G R E S H A M
COLLEGE

THE MATHEMATICAL ABILITY OF SCHOOL LEAVERS

Gresham College, Barnard's Inn Hall, Holborn, London EC1N 2HH

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FOREWORD

Peter Nailor
Provost, Gresham College

The quality and level of scientific training in the UK has caused concern for many years, as has the number of students going on to study scientific subjects at university. Gresham College can play a unique role in encouraging debate on such topics: it is simultaneously an institution of further and higher education and a neutral venue.

This booklet is a summary of a seminar held at the College on 8 December 1995, organised by Prof. Ian Stewart (The Gresham Professor of Geometry) and Dr David Tall, both at the University of Warwick. The seminar was an initial response to the document *Tackling the Mathematics Problem* published jointly by the London Mathematical Society, the Institute of Mathematics and its Applications, and the Royal Statistical Society. This document presented evidence for a decline in the mathematical skills of school leavers, despite the increasing numbers of A-level passes, and summarised the consequent problems facing universities and other higher education institutions faced with the declining mathematical skills of beginning undergraduates.

In line with Gresham College's neutral stance, this is not the place to comment upon the substance of that report: such comments will be found in the discussions and conclusions of the seminar which are laid out in this booklet. It is, clear however, from the heat of some of the recent exchanges between university mathematicians, on the one hand, and mathematics educationalists and teachers on the other, that some kind of common forum for discussion is needed. I hope that this seminar will help, in the short term by providing such a forum, if only for a day, and, perhaps, in the longer term, through some concrete steps that might be taken as a consequence of what is said.

The problem of scientific training, and in particular mathematical skills, is too important to be tackled in a fragmented and adversarial manner. The objective should be not to blame any particular group, but to find practical ways to raise the general level of mathematical skills while at the same time making it possible for the most talented students to develop their abilities to the utmost and put them to use for the benefit of us all.

Opening Remarks

Ian Stewart
Gresham Professor of Geometry

I'd like to welcome you all to Gresham College and thank the College for organising, housing and funding this meeting.

In my opening remarks I don't want to prejudice the discussions: I'll prejudice them later on. I came here to learn what is going on. We all get an incomplete picture of what seems to be a problem. The immediate cause of this meeting was an article by Peter Saunders in the *Guardian* pre-releasing the LMS/IMA/RSS joint report, pointing out that seen from the university perspective, not just from mathematics departments but from engineering, physics, science and business studies departments, where students are expected to have a certain level of mathematical skills, there was a rather strong feeling that somehow today's students are less well-prepared than they used to be.

There are all sorts of explanations for this. One is that everybody in the universities is getting old, and therefore can't remember how abysmal they were themselves. That's certainly part of it. So it's important to disentangle the reality from strongly-held but perhaps erroneous beliefs. On the other hand there does seem to be some sort of change; one of the signs is that over the last few years various subjects have become less and less mathematical, which really ought not to have been. Biology was traditionally the science for people who didn't like mathematics; chemistry followed. Chemistry used to demand A-level mathematics from its entrants, and now it doesn't. One indirect consequence of this is that the British chemistry degree is now not recognised on the continent as a professional qualification for chemists.

Physics is starting to follow the same line. One of the signs of trouble is that physics is now a 4-year course, or at least that a 4-year degree is now available to students. Mathematics nation-wide has also followed on to a 3-year/4-year structure. The argument is that in three years we can no longer bring students to the level that we used to. We need to spend six months at the beginning teaching them things they used to know when they came, and then we've got six months at the end of the 4-year course to actually take them further. And we now have a bachelor's degree which contains less than it used to.

This is the kind of step that universities have had to take to deal with the problem as they perceive it — and they do perceive it as a problem. That may be because it's only the universities that have got a problem. It may be because it is a sign at university level of a much wider malaise. What I'd like to do, and what this meeting is about, is to discuss this whole thing at all levels and try to come to some kind of consensus. We've got a very broad audience here, and there will be a lot of time for discussion of the issues.

WHAT IS MATHEMATICS, ANYWAY?

Peter Saunders
Department of Mathematics
King's College, London

Unlike the other speakers you will hear today, I am not professionally involved in the study of education. I do of course teach, so I have many years of professional experience in trying to put mathematical ideas across to young people, but my research interests are well away from the field of education. If I didn't think there was a serious problem that needs urgent attention, I wouldn't have got involved in the question of maths in schools and I wouldn't be here today. I have lots of other things to do — until recently I would have said lots of *better* things to do — and you can take the fact that I and others like me are spending a lot of time and energy on this issue as a measure of how serious we think the problems are.

For a long time, we've been noticing a clear decline in the mathematical ability of the students who were coming up to university. This had been going on for a long time, but it seemed to be accelerating. Naturally those of us who teach in mathematics departments were the first to be conscious of it, but recently there have been complaints from engineers, physical scientists, and even biologists. They are finding it increasingly difficult to teach their own subjects to students whose mathematical background is so weak. And for them it's not a question of anything esoteric: just the other day, a physiologist was describing to me how his students couldn't interpret the results of a simple experiment on muscle fibre because they knew practically no co-ordinate geometry and so couldn't infer anything about the process from the way the points lay on the graph.

Naturally I wondered why this had happened, and it occurred to me to talk to educationalists to find out. Now everyone knows that schools are under pressure. It is a lot harder to be a teacher now than it was in the past. The schools are under-resourced, there aren't enough qualified maths teachers, more students are staying on, discipline – both in the sense of order in the class and making sure that homework is done – is harder to enforce, teenagers have more money and more things to do, maybe the spirit of the age is hostile to a subject like mathematics... There could be all sorts of reasons, and I was interested to know which were seen as the crucial ones.

What I wasn't expecting was the answer I actually got. Today's students, I was told, are not less well prepared than their predecessors, they are better. They may not be quite as good at certain kinds of manipulation, and they may not know so many facts, but they are better at problem solving and finding things out for themselves. They really understand mathematics; it's not just a collection of recipes. They also know about computing, which 'you didn't have to learn when you were in school'.

Now this just didn't tally with my own experience. Well, most of it didn't. I'll agree about not being as good at manipulation and not knowing so many facts. But as for the things they are supposed to be better at, my experience is that they are worse. Many students have no idea how to solve a problem that involves several steps unless they are led through it. They lack what we can call mathematical sophistication, which includes a reasonably clear idea of what a proof is. They expect an inordinate amount of spoon-feeding. After more than a century of managing without, my Department now monitors the first years' homework and hauls them in for a talking to if they are getting behind. And as for computers, while there is a minority who know more than I do, most of any incoming class know very little beyond how to turn one on and put the games disk in.

How can it be that we have become so worried while the educationalists were so pleased with the way things were going? Was it just our imagination? No, I don't think so. The evidence is too strong. For a start, mathematicians have not just complained, they have acted. We have all made our courses deliberately easier so that today's students can cope with them. I can assure you, we didn't do that lightly. Analysis, which demands the most mathematical sophistication and also a good grounding in calculus, has moved from the first year to the second. Even Cambridge has finally given in, which tells you that it isn't just a question of having to scrape further down the barrel. Most mathematics departments in the old universities have introduced the four year degree, which, frankly, does little more in four years than was once done in three.

How can we square this with the educationalists' view that everything is better than before? Well, let's see. First of all, they seem to take as their main criterion the improved A-level and GCSE results. Only no one seems to have been monitoring the exams, which makes them hardly a reliable measure, especially given all the pressures including (but not only) market forces. The only serious study I have seen, reported by Carol Fitz-Gibbon[1], suggests a sharp decline in standards.

Actually, I am appalled that there hasn't been any monitoring of standards, especially when there have been so many changes. I'm not one of those who would die on the barricades for a gold standard (and the experience of the British economy between the Wars ought to warn us about the dangers in sticking to an inappropriate gold standard anyway), but at least we ought to know what is going on.

There are also complicating factors, like the expansion in numbers and the decrease in popularity of further maths at A-level. Again, I don't want to go into that except to acknowledge these exist but that they aren't anywhere near enough to account for the discrepancy. For one thing, if they were, privileged places like Cambridge wouldn't be affected, as my colleagues there assure me they have been. And if the problem were only further maths, I wouldn't expect to hear the same complaints from physics and engineering: most of their students never did further maths anyway.

The more I looked into the problem, the more it became clear to me that there was a fundamental disagreement about what mathematics students should learn, and even about what mathematics is. And if we don't agree on that, no wonder we can see the outcomes of school mathematics very differently.

There are lots of ways in which this can be seen. For example, there is a great reluctance to do anything without an immediate context. Apparently you can't ask what 3+4 is anymore, you have to say 'Mae-Wan wants to know what 3+4 is. What answer should she get?' Does this really make the subject more relevant? And questions about fractions all seem to be about pizzas, and are often accompanied by pictures which allow you to find the answer by counting.

Now this betrays a fundamental misunderstanding of what mathematics is. Mathematics is an abstract subject. It is about patterns, but it is not about patterns of pizzas or patterns of chairs. It is about the patterns themselves. The idea is that once you have understood the pattern, every time the same pattern arises, you are there. Once you know that 2+3=5, you know that if you have two

[1] Talk at the British Association meeting, Newcastle, September 1995.

pizzas in one box and three in another then you have five pizzas. If you have two pounds and three pounds you have altogether five, and if you have the product of a two-dimensional space and a three-dimensional space you have a five-dimensional space.

Equally, once you learn what a derivative is, you can understand the velocity and acceleration of cars and rocket ships, but you can also understand rates of chemical reaction, the growth of populations, and marginal utility as well.

We don't help this by refusing to acknowledge the abstraction. It's not that children find it hard to learn that 2+3=5; it's that the idea has got about that we always have to tart it up in terms of pizzas. But do we? Real life examples certainly have their place. They can motivate: not many children, or adults either for that matter, are keen on learning an abstract subject that seems never to relate to reality. It's also often easier to start with something concrete: when I'm first teaching a class about derivatives I talk about rubber hoses across the road to estimate the speed of passing cars.

