQ propagating more. Shame!"
• "Your answer to the question is in error. But if it is any consolation, many of my academic colleagues also have been stumped by this problem."

To which Marilyn replied that her answer was correct. Here is one of the several different explanations that she gave. Replace 'Andromeda' by 'auto', 'gorgon' by 'goat', and 'cave' by 'door' to get her nomenclature. Suppose (for sake of argument) that you choose cave 1. (You can check that the same kind of thing happens if you choose doors 2 or 3. Anyway, what's in a number?) There are three possibilities. Each is equally likely because, as Pigasus has told us, the chance of Andromeda being in any particular cave is the same in all cases. Outline typeface shows which cave (or caves) Pigasus can point to.

Perseus's strategy: never swap

CAVE 1	CAVE 2	CAVE 3	
Andromeda	gorgon	gorgon	**win**
gorgon	**Andromeda**	gorgon	lose
gorgon	gorgon	**Andromeda**	lose

Pigasus's strategy: always swap

CAVE 1	CAVE 2	CAVE 3	
Andromeda	gorgon	gorgon	lose
gorgon	**Andromeda**	gorgon	**win**
gorgon	gorgon	**Andromeda**	**win**

There's no escaping it: Perseus's strategy wins only one time in three, whereas Pigasus's wins two times in three.

To which yet more university and college types replied:
• "Your answer is clearly at odds with the truth."
• "May I suggest that you obtain and refer to a standard textbook on probability before you try to answer a question of this type again?"
• "How many irate mathematicians are needed to get you to change your mind?"
• "I am in shock that after being corrected by at least three mathematicians, you still do not see your mistake."
• "Maybe women look at math problems differently than men."
• "You're wrong. But look on the serious side. If all those PhD's were wrong, the country would be in very serious trouble."

Marilyn reported receiving "thousands of letters, nearly all insisting that I'm wrong, including one from the Deputy Director of the center for Defense Information and another from a research statistician at the National Institutes of Health." In all, 92% of letters from the general public disagreed with her and sided with Perseus; the same was true of 65% of the letters from universities.

4 So Who's Right?

Are Marylin vos Savant and Pigasus right, or does that honour fall to Perseus and several thousand Ph.Ds? Before I attempt to settle this thing once and for all (with some trepidation, anticipating precisely the same kind of mailbag), let's go back to computer experiments. You may feel that 20 tries is too small to be convincing. I warn you, you may be grasping at straws; but let's try a more substantial trial. I'll report my results: you may beg to doubt their veracity of you wish. Write your own simulation if you want to; but do be careful to follow the prescribed sequence of events. If you build in short cuts of your own you may end up begging the question.

I ran the problem on a computer, making 100,000 trials. Perseus's strategy gave the correct answer in 33,498 trials, but the wrong answer in 66,502. With Pigasus' strategy the numbers were the other way round. The corresponding probabilities of 0.33498 and 0.66502 are convincingly close to Pigasus's claimed values of 1/3 and 2/3.

Marilyn — and Pigasus — are definitely *right.*

As Marilyn remarked: "math answers aren't determined by votes." Still unconvinced? She offered one final argument that may make you rethink. Suppose there are a *million* caves. Perseus chooses number 1. Pigasus points to 999,998 others, each containing a gorgon. Recall that Pigasus *knows where Andromeda is*; and he can't point to that cave. Apart from your choice, you observe that Pigasus strenuously avoids cave number 777,777. Which do you think is more likely: you chose the right cave (out of a million) or you got it wrong and Pigasus's behaviour is offering a rather strong clue? But if you sided with Perseus before, then by the same argument you have to claim that there are only two caves that matter (1 and 777,777) and each has an *equal* chance of being right. If you do believe that, I've got a great car to sell you, owned by a little old lady who only drove it to the beach when the gorgons weren't around...

5 Conditional Probability

Look, if you don't believe *proofs*, you don't believe *experiments*, and you don't believe *analogies* or *counterexamples*, what *will* you believe?

Fun, isn't it.

The trouble arises because we are dealing in *conditional* probability: the probability of something happening *given that something else already has*. Conditional probabilities are not especially intuitive. The basic point is that Pigasus's choice of cave *depends on what Perseus has chosen.* If Perseus has chosen correctly, then Pigasus has a free choice of the other two caves; but if Perseus is wrong (which is twice as likely) then Pigasus has *only one choice*, and gives the entire game away.

The argument that with only two caves left, Andromeda is equally likely to be in either of them, is correct if Pigasus chooses *first*; but wrong if Pigasus's choice has to depend upon what Perseus already chose. It's one occasion when the order in which the choices are made matters. This often happens with conditional probability.