But the power of mathematics is precisely its abstraction. While real examples are useful for motivation and for getting started, it is important to move to the abstract. Yet those in charge seem bound and determined to hang on to context almost at all costs. Why? I suspect that it's because they have never thought the thing through. They haven't considered that context is largely a means to an end, like scaffolding or water wings.

The greatest disagreement is probably about proof. It seems to have gone from the schools with the demise of Euclidean geometry. I've been accused of being ignorant for complaining about the demise of proof, yet the only place I can find the word in the National Curriculum is at the end of the further material for Key Stage 4, where section 2 on Using and Applying Mathematics ends with '... leading to notions of proof', whatever that means.

This doesn't strike me as implying that most children are going to see very much about proof, and certainly most universities find that the majority of the intake are not at all clear about what is meant by proving things. That's the main reason analysis can't be taught in the first year any more.

I've been struck that when I've spoken with educationalists about this, they tend to accuse me of being one of the cultural restorationists who want to preserve the elitist canon, of which Euclid was the very epitome. ('Cultural restorationist' is a term of abuse used to describe people like me who would rather teach students an efficient technique that works instead of presenting them with learning opportunities so they can spend a great deal of time deriving incorrect methods for themselves. I gather from yesterday's papers that this position may be softening a bit.) In any case, I find it an interesting charge; let me explain why.

There has been, and I believe still is, a big debate in English literature, at least part of which concerns the 'canon'. Must every schoolchild read Shakespeare and every university student Chaucer and Beowulf? Part of the argument is about whether an English curriculum is supposed to inculcate in pupils the common culture of our society, which of course presupposes that there is an agreement about what that common culture is or ought to be. Many American universities, incidentally, used to have courses that were irreverently known as 'Plato to NATO', which may give you some idea of what the issues were.

It seems that the educationalists in mathematics didn't want to feel left out, and they saw Euclid as the mathematical equivalent of Beowulf. So out it went. Unfortunately, what they didn't seem to realise — and I've got this from talking to some of them — was that Euclid was where many pupils got a feel for geometry, including graphs, where they learned how to approach problems whose solution is not obvious at first glance, and where they learned about proof.

You may call it part of the canon if you like, but it was serving a real mathematical purpose, and the drive to get rid of it was mostly due to a misunderstanding. It's interesting that only recently a retired maths adviser, that is to say someone who was very important in influencing how mathematics was taught in schools, criticised people like me for regretting the loss of Euclidean geometry when its foundations had been demolished a century ago[2]. I'm not sure whether he means by Lobachevsky or perhaps much later by Gödel, but in either case he's missed the point, and in a way which rather worries me, coming from one in his position.

It's not a question of a sacred text to be revered or overthrown, depending on the outcome of a theological debate. Actually, the discovery of non-Euclidean geometry made the difference between mathematics and science a lot clearer than it had been before, so you could argue that it strengthened, not weakened, the case for teaching Euclid. It's a valuable tool, and if you throw it away you had better find something else that will do the same things, assuming, that is, that you understand what those were.

Proof seems to have been pretty much pushed aside and replaced by investigation, and the search for patterns. I'm all in favour of the search for patterns, though I can't say that my present students are better at it than their predecessors. But the search for patterns is something that goes on in all of science and in other fields as well. What is distinctive about mathematics is proof. Mathematics is where you learn that the fact that the first dozen or so integers are less than 60 doesn't prove that all of them are. And this is being thrown aside. What's especially worrying is that we are meeting more and more teachers who don't even know what a proof is themselves. "But 35 pupils have all measured the angles of the triangles they drew and they've all added up to 180°. What more proof do you want?" was just one example. I myself have interviewed a very senior mathematics teacher who clearly had no idea what a proof is.

By the way, I have heard it claimed that proof has been dropped because hardly any students understood it anyway. I really find that hard to believe. I was educated in the Canadian Province of Ontario, and I went to the nearest non-selective coeducational comprehensive school because that's what everyone did; the only choice you had was between Catholic and Protestant schools. Like most Ontario secondary schools at that time we had a more or less voluntary division into what you would have called grammar and secondary modern streams. I recommend that system, by the way; it allowed for separate curricula without subjecting young children to the trauma of the 11-plus, and while it wasn't perfect it made it less difficult to correct mistakes.

Now about 40% went into the academic stream. Yes, I do mean 40%. Last time I was home I got out my school yearbook and checked it. It is, you will notice, about twice the proportion that passed the 11-plus and very nearly the proportion that we now expect to go on to higher education. You might think about that.

Anyway, it was 40%, and we all did Euclid in Grade 10, at say age 16. Not everyone was very good at it, but I don't recall that any of my classmates were unable to grasp what was going on. It certainly didn't have a reputation as an especially difficult subject, and I think most pupils actually preferred it to algebra. So unless Canadian children are inherently very much more intelligent than English children, which I doubt, I really can't accept that the idea of proof is beyond all but the tiny minority who are going to go to the highest levels of the National Curriculum.

[2] Quoted in Peter Reynolds, 'That impostor 1.4142136', *Maths Gazette*, Nov.1995, p.535.

Proof is not a sort of optional add on. It's at the core of mathematics. What's more it's not just for mathematicians. One of the reasons for teaching mathematics at all, beyond the basic numeracy (which a lot of children don't seem to get either, if I am to judge by supermarket checkout staff) is that it is probably the best place to learn what is meant by a proof. Not teaching that in school creates problems for maths teaching at university, but more importantly, it means that those young people who do not study mathematics at university — which is most of them — will never learn what the word means. I don't think it is a coincidence that it is since the demise of Euclid in the schools that the verb 'to refute' has come to mean 'to deny in a loud voice.'

Let me go on to a different topic, problem solving. This is something we are told the students are better at. And indeed, I recently heard a senior teacher justify what was going on by patiently explaining that nowadays we don't so much teach students mathematics as how to apply mathematics. So that's all right then.

No it's not. Actually I feel very strongly about this, because I am an applied mathematician, and, what is more, not one of those who does what looks very much like pure mathematics. I do mathematical modelling where the hardest part is usually the modelling rather than the mathematics. And the plain fact is that you can no more apply mathematics without knowing it and understanding it than you can do pure mathematics without knowing and understanding what came before.

The trouble is that you don't solve problems by an algorithm and in a straight line. You often have to work backwards or sideways or whatever, and for that you need knowledge. It's only when you *know* that $\sin^2+\cos^2=1$ that it occurs to you that where you see $\sin^2$ you ought to look for a matching $\cos^2$, or, if there isn't one, see if you can fiddle one out of somewhere. Without that, you have to be led step by step each time, which is what we see on exam papers now and is why many so students never learn to solve problems on their own.

Nowadays pupils are now given formula books on the exams. I wouldn't object to that if I thought it was only to save them losing out because they've forgotten some important formula, but more and more we are finding that students are not bothering to learn the formulae at all. The result is that they never get them into their heads and so they can't use them in problem solving. Which makes it very difficult for them to learn how to solve problems. It's something like having to learn all the various road signs in order to pass the driving test. There is, to be sure, a complete list in the Highway Code, but when you are driving at high speed down the M1 you need the information in your head, not in the glove compartment.

I can't finish without showing you the following problem, which comes off a draft Key Stage 3 exam I was once shown. The idea is that students are supposed to be able to use Pythagoras' theorem. Not prove it, mind, just use it.

I don't actually object to that. It's probably a bit too early to be proving Pythagoras' theorem, and I'm not one of those puritans who insists you can't use something until you've proven it. Anyway here is the question:

5.

(a) You cannot draw a right-angled triangle with edges 8cm, 12cm and 18cm.

Use Pythagoras' theorem to show why.

Now in the old days, you would have been asked whether 8,12 and 18 could be the sides of a right angled triangle. You would have had the tiny bit of abstraction that there wouldn't have been units, so the same work would deal with 8 in, 12 in and 18 in, or 8 km, 12 km and 18 km, but that's not important here except as a symbol.

Things have changed. First you are given the clue that the answer is negative. That too probably doesn't make a lot of difference here, but again it is typical and many problems are a lot easier if you are given the answer. Second, and this one is important, you are *told* to use Pythagoras' theorem, which means that you are not actually solving a problem at all. You don't have to work out how you could show what is required, you don't have to think of Pythagoras' theorem, and you don't have to know that Pythagoras' theorem applies to right angled triangles (and not to others).

But I haven't shown you the whole of the page this problem is on. At the bottom is a right angled triangle with its sides marked x,y,r, the relation $x^2+y^2=r^2$ and, for those who haven't got the hint even yet, the words 'Pythagoras' theorem'.

What exactly is left for the student to do? You don't even have to know what a triangle is.

Now anyone who sets examination questions knows that it is all too easy to set a question that turns out not to test what it was meant to. But when I explained why I thought this was a bad question, no one acknowledged my point. On the contrary, they responded that I hadn't understood that the curriculum requires that pupils should be able to *use* Pythagoras' theorem in solving problems, and that's what they were doing. Yes, but only in a very restricted sense of the verb 'to use' and the expression 'solving problems'. It's like saying you can use a metal-working lathe if someone else does all the setting up and you just go over and press the button labelled ON.

I have to say that while I am disappointed that anyone could set such a question, I am astonished that anyone would want to defend it. Mind you, I may have been a bit naive, because this question from a recent A-level paper[3] isn't much better:

2. Fig. 1 below shows a circle with centre O and radius r. Points A, B and C lie on the circle such that AB is a diameter. Angle $BAC = \theta$ radians.

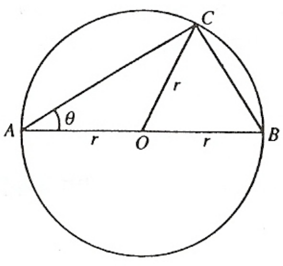


Fig. 1

(*a*) Write down the size of angle AOC in terms of θ. **[1]**

(*b*) Use the Cosine Rule in triangle AOC to express AC^2 in terms of r and θ. **[2]**

(*c*) By considering right-angled triangle ABC, write down the length of AC in terms of r and θ. **[1]**

Deduce that $\cos 2\theta = 2\cos^2\theta - 1$. **[1]**

There isn't a lot left for the student to do here, bearing in mind that there is a formula book supplied. No wonder only three students out of 55 (most of them with good A-levels) who had come to read mathematics at a university could solve the third of these problems:

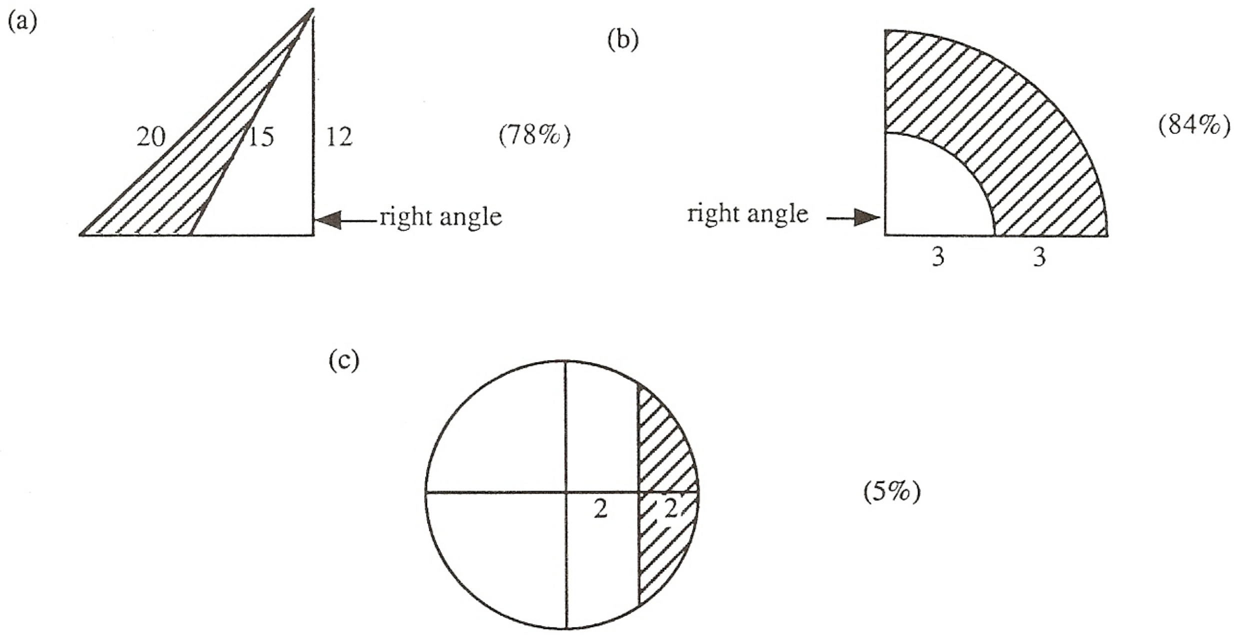
[3] Oxford A-level Mathematics Paper 51, Module 5: Pure 1, 23 June 1995

1. Solve the equation $x^2 - 4x = 0$. (75%)

2. Factorise the following expressions as far as possible:
(a) $2x^2 - x - 3$ (78%) (b) $x^2 - 9y^2$ (73%)

[Note: of the 6 students who managed to factorise neither of these quadratics, 3 had achieved a grade B at A-level.]

3. Calculate the areas shaded in the diagram, leaving your answer in terms of π where appropriate.



Far more of them could manage the first two, which are at about the same level in terms of the mathematics that is used. The difference is that in the third one you have to do a very small amount of thinking and maybe even draw in a couple of lines that haven't been marked on the figure for you.

Problem solving means looking at a problem and figuring out what to do. It doesn't mean following a detailed set of step by step instructions and applying formulae which have been conveniently provided just where you need them. Being able to go through the motions of carrying out a task may be the sort of thing Keith Joseph had in mind when he used the example of the driving test, and it seems to be a lot of what informs NVQs, but it isn't applied mathematics, any more than you can have pure mathematics without abstraction, precision and proof.

I could go on, but time is running out and it's too depressing anyway. What is to be done? That's hard to say, though for a start we certainly need more qualified teachers and we need to relax the pressure on them so they can use their qualifications and initiative *within* a framework but not be dominated by it. In Ontario we had a Provincial Curriculum, and in my day some outside assessment, though nowhere near as much as in England and Wales. We also had a lot of good teachers, and while they taught to the curriculum and prepared us for the exams, they didn't show any signs of feeling constrained.

Whatever we may think about the present curriculum, if you have to assume that all that gets taught is what is in the curriculum (worse, in the 'real' curriculum that is specified by the assessment procedures) then you're never going to get good education. That's one reason why you need qualified teachers. The other is that if you have too many unqualified teachers, you have to design a curriculum that they can deliver, and while I'm not always very sympathetic to those who have been determining the curriculum, they really have been set an impossible task.

Obviously, I'm not happy about some of the present curriculum, by which I include what is taught in the 6th form. I don't think it respects the nature of mathematics enough, and I believe this is bad for the majority of students, not just those who will study mathematics at university. There is a worrying tendency to think that the subject doesn't matter and that people who really understand the subject are not needed as part of — I said 'part of' — the team that decides what will be taught and how. Educationalists don't just believe that about the schools, by the way. Just to show you, here is a quote from a member of the Centre for Higher Education Studies at London University[4]:

To say 'I am a Latin scholar therefore I can teach', is like saying 'I am a Latin scholar, therefore I can drive a heavy goods vehicle'.

In other words, understanding your subject is totally irrelevant to teaching it. That's precisely the attitude I would like to see eliminated from education.

The recent joint report of the three mathematical learned societies, while it made a number of suggestions for improvement, had as its firm recommendations only that there be proper committees set up with an overview of mathematics from primary school to university and that there be a significant (not *dominant*) university presence on these. I think that was wise: the problems are serious and not readily solved and we have to work together. University mathematicians are well placed to see some of the problems and to make suggestions, and we ought to be involved in finding the answers, but we are under no illusion that we can find them by ourselves.

When the House of Commons Select Committee on Education reported last summer on science education at Key Stages 2 and 3, they ended their report[5] with the following paragraph:

It is clear, as we stated at the beginning of our Report, that the education system needs time to absorb the revised National Curriculum, and time is needed for this revised Curriculum to bed down. However, it is obvious from the commentary provided by our witnesses that any future review will have to examine very carefully the nature of the science and technology curricula, their structure and the kind of understanding of science and technology they aim to portray, if future generations of children are to gain a proper understanding of science and technology and how they affect their lives.

We have also to examine very carefully the nature of the mathematics curriculum. Not just A-level, not even just 14-19, but the whole curriculum. And the people who do that had better have a clear idea of what mathematics is. It won't be an easy task, especially since we have to provide for pupils with a wide range of abilities who will be going on to do many different things. Since I imagine all of you here have benefited from the sort of education that included such ideas, I think you will understand what I mean when I say that the condition I am talking about may not be sufficient but it is certainly necessary.

[4] Cari Loder, quoted by Jennie Brookman, *THES*, 16 April, 1993, p.1

[5] House of Commons Education Committee, Session 1994-95, Fourth Report: *Science and Technology in Schools*. HMSO. (paragraph 89)

A-Level: Some Considerations

Geoffrey Howson
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The problem in writing a paper for a meeting such as this is that it is not easy to separate out concerns, constraints, aspirations and proposals for action into watertight compartments. There are many inter-relations and so what I have to say will doubtless impinge on other contributions. However, I thought it might be valuable if I were to concentrate on certain quantitative and comparative aspects.

Peter Saunders has just spoken about the position as universities view it: to sum up, that entrants to university courses making relatively heavy mathematical demands lack needed technical fluency, the confidence and experience to tackle multi-step problems and have serious misconceptions concerning the nature of mathematics: in particular, the central position in it of deductive reasoning and proof. These claims have been echoed by others e.g. in a recent report of the Engineering Council. Standards, it is asserted, have fallen. What I wish to do is to consider this assertion in the light of data and of experiences in other countries.

The State of A-level Mathematics

Perhaps a convenient starting off point would be the position thirty years ago. Harold Wilson had taken office in 1964 pledging to develop the country's technological strength and future. As part of that aim he established in 1965 a committee under the chairmanship of Fred (now Lord) Dainton to study 'the flow of candidates in science and technology into higher education'. The committee reported in 1968 and expressed its concerns that insufficient sixth-formers were studying science and mathematics A-levels, that the majority of students post-16 studied no mathematics, and that there was an evident swing developing away from science subjects. One of the major reasons for this, it was claimed, was the shortage of teachers who in the students' formative 11-16 years could fire their enthusiasm for the subjects. The committee made several recommendations, including inevitably that the A-level system be replaced by one which did not encourage such premature specialisation, but rather gave students a broader-based post-16 education.

Much has happened since 1965. The education system has moved to being largely comprehensive and frequently a new 'break' has been introduced into the system at 16. No longer has the A-level mathematics student almost invariably been educated from 11 or 13 in a selective school where he or she was probably taught by those same teachers who will now be in charge of the A-level classes. Today's sixth-form college teachers no longer have the opportunity to spot and nurture mathematical talent in those pre-16 years and to develop it with the possibility of A-level firmly in mind. More generally, in the last thirty years the total number of students enrolling for full-time post-16 and higher education has increased enormously.