Really, it's very simple. Pigasus not only had the right strategy, he had the right argument in its favour. Let's take it again, very slowly.

Do you agree that, when Perseus first chooses a cave, his chance of being right is 1/3?

I hope so, since that was one of the conditions of the problem (Pigasus told us they are all equally likely). Fine. Suppose Perseus sticks to the 'never swap' strategy. Then he can't alter that probability, because he doesn't *do* anything. He could stuff wax in his ears and not listen to Pigasus's siren song and it wouldn't make the slightest difference.

So the idea that his chances are fifty-fifty is definitely wrong. And with a million caves, the 'never swap' strategy gives him only a one in a million chance of being right.

Do you agree that if Perseus is wrong, and if Pigasus narrows the remaining possibilities down to one other cave, then Perseus must get it right if he swaps?

Again, I hope so. Andromeda is in there somewhere. She's not in any cave named by Pigasus; and if Perseus is wrong, she's not in the cave he chose either. In which case, whenever Perseus's strategy 'never swap' gets the wrong cave, Pigasus's 'always swap' must get the right one. So the probability of 'always swap' being right is 1-1/3 = 2/3.

With a million caves, Pigasus's strategy gets it right with probability 0.999999.

6 An Information Theoretic Approach

Maybe saying things with a different form of words will help. It's like this: Pigasus gives Perseus some useful information: namely, a cave in which Andromeda is *not*. This ought to be something that Perseus can use to improve his chances: that's what information is for. But he can't use the information if he never changes his choice, because his choice was made *before* the information became available. This doesn't prove he ought to swap; but it does imply that never swapping can't be the best strategy.

If you still don't believe me, get a friend, and try it with a sugar lump for Andromeda and cups for the caves. Play one strategy for 100 games; then switch to the other. For comparison, get your friend to reveal one empty cup *before* you choose. Now your chances *will* be fifty-fifty.

But please, don't write to me about it! In Marilyn's case things got totally out of hand. There were stacks of mail, phone calls, fax messages, wild accusations. After a few days, though, there were also some embarrassed retractions. I doubt that the perpetrator of sexist sarcasm was among them: people like that seldom recognise when they're wrong. But if the mathematician who advised her to "refer to a standard textbook on probability" had done so himself, and turned to the chapter on conditional probability, he might very well have come across a discussion of precisely this problem.

With Pigasus upheld.

"When reality clashes so violently with intuition,", says Marilyn, "people are shaken." True. But the really dreadful truth, I'm afraid, is that the entire problem is an old chestnut, one which *every* probabilist should have run into, in one form or another, during their careers An equivalent problem is in J.Bertrand's *Calcul des Probabilités* of 1889. It is usually known as Bertrand's Box Paradox, and as Eugene Northrop observes in his *Riddles in Mathematics*, it "has been used as an illustrative example in almost every subsequent textbook". A version involving three prisoners and a prison governor is described in detail by Martin Gardner in *More Mathematical Puzzles and Diversions*. So not only did many of Marilyn's critics let their sense of outrage and hyperbole run away with their good taste: they didn't do their homework.

Now that we know that Perseus can indeed improve his chances by using the information kindly provided by the faithful Pigasus, it's worth asking whether any other strategy will perform even better. For example, suppose he either swaps or not, with probability p. Can this improve on the 2/3 chance of success? Or is there some other strategy that does? I leave these problems to you: my answers, for what they are worth, are summarised below.

One last letter. I'll give the author's name because there are no blushes to spare. "Dear Marilyn, you are indeed correct. My colleagues at work had a ball with this problem, and I dare say that most of them — including me at first — thought you were wrong. *Seth Kalson Ph.D., Massachusetts Institute of Technology.*" To echo Marilyn vos Savant: "Thank you MIT. I needed that!"

Answers

The strategy of randomly swapping with probability p has a chance of success of

$$\frac{2}{3}p + \frac{1}{3}(1-p) = \frac{1}{3} + \frac{1}{3}p$$

which is largest when p = 1, so Pigasus's strategy can't be improved on in that manner.

The best strategy of all is 'only swap when you're wrong', which has probability 1 of success, but seems impossible to implement without the services of an oracle.

Unless probability theory and information theory are completely wrong, Pigasus's strategy is the best available.

FURTHER READING

Martin Gardner, *Mathematical Puzzles and Diversions from Scientific American*, Bell & Sons, London 1961.

Martin Gardner, *More Mathematical Puzzles and Diversions from Scientific American*, Bell & Sons, London 1963.

Eugene P. Northrop, *Riddles in Mathematics*, Penguin Books, Harmondsworth 1960.