What, then, are the quantifiable results so far as mathematics is concerned? As is often the case, data can be used to support most cases one wishes to make. The 'optimist' (e.g. the Government or

Quango spokesperson) will refer to an increase of some 60% in the number of passes in single subject A-level mathematics. Others might argue that the number of passes is a function of policy and not of standards: for in 1965, 30% of A-level students were destined to fail willy-nilly as a consequence of the 'norming' standards then in force. Perhaps then we should look rather at the rise in the number of entries. These have risen by about one third compared with a rough doubling in the total number of A-level entries over the same period. (At least mathematics has shown some improvement in entries: physics has not been so fortunate; its entries have dropped by about 20%!) But more significant, so far as universities are concerned, is the fact that whereas in 1965 over a third of A-level mathematics students took two A-levels in that subject ('Pure Mathematics' and 'Applied Mathematics', or 'Mathematics' and 'Further Mathematics') now less than a tenth do so. Moreover, and this is vital for subjects such as physics and engineering which overwhelmingly attract male students, the number of males attempting mathematics A-level has shown a small decrease in the last thirty years. Add the fact that in 1965 a fair proportion of students would have studied 'Additional Mathematics' O-level (including the calculus) pre-16, for which there is no modern equivalent, and that now some A-level students come from 'further down' the 16+ ability range than previously, and the extent of the differences facing sixth form students and teachers becomes more apparent.

The swing which Dainton remarked upon has now developed even further: in 1965 38% of A-level students studied only science and mathematics (i.e. represented the tradition of the old 'Sixth-form Science' divide). By 1993 the percentage had dropped to 16% and recent DFEE data suggest that this percentage has decreased even further. Students are clearly opting for a 'more balanced' curriculum. Yet, regrettably, 'more balanced' when selecting only three subjects can too often only mean 'non-coherent' or 'mutually independent'. The support and motivation which, say, physics and mathematics traditionally provided for each other can no longer be assumed.

All these, then, are important factors to take into account when considering 'standards'. What to me is readily apparent – and, indeed, in view of the data I have presented so far, would be miraculous were it not the case – is that schools are not educating so many to so high a level in mathematics. It could be claimed that we are educating more students to a slightly lower level and that this is sufficient compensation.

Indeed, those familiar with my writings and pleadings ten years or so ago will see in this last sentence the concept which I then termed 'yield'. It was perfectly clear in the 1980s that insufficient students were being attracted to or retained in A-level Mathematics courses. Subject pair comparisons showed that A-level Mathematics was a 'harder' subject than most others and that students with B and C grades in the subject at O-level had considerable difficulty in turning these into A-level passes.

This led me to argue for:

(a) reducing the overall standard of A-level to make it more comparable with that of other subjects;
(b) introducing differentiated papers (as in GCSE) so that high standards could still be maintained for the most able.

However, an important quid pro quo for (a) would have to be a greater uniformity of content amongst A-level syllabuses.

The idea of 'yield' i.e. taking into account the total amount of mathematics learned by the age cohort, rather than merely what was learned by the top few per cent of students, I picked up from Canadian sources. The approach of reducing standards to increase yield was already being followed in France as a consequence of a report from the Collège de France claiming that students were being deterred from following scientific and technological careers because of what were deemed to be the excessive mathematical demands of the baccalauréat. As can be seen, the result in France was that numbers taking the most mathematically demanding of the baccalauréats roughly doubled in nine years while A-level passes declined by 8% and entries by a quarter. (See Appendix 1)

In England the problems were given no official recognition. The 'gold standard' of A-level was invoked: as if by that time, or indeed at any time, it had serious meaning. One board produced a syllabus which offered differentiated papers. Others claimed that these were not required, that they could produce papers which could both enable potential E-grade students to demonstrate 'positive achievement' and also test and stretch the most able. Events since then have only confirmed my belief that such a claim is entirely spurious.

The position was further complicated by the introduction, following the Education Reform Act, of simplistic league tables which attempted to allow outsiders to judge the efficiency of a school on the basis of the number of A-level passes it achieved. The results have been far from satisfactory:

(a) modular A-levels have been taken up with enormous enthusiasm: not necessarily because of their mathematical and pedagogical merits (and these certainly exist - and, in my view, potentially out-balance any deficiencies) but because those who fail to attain an A-level are effectively discounted;

(b) schools began to look around for 'easier', more 'generous' boards and syllabuses: the initial stimulus often coming from head teachers and principals rather than heads of department. (Thus the number of entrants for what I view as the most intellectually demanding mathematics A-level dropped from over 5000 in 1990 to just over 2000 in 1994, and is forecast to be about 1200 in 1996.)

The deleterious effects of the simplistic tables and the failure to meet the original claims of the National Curriculum of providing all pupils with an entitlement to an agreed curriculum and appropriate assessment is shown by the evidence, revealed by recent SCAA-sponsored research, that some 11-16 schools are not entering any pupils for the higher tier papers in GCSE mathematics. Pupils can obtain Bs and Cs (even occasionally As) on intermediate papers and such grades will suffice for league table purposes even though the work on which students have been examined will not form a suitable foundation for A-level mathematics.

Of course, searching for 'easier' boards can soon develop into looking for 'easier' subjects. Appendix 2 gives comparative figures showing how GCSE grades in English and Mathematics in 1992 converted into A-level grades in 1994. Let us consider advising a student with B in Mathematics and C in English on what A-level course he or she should follow. First, the probability of passing in English (0.78) would be greater than in Mathematics (0.68); moreover, the median grade would be D in both. Yet in mathematics fewer students obtained grades B and C at GCSE than they did in English. The consequences of such comparisons are demonstrated by the fact that, so I am told, one of the country's leading girls' schools only allows students to enrol for A-level mathematics if they obtained an A* at GCSE: at least that should ensure that their keeping on with mathematics does minimal damage to the school's league table position! Something serious, affecting the country's scientific and technological well-being, has gone adrift.

Chris Belsom will tell of one initiative taken to make A-level mathematics more attractive, retentive and rewarding. Other serious long-term curriculum initiatives were carried out by the Mathematics in Education and Industry (MEI) Project and under the aegis of the Nuffield Foundation. These have had some success but have not to date materially increased the overall numbers entering for mathematics A-levels. (Appendix 1). Subject pairs comparisons would suggest that there has been some lowering of relative 'standards', but the hoped-for increases in 'yield' have yet to appear. *It remains the case that, so far as I know, no other developed country has a smaller percentage of its 17 year-old cohort following the country's specialised mathematics option.* Moreover, the recent movements have led to a reduction in content in A-level syllabuses but not to any greater homogeneity. (See Appendix 3). Universities can only assume a reduced common foundation of mathematical knowledge on which to build from 1996 onwards, and there is no clear indication that there will be a considerably increased number of students who possess even this limited, shared foundation of mathematical knowledge. Add to this that attempts since 1965 to increase the amount of serious mathematics teaching available to non A-level students have been muted and to a great extent (e.g. AS mathematics) irrelevant, and we see that overall we cannot claim that the foundations for the further study of scientific and technological subjects are really in a better state now than when Lord Dainton reported. Let me just remind you of the varied provisions for post-16 mathematics education in 1994 France (See Appendix 4). (Since then the baccalauréat programme has been slimmed down and some of the emphasis on mathematics removed. Recall, also, that French 16-18 year-olds have over 50% more timetabled hours than do their English peers.)

The major structural problems

If one compares the provisions for mathematics in Years 12 and 13 in France and England then the following points are immediately apparent:

(i) far fewer English students are taking courses in mathematics having clearly prescribed and nationally-agreed syllabus content and assessment procedures;
(ii) in England available courses are not sufficiently well differentiated according to the needs, attainments and aspirations of students;
(iii) the à-la-carte A-level courses which English students construct for themselves on the narrow basis of three or four subjects lack the coherence of the more broadly-based courses followed by students elsewhere.

Major existing organisational problems

(i) the lack of homogeneity of A-level mathematics courses,
(ii) the effects of 'market forces',
(iii) a lack of clarity of pedagogical and mathematical thought and purpose to be seen in some syllabuses,
(iv) a failure to ensure the technological aids now available are always used sensibly and effectively,
(v) a failure to ensure the quality of examination procedures,
(vi) the lack of sound 16+ foundation on which to build.

[Note that (i), (ii) and (v) apply equally at 16+, (iii) also refers to the national curriculum about which David Tall will speak, and, together with (iv), presents an even greater problem pre-16+, and recently the Secondary Heads have raised doubts about the adequacy of the 11+ foundation (cf (vi).]

Desirable actions

1 The present A-level should be replaced by one based either on the baccalauréat model or

one in which students have to opt for more subjects (as suggested by Higginson, but with some constraints on choice in order to achieve some sensible form of coherence of curricula).
[Note, however, that this would not remove the problem of students entering courses with very different levels of attainment. Is it possible, in fact, to have a satisfactory, economical system in which teachers and the system itself have responsibility for coping with the varied entrance qualifications of students, rather than one in which the responsibility of meeting certain entrance requirements is placed firmly on the students e.g. the need to repeat a year if certain pre-conditions are not met?]

2 The provisions for mathematics teaching post-16 should be re-examined in order to ensure that different needs and aspirations are met. However, constraints imposed by finance and the teaching force must not be ignored.

3 The current examination system with its multiplicity of competing boards must be replaced. There is a great need to ensure greater comparability, increased homogeneity of content, a tighter control of standards and, above all, higher quality examining. This can only be done by concentrating our resources much more. The examination system no longer operates in the best interest of education.

4 Steps must be taken to decrease the seemingly increasing gap between maintained and independent schools so far as aspirations and attainment in mathematics and the physical sciences are concerned.
5 There is a clear need to reconsider mathematics teaching 5-16 and, in particular, to see how the needs of potentially high-attaining students in mathematics and science (i.e. those in the top quartile) can be more adequately met. GCSE must provide a sounder foundation for post-16 studies than it presently does.

6 Urgent consideration must be given to improving the professional qualifications of the teaching force through the provision of training, help and advice, and to raising their levels of expectation concerning students' attainment.

APPENDIX 1

The table below shows the changes in numbers taking A-level Mathematics and Further Mathematics. It shows, by way of contrast, the numbers in France passing the Series C Baccalauréat.

	0-Level A-C, CSEl or GCSE A-C two years previously	Single A-level entries	Single A-level passes	Double A-level entries	Size of English 18 yr. old cohort	Baccalauréat Series C passes (Metropolitan France)
1960	81,500 E&W	29,000 E&w	19,200 B&W	10,500 E&w	550,000	17,100
1970	150,400* E&W	50,200 E&W	33,800 E&W	14,200 E&W	628,000	21,400
1980	224,000* E	67,500 E	45,400 E	13,400 E	761,000	32,700
1985	259,400* E	75,800 E	52,100 E	11,900 E	764,000	33,500
1990	243,500 E&W	69,500 E&W	50,300 E&W	6,900 E&W	679,000	56,800
1994	260,600 E	56,900 E&W	48,000 E&W	5,400 E&W	557,000	64,600**

E = England
E&W = England and Wales

* = these numbers are inflated since some students entered for and obtained both an O-level A-C and a CSE 1.
** includes Antilles, Guyana & Reunion (very small numbers).

APPENDIX 2

GCSE Maths (English) Grades of 17 year-olds taking single A-level Mathematics in 1994

	GCSE Grade	
A-level grade	B	C
A	4 (6)	2 (1)
B	11 (17)	5 (5)
C	17 (27)	10 (16)
D	19 (27)	15 (28)
E	17 (15)	18 (28)
NUX	32 (8)	49 (22)
Totals	12,009 (19,122)	3310 (8,158)

The data in brackets refer to grades for English Language at GCSE and English Literature at A-level. (English Lit/English Lit results were very similar.)

Thus, of those obtaining a B grade in mathematics who went on to take mathematics A-level, 32% failed. In comparison, only 8% of those with a B in English GCSE failed English A-level. It will also be noted how many more B and C grade candidates opted to study A-level English than A-level Mathematics.

APPENDIX 3

We present a summary of the extent to which new A-level syllabuses for 1996 cover those topics in the previous A-level core which are not in the new A-level core. The analysis is presented in the form of a table. This may conceal some detailed distinctions, but is done for ease of reference which a more discursive treatment would not provide. Each topic is indicated by a simple Y or a dash, showing whether or not the topic is part of the compulsory element of the board's syllabus, taken by all the candidates irrespective of what award the are entered for, or which combination of modules they take. In some cases a topic may be an optional part of the syllabus, but it cannot be assumed to have been covered by all candidates. This appendix is a slightly revised version of the one to be found in the recent LMS/IMA/RSS report.

Topic	A	B	C	D	E	F	G	H	I	J	K
Rational functions	Y	–	Y	–	–	Y	–	Y	–	Y	–
Partial fractions	Y	Y	Y	–	–	Y	Y	Y	–	Y	Y
Binomial for $\|x\| < 1$	–	Y	–	–	–	–	Y	Y	Y	–	–
Six trigonometric functions	Y	Y	Y	–	–	Y	Y	Y	Y	Y	Y
Sine and cosine rules	Y	Y	Y	Y	–	Y	Y	Y	Y	Y	Y
$a \cos q + b \sin q = r \cos(q+a)$	–	Y	–	–	–	Y	Y	Y	Y	–	–
General soln of trig equation	–	–	–	–	Y	Y	–	–	–	–	–
Small trig approximations	–	Y	–	–	–	Y	–	–	–	–	–
Inverse trig functions	–	Y	Y	–	Y	Y	Y	Y	Y	–	Y
Implicit differentiation	–	–	Y	Y	Y	Y	Y	Y	–	Y	Y
Parametric differentiation	Y	–	Y	Y	Y	Y	Y	Y	Y	Y	Y
Normals	Y	Y	Y	–	Y	Y	Y	Y	Y	Y	–
Small increments	–	–	–	–	–	–	–	–	–	–	–
$\int 1/(1+x^2), \int 1/\sqrt{(1-x^2)}$	–	–	Y	–	–	–	–	–	–	–	Y
Volumes of revolution	–	–	–	Y	–	Y	Y	Y	Y	Y	–
Vectors	–	Y	–	Y	Y	Y	Y	Y	–	–	–
Scalar product	–	Y	–	Y	Y	Y	Y	Y	–	–	–
Vector equation of a line	–	Y	–	Y	Y	Y	Y	–	–	–	–

A: AEB
B: MEI (O & C)
C: NEAB
D: NUFFIELD (OXF)
E: OXFORD
F: O & C
G: SMP 16-19 (NEAB)
H: UCLES (Linear)
I: UCLES (Modular)
J: ULEAC
K: WJEC

APPENDIX 4: FRENCH BACCALAUREAT, 1994

		Entrants	Passes	Hours of study per week		
General Stream				*YII*	*Y12*	*Y13*
A	Arts-Maths	45,516	32,762	4	5	5
B	Economics-Social	96,900	64,481	4	5	5
C	Maths-Physics	75,291	64,572	4	6	9
D	Maths-Biology	87,202	66,264	4	6	6
Technical						
E	Science-Technology	14,000	10,577	3	6	9

(Other streams lay less emphasis on mathematics)

A View From The Schools

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I am pleased to be able to make a contribution to this debate, as I feel most strongly that the voice of practising schoolteachers should be heard. Although I teach in an independent school, my experience both as a Chief Examiner and in curriculum development is extensive, and I believe that what I am going to say has relevance for all schools.

I do not share the much-publicised negative view of school mathematics taken by the press and others. I am a great enthusiast for school mathematics: I think there is a great deal of positive and productive work which takes place, and the environment for students in the 6th form in mathematics is stimulating, varied and challenging. I would not wish to give the impression however, that I am not conscious of some of the problems that the LMS report has highlighted. On the contrary, on most days I observe algebraic difficulties in the work of students; hardly a day goes by when I don't see a student who will simplify $\sqrt{(x^2-a^2)}$ to x-a, or who is convinced that $(-4)^2$ is -16, because the calculator says so! These are common problems, the sort the press loves to emphasise. I prefer to dwell on the more positive things I see! These problems have always existed; the question is whether they are any worse now than in the past. We have to work hard to overcome these basic difficulties – that is the challenge of teaching.

The LMS report highlighted the following three major concerns in its recent report, and these three issues provide the background to what I am going to say:

1. a serious lack of essential technical facility - the ability to undertake numerical and algebraic calculation with fluency and accuracy;
2. a marked decline in analytical powers when faced with problems requiring more than one step;
3. a changed perception of what mathematics is - in particular of the essential place within it of precision and proof.

Challenging the most able

There was discussion this morning about proof. I believe that the National Curriculum has actually raised the profile of proof. On the other hand, the lack of geometry was cited in the report as one reason why proof was less well understood than it once was. One of the reasons that geometry declined in school mathematics was that we had lost a rationale for teaching it! If any flavour of proof actually came through in geometry, it was in a sense almost as an afterthought. Proof was seldom explicit in the teaching of geometry and became less so as time went on, and the subject fell into disrepute. Ironically, able students actually enjoy geometry, and there still is geometry in the National Curriculum. I suspect that the teachers here continue to teach geometry, and to emphasise the proper aspects of proof within it. Part of my own departmental policy document emphasises this important aspect of mathematics, and I am sure much of its spirit would be supported by other teachers:

> *Although the top sets will move through much of the material of the National Curriculum more quickly than their peers, it is not our policy to race through it as quickly as we can. Rather we seek every opportunity to challenge the most able, with the teaching of the National Curriculum to greater depth, particularly in algebra, the teacher seeking at all times to develop greater powers of abstraction and generalisation, of logical proof, and of problem-solving. Exploratory and extended pieces of work are another way in which the most able will deepen their knowledge. The distinguishing feature of the work of the most able must be a greater emphasis on problem solving itself, and on mathematical rigour and mathematical proof. It is the nature of proof that distinguishes mathematics from almost all other fields of human knowledge, and it consequently must play the highest role in the education of the most able.*

I see it as the role of the Head of Department to ensure that proper emphasis is laid on such central themes – it is a vitally important role. It is clear, I hope, that we are very much concerned with matters such as proof – at least as much as we ever were. I do not believe that the National Curriculum has lessened this concern, indeed its explicit mention within the curriculum has raised awareness. It is the responsibility of teachers to ensure that they take proper account of this requirement.

Mathematics is a core curriculum subject, and as such has a very privileged position at the heart of the school curriculum – and rightly so. There is huge curricular support for mathematics teachers by a number of organisations; the importance of this in raising expertise should not be underestimated. A disadvantage, however, is that this conspicuous position leaves us very exposed to criticism, and mathematics often carries the burden of public concern for more general problems. This is not to diminish the problems, but just to observe that there may be some more general causes, not necessarily confined to mathematics itself.

The second point, of particular relevance to the University debate, is that the most able must be suitably challenged – this is something which must not be overlooked. It is my unequivocal view that this challenge is the responsibility of teachers in the classroom. The lack of such teaching is of far greater consequence than any perceived deficiencies in the National Curriculum. I believe therefore that a major requirement is for us to strengthen the quality and effectiveness of the teaching force in mathematics in every way that we can. We must also be careful not to damage the morale of teachers and pupils as a consequence of our legitimate concern about standards, and our criticism of what is currently being achieved. Raising the effectiveness of the current teaching force will require good in-service support, and effective and worthwhile curricular materials. Most importantly, we must increase the number and quality of those entering the profession. The universities have an important role to play here. Although other factors influence whether graduates enter teaching or not, the universities are educating those that do. It seems to me that many graduates entering teaching have neither a coherent view of mathematics nor a sufficient understanding of its cultural and historical background. Although this is acquired in part by the process of teaching itself, students do need time on their university courses for reflection, and teaching which makes the subject more coherent. Failure in this respect means that classes they teach in schools are less rich than they could be, and the subject appears less interesting than it should.

Calculators and Computers
When I was reading the LMS report, on my desk landed a little brochure, which I am sure all who are teaching mathematics will have seen by now – it describes the latest Texas Instruments calculator, the TI92! For those who haven't seen the TI92, it will now perform every technique currently taught in school mathematics! It can do all the algebra and calculus that we currently teach – it even has the equivalent of Cabri geometry installed! Beginning with the introduction of hand-

held calculators in the 1970s – which transformed the teaching of arithmetic – we saw calculators rapidly extend their capabilities to cover fractions, then (in a quantum leap) graph-plotting, and now algebra and calculus! This steady growth in the range of tasks performed by calculators has huge implications for teaching in schools which we need to consider most carefully. I certainly welcome, (although I was disappointed to learn of it only from the front page of *The Times*), that assessment of younger pupils is going to include some papers in which calculators are not to be used. I regard this as a step in the right direction. We are all aware that there are great advantages for the teaching and learning of mathematics with calculator and computer technology. Many students do very clever things on calculators, and a great deal of mathematical thinking goes into what they do. Too often, however, they will use a calculator to do the most trivial of tasks. I believe that, improperly used in the learning process, calculators can have a detrimental effect on the understanding of some basic skills and ideas, and that, while encouraging their use in the learning of mathematics, we must be conscious of how they are used and of possible consequences. In examination work also you have to be very careful now what you ask students to do! On the positive side, the 16-19 mathematics A-level writers chose to make the graph-plotting calculator an integral part of the new course. This radically affected the way we look at the calculus, and it has also dramatically affected the way we teach and examine graph-plotting. Old-style questions, such as, "x2/(x-1)(x+1), plot this graph, 5 marks." have to be reappraised. Either you don't set a question like this at all or you recognise that there are worthwhile skills in setting the calculator up to answer it. What has happened is almost certainly that while schools are still teaching curve sketching skills – plotting the graphs of rational functions, finding asymptotes and discontinuities etc. – there is a natural tendency , human nature being what it is, for students to resort to the graphic calculator to do all the practice examples that you set them to do! The calculator has had a huge effect, both positive and negative. So, great implications, but friend or foe? I will leave you to decide. Whatever you do decide of course, we simply have to find effective ways of using the calculator in our teaching programmes, and of encouraging our pupils to use them as ladders rather than as props!

Computers also have dramatically changed the way many topics are taught and have added considerably to the interest of mathematics in schools, providing an extra dimension to teaching, especially in the 6th form. Once again, students do very good work using spread-sheets in particular, and some will be confident in programming both computers and calculators. These are important new skills.

The National Curriculum

I would like to talk now about the National Curriculum. It is certainly a very worthwhile attempt to provide a coherent structure for school mathematics from 7 through to 16. I suspect, however, that an obsession with levels in the National Curriculum was a distraction, and for many teachers was a bureaucratic nightmare. An all too literal interpretation fragmented the subject, and as an examiner I could see that this also distorted examination papers. We must take a much more holistic view than the level structure has actually led to. I accept the comment in the LMS report that better students need to be given more than one-step problems to solve, and I think that curriculum developers especially should take this into account. However I do not believe that we should have the situation – which we presently have – where mathematics examinations are much more difficult than other GCSEs and in particular other A-levels. Where examinations have been so inaccessible to the majority of students that they were not properly rewarded for their efforts, quite sensibly they will choose to look elsewhere for their A-levels. This has resulted in generations of adults having a very negative view of mathematics – a situation which has not served university or teacher recruitment well.

I am a supporter of coursework as a part of the National Curriculum in mathematics! Despite some obvious dangers, it does provide a valuable opportunity for investigative and exploratory work. The idea that teachers in schools are allowing students to discover mathematics for themselves all of the time is, of course, simply to parody the real situation. What we do encourage, at appropriate times, is for students to explore ideas for themselves. Many students, especially brighter students, will progress from 'Yes, now I can see this' to 'How can I prove this result for myself?' Coursework of this kind provides an environment where the need for proof arises naturally, and sometimes leads students to try to prove their own results, which gives it added significance for them.

I believe that there should be a single national GCSE, with the proviso that there is sufficient accommodation for challenging new curricular developments, and some choice of coursework options. I really don't see any fundamental difference between the examinations of one board and those of another, and believe the multiplicity of boards and exams to be a huge dilution of effort, as well as a cause for concern over comparability of standards which we could well do without.

Post-16 Mathematics

I would like to discuss post 16 mathematics, where I have been most involved in the last 6 or 7 years. At this level there have been a number of major curriculum developments that have addressed the problem of low uptake of mathematics. These have been put together with considerable care and research by groups of practising teachers who are certainly aware of many of the problems we have been talking about, and raised in the LMS report. The major of these developments have been those of MEI, Nuffield, and SMP(16-19 Mathematics). I would like to point to two achievements here.

First, more students have been encouraged to study beyond GCSE in the majority of schools running the new courses. I believe these are new students, and not students moving from one exam board to another. In my own school, for example, we have doubled the number of students doing mathematics post 16 in 3 years – and we need to do this nationally. As the pool of students taking mathematics post-16 increases, then more will be encouraged and interested enough to go on to read mathematics at university (although we should not forget that there are many other attractive options and courses for them now – more than ever before!). In any event these students will be more able to cope with other subjects that they may encounter at university or in employment. Teachers comment that there is increased motivation in their students, and a greatly reduced drop-out rate.

Secondly, AS level in Further Mathematics is now available in 16-19 Mathematics and in MEI, and is an extremely attractive option. I believe the 16-19 Further Mathematics AS level in particular is a good course, and I would have thought the universities would have valued it. What's more, the resource material that supports this course has allowed more centres to offer it than could have done so before. It has made Further Mathematics an attractive option in the 6th form, and there are signs in my school, and I think probably in others, of greater interest in university mathematics. Finally, and I want to stress this, the schemes are popular with teachers, with students and parents.

I am concerned that universities may not really have come to terms with the fact the most students are now reading mathematics with a single maths A-level, and many of them with moderate grades. I hope that the universities have really accepted the implications – that their students are arriving with only half as much mathematics as they did a few years back, and possibly even less. I have serious concerns about students who have got a grade C or below at A level going on to university to read mathematics. If such students are to be accepted then it is important that their obvious potential difficulties are recognised, and that the universities have appropriate support structures in place to accommodate them. The school-university link is absolutely crucial to the current debate. Universities should be encouraged to contribute to mathematical education in the 16-19 age range – there is a clear and obvious need for much greater co-operation between the schools and the universities. A more formal linkage should be established – in each university, there should be someone responsible for the first year of the mathematics degree course, whose job it is to see that the course follows on naturally from A-level. We should have a university degree structure that takes full account of what students have and have not learned at A-level. It should be the job of such a person to know what is going on in school mathematics and to find ways to contribute to it.

Conclusions

I would like to conclude by summarising some of the points I have made and by suggesting some of the things we might do to improve the situation.

I stressed at the beginning that the quality of mathematics education in the schools is a function of the quality of the teaching force. I believe therefore most strongly that we need to enhance the quality of the teaching profession as a matter of some urgency. We must find ways to overcome the recruitment problem – giving pupils a much better experience of mathematics in schools will naturally help. We must also provide good quality in-service support for those presently teaching. We also need to employ those teachers we have to best effect, to try to ensure that they are kept in the classroom as much as possible (and that their career prospects are not diminished as a consequence).

Suitable timetabling arrangements are also important. Mathematics needs a timetable adapted to its own special requirements, and this should not be over-ridden by general school policy which says, for example, that every class is to be a double class, an hour and a half to an hour and forty minutes long. Such policies lead to real problems: you do not see students enough times each week, and I am sure that less is achieved in one eighty minute session than in two forty minute ones. I believe setting by ability makes the teaching (and learning) of mathematics more manageable and more effective. I have doubts that most mathematics teachers, ordinary mortals that we are, are able to teach successfully in classes of widely mixed ability.

We must examine very carefully what *are* the important skills now, particularly in view of changes in calculator and computer technology. We should take the opportunity to reassess what school mathematics *is* about. It is the role of the Head of Department to make sure that appropriate emphasis is made in teaching programmes. As far as calculators are concerned, we must recognise that they have made some skills redundant, and we must be prepared to abandon these rather than artificially sustain them. We should not criticise students who do not have these skills, and we should encourage other, newer, and perhaps more worthwhile ones that they have acquired.

I have mentioned in-service provision. I believe that all newly promoted heads of department should be offered training. HMI used to, and possibly still does, run some excellent courses, but I'm afraid their advisory service has diminished somewhat as they concentrate on inspection. This is to the great detriment of teaching. Although I don't disapprove of inspection itself, we do need more good mathematics advisory teachers.

There probably is a need for a national steering group for mathematics – a National Mathematical Council perhaps – with at least university and school representation. Years ago, when the Standing Conference on University Entrance used to designate what was then the common core, to my held calculators in the 1970s – which transformed the teaching of arithmetic – we saw calculators rapidly extend their capabilities to cover fractions, then (in a quantum leap) graph-plotting, and now algebra and calculus! This steady growth in the range of tasks performed by calculators has huge implications for teaching in schools which we need to consider most carefully. I certainly welcome, (although I was disappointed to learn of it only from the front page of *The Times*), that assessment of younger pupils is going to include some papers in which calculators are not to be used. I regard this as a step in the right direction. We are all aware that there are great advantages for the teaching and learning of mathematics with calculator and computer technology. Many students do very clever things on calculators, and a great deal of mathematical thinking goes into what they do. Too often, however, they will use a calculator to do the most trivial of tasks. I believe that, improperly used in the learning process, calculators can have a detrimental effect on the understanding of some basic skills and ideas, and that, while encouraging their use in the learning of mathematics, we must be conscious of how they are used and of possible consequences. In examination work also you have to be very careful now what you ask students to do! On the positive side, the 16-19 mathematics A-level writers chose to make the graph-plotting calculator an integral part of the new course. This radically affected the way we look at the calculus, and it has also dramatically affected the way we teach and examine graph-plotting. Old-style questions, such as, "x2/(x-1)(x+1), plot this graph, 5 marks." have to be reappraised. Either you don't set a question like this at all or you recognise that there are worthwhile skills in setting the calculator up to answer it. What has happened is almost certainly that while schools are still teaching curve sketching skills – plotting the graphs of rational functions, finding asymptotes and discontinuities etc. – there is a natural tendency , human nature being what it is, for students to resort to the graphic calculator to do all the practice examples that you set them to do! The calculator has had a huge effect, both positive and negative. So, great implications, but friend or foe? I will leave you to decide. Whatever you do decide of course, we simply have to find effective ways of using the calculator in our teaching programmes, and of encouraging our pupils to use them as ladders rather than as props!

Computers also have dramatically changed the way many topics are taught and have added considerably to the interest of mathematics in schools, providing an extra dimension to teaching, especially in the 6th form. Once again, students do very good work using spread-sheets in particular, and some will be confident in programming both computers and calculators. These are important new skills.

The National Curriculum

I would like to talk now about the National Curriculum. It is certainly a very worthwhile attempt to provide a coherent structure for school mathematics from 7 through to 16. I suspect, however, that an obsession with levels in the National Curriculum was a distraction, and for many teachers was a bureaucratic nightmare. An all too literal interpretation fragmented the subject, and as an examiner I could see that this also distorted examination papers. We must take a much more holistic view than the level structure has actually led to. I accept the comment in the LMS report that better students need to be given more than one-step problems to solve, and I think that curriculum developers especially should take this into account. However I do not believe that we should have the situation – which we presently have – where mathematics examinations are much more difficult than other GCSEs and in particular other A-levels. Where examinations have been so inaccessible to the majority of students that they were not properly rewarded for their efforts, quite sensibly they will choose to look elsewhere for their A-levels. This has resulted in generations of adults having a very negative view of mathematics – a situation which has not served university or teacher recruitment well.

Can All Children Climb The Same Curriculum Ladder?

David Tall
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This presentation presents evidence that the way the human brain thinks about mathematics requires an ability to use symbols to represent both process and concept. The more successful use symbols in a conceptual way to be able to manipulate them mentally. The less successful attempt to learn how to do the processes but fail to develop techniques for ***thinking*** *about mathematics through conceiving of the symbols as flexible mathematical objects. Hence the more successful have a system which helps them increase the power of their mathematical thought, but the less successful increasingly learn isolated techniques which do not fit together in a meaningful way and may cause the learner to reach a plateau beyond which learning in a particular context becomes difficult.*

Introduction

At the age of 16 the number of children passing their General Certificate of Secondary Education at grades A, B, C is currently increasing year by year (although there is a worrying trend that the number with the lowest grades is remaining stubbornly stable). At age 18 there is an improving spectrum of passes at A-level, although mathematics is becoming a less popular subject. Despite an apparent trend for top students to get better marks in school examinations, university lecturers claim that the students arriving at university lack basic skills. In particular:

(i) They lack fluency in arithmetic and algebraic skills,

(ii) They are less able to solve problems involving several steps,

(iii) They do not perceive the need for absolute precision and proof in mathematics.

The contrast between the apparent success as seen from a school perspective yet failure from a university perspective has led to unseemly accusations flying in all directions. My own perception of the phenomenon is that the two viewpoints are focusing on different things and arguing at cross-purposes. To be able to unravel the conundrum, we need get an insight into what is happening when individuals learn mathematics and begin to 'think mathematically'. By doing so it is hoped that some light can be shed on the situation.

Let us begin to look at some fundamental questions to see how it is that individuals learn to use mathematics in a powerful and productive way.

Cognitive considerations

The human brain is a huge simultaneously processing system. To be able to make conscious decisions using such a mechanism requires the individual to filter out inessential detail and to focus attention on the important essentials.

> The basic idea is that early processing is largely parallel – a lot of different activities proceed simultaneously. Then there appear to be one or more stages where there is a bottleneck in information processing. Only one (or a few) 'object(s)' can be dealt with at a time. This is done by temporarily filtering out the information coming from the unattended objects. The

> attentional system then moves fairly rapidly to the next object, and so on, so that attention is largely serial (i.e., attending to one object after another) not highly parallel (as it would be if the system attended to many things at once).
>
> (Crick, 1994, p. 61)

The process leads to the sensation in which the conscious mind focuses attention on things of current interest, manipulating them in the mind, then passing on to related ideas which occur in the conscious train of thought:

> There seems to be a presence-chamber in my mind where full consciousness holds court, and where two or three ideas are at the same time in audience, and an ante-chamber full of more or less allied ideas, which is situated just beyond the full ken of consciousness. Out of this ante-chamber the ideas most nearly allied to those in the presence chamber appear to be summoned in a mechanically logical way, and to have their turn of audience.
>
> (Galton, *Inquiries into human faculty and its development*, 1883)

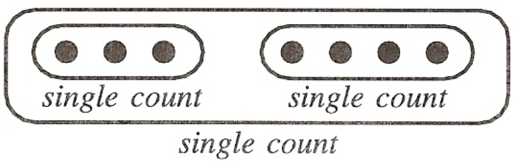
This limited focus of attention means that thinking is enhanced by two complementary processes:

- *compressing* data to fit in the focus of attention,
- *linking* data in the brain to be able to bring other ideas in the huge long-term memory into the focus of attention for processing.

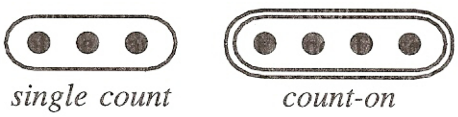
In mathematics, compression of data occurs in various pragmatic and elegant ways. For instance, we may draw diagrams to represent information succinctly, use words to stand for complex ideas, or mathematical symbols to to represent problem statements that can be manipulated to produce solutions. The last of these proves to be particularly powerful and subtle.

The problems that occur later in school and university begin much earlier, starting with the way that the processes of simple arithmetic develop (or even before). In level 1 of the National Curriculum, children are expected to be able to handle addition of numbers up to ten by the process of 'count-all', counting a set of objects, then another set, and putting the two together to count them all to obtain the sum. Subsequently various other techniques are developed showing greater efficiency:

- **Count-all**: a succession of three simple counting processes, 3+4 is '1, 2, 3', '4, 5, 6, 7', then count '1, 2, 3, 4, 5, 6, 7', usually pointing at specific objects,



- **Count-both** is two counting processes, a simple count, eg '1, 2, 3', then a count-on '4, 5, 6, 7', usually using some technique (eg four fingers) to keep a tally of the count-on,

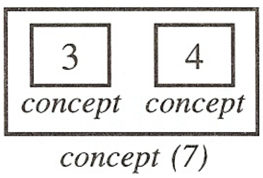


- **Count-on** is a concept ‘3’ and a process to count-on 4 after 3 as ‘4, 5, 6, 7,’

- **Count-on from larger** is a variant of count-on which shortens the counting process,

(This is more spectacular where the numbers differ greatly in size, for instance, computing 2+9 by counting on 2 after 9 rather than 9 after 2).

- **Known fact** regards the symbols as number concepts and recalls the result as another number concept, ‘3+4 is 7.’



- **Derived fact** uses the number facts themselves as manipulable mental objects, operated on them to give new facts, eg ‘3+4 is one less than 4+4=8, so it is 7.’

(Fuson, 1992,;Gray & Tall, 1994)

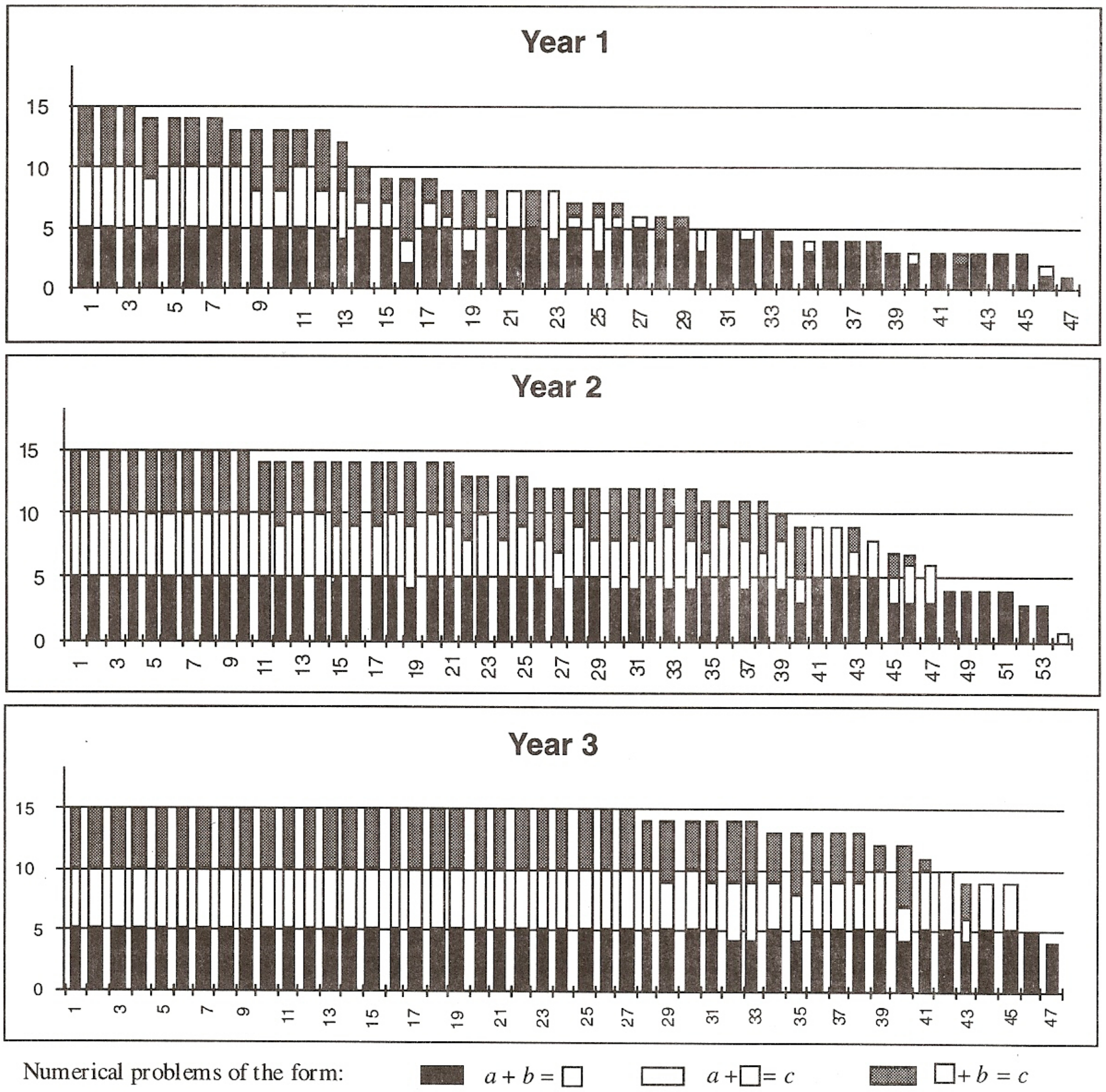
As children develop they initially interpret symbolism in process terms so that the symbol 3+2=5 means ‘3 plus 2 *makes* 5’, and the equation 5=3+2 lacks meaning because 5 does not *make* 3+2. The initial introduction of what adults may see as ‘algebra’ in solving equations such as

$$4+\square = 7,$$

$$\square + 3 = 7.$$

represents completely different problems for children. At a certain stage the first may be read ‘How many do I count-on after 4 to get to 7?’ and the second is much harder if it is read as ‘Which number do I start at to count on 3 and reach 7?’.

The following figure shows the results of a test given to children in years 1, 2, and 3, involving such problems (Foster, 1994):



Children's performance on arithmetic problems

Each bar represents an individual child. The three parts of each bar represent the number of correct responses to five written arithmetic problems (fifteen in all) in the form:

: $a+b = \square$ (1)

: $a+\square = c,$ (2)

: $\square + b = c.$ (3)

This reveals a familiar statistic—as the children get older, the performances get better, which is the basis of the idea of successive levels of attainment which children reach. But this is a grossly oversimplified interpretation. If the classes are divided into thirds as 'higher', 'middle' and 'lower', then qualitative differences are revealed in the spectrum of performances.

Year 1	(1)	(2)	(3)
Higher	99%	*81%*	*83%*
Middle	86%	29%	29%
Lower	59%	5%	1%

Year 2	(1)	(2)	(3)
Higher	100%	96%	96%
Middle	93%	*76%*	*80%*
Lower	74%	38%	20%

Year 3	(1)	(2)	(3)
Higher	100%	100%	100%
Middle	100%	98%	98%
Lower	93%	73%	53%

Percentage of correct responses in each category by year at different levels of attainment

Apart from tiny reversals shown in italics, the performances are all ordered (1)≥(2)≥(3). In year 2 the higher attainers reach almost 100% whilst the middle attainers do not do so until year 3. The higher and middle attainers all show (2) approximately equal to (3), but the first year lower attainers cannot do items (2) or (3) and those in years two and three continue to show a difference between each of the three categories in the order (1)>(2)>(3).

It is not simply a question that the lower attainers do the same thing but slower. The *marks* improve, but the *methods* at extremes of the spectrum may be very different. Whilst the five-year olds who succeed are already beginning to use their number facts in a flexible manner as both processes and concepts, those who first succeed at a later age are more likely to rely on procedural counting. Those nearer the top end of the spectrum are developing a way of thinking which compresses knowledge into flexible symbolic form suitable for more abstract developments. Those nearer the bottom end are focussing more on procedures on physical objects which lock them in a mode from which abstract thought is much more difficult, even, perhaps, impossible.

The notion of procept

It often happens that a mathematical process (such as counting) is symbolised, then the symbol is treated as a mathematical concept (such as number) which is then manipulated as if it were a mental object. Here are just a few examples:

symbol	*process*	*concept*
1, 2, 3, ...	counting	number
3+2	addition	sum
−3	subtract 3, 3 steps left	negative 3
3/4	division	fraction
$3+2x$	evaluation	expression
$v=s/t$	ratio	rate
$\sin A = \frac{opposite}{hypotenuse}$	trigonometric ratio	trigonometric function
$y=f(x)$	assignment	function
dy/dx	differentiation	derivative
$\int f(x)\,dx$	integration	integral
$\lim_{x\to 2} \frac{x^2-4}{x-2}$, $\sum_{n=1}^{\infty} 1/n^2$	tending to limit	value of limit
$\sigma \in S_n$	permuting $\{1, 2, \ldots, n\}$	element of S_n
solve(f(x)=0,x)	solving an equation	solution of equation

Given the wide distribution of this phenomenon of symbols evoking both process and concept, it is useful to provide terminology to enable it to be considered further.

Definition: An ***elementary procept*** is the amalgam of a ***process***, a related **concept** produced by that process and a **symbol** which represents both the process and the concept.

Definition: A ***procept*** consists of a collection of elementary procepts which have the same object (Gray & Tall, 1994).

These definitions are of a cognitive concept to model what seems to happen in the cognitive structure of the individual. The mathematically oriented child does not only think that 2+3 "makes" 5, but that 2+3 *is* 5, as are 4+1, 7–2, $^{10}/_{2}$, and so on. The procept "5" grows to include all these different ways of making five, so that it becomes cognitively richer as the child grows. It is this cognitive richness that gives the individual power as a mathematician. It represents the way that individuals seem to use symbols to give great *flexibility* in thinking – using the *process* to *do* mathematics and get answers, or using the *concept* as a compressed mental object to *think about* mathematics – in the sense of Thurston:

> I remember as a child, in fifth grade, coming to the amazing (to me) realization that the answer to 134 divided by 29 is $^{134}/_{29}$ (and so forth). What a tremendous labor-saving device! To me, '134 divided by 29' meant a certain tedious chore, while $^{134}/_{29}$ was an object with no implicit work. I went excitedly to my father to explain my major discovery. He told me that of course this is so, *a*/*b* and *a* divided by *b* are just synonyms. To him it was just a small variation in notation. (Thurston, 1990, p. 847)

Perhaps the reason why mathematicians haven't formulated the definition of a procept (as they have done other profound simplicities such as "set" and "function") is that mathematicians seem to *think* in such a flexible ambiguous way often without consciously realising it, but their desire for final precision is such that they write in a manner which attempts to use unambiguous definitions. This leads to the modern set-theoretic basis of mathematics in which concepts are defined as *objects*. It is a superb way to systematise mathematics but is cognitively in conflict with developmental growth in elementary mathematics where arithmetic and algebraic processes *become* mathematical objects through the form of compression called encapsulation. In this way the viewpoint of the mathematician is not synonymous with the needs of the child in developing mathematical knowledge.

The more mathematically oriented children seem to develop this flexibility early on, the less mathematical do not and this imposes a great strain on their focus of attention, causing them to fall back on rote-learning of procedures which will *do* mathematics at the time, but are far less likely to be suitable for holding in the small focus of attention to reflect upon and build more sophisticated ideas.

I hypothesise that greater mathematical success comes not from remaining linked to the perceptions of the world through our senses, but through using the symbolism that is especially designed for *doing* mathematics and for *thinking* about it. We have only so much conscious focus of attention to use at any one time and rely on the richness of internal connections to build up the flexible characteristics of the mathematical mind. Those who use internal conceptual connections to fit together mathematical processes and concepts in a flexible manner have a greater chance of success than those who burden their already strained memory with long procedures needing concrete support.

Fractions

If the difficulty of compressing counting procedures into number concepts is unresolved, the problem grows still larger in later developments. For instance, in the study of fractions, a child who does not have flexible knowledge of arithmetic of whole numbers will find it more difficult to coordinate the notion of equivalent fractions and a child who sees a fraction 2/3 as a process 'take three equal parts and select two of them' is going to have great difficulty in operating on fractions. For instance if 1/2 + 2/3 is interpreted as 'divide something into two equal parts and take one of them and add this to the result of dividing something into three equal parts and take two of them', then it is hardly likely to make sense to someone already at the limit of their cognitive capacity.

Thus it is that fractions prove difficult for some and impossible for many and so have no place in a democratic curriculum on a ladder which all will climb. But the more successful students, who would find such things cognitively simpler, may be denied adequate exposure to the topic to give it meaning.

The beginnings of algebra

Algebra suffers in the beginning from the difficulty many children have with meaning. For instance, if an arithmetic expression such as 2+3 is considered as performing a sum to get an answer, then an algebraic expression such as $2+3x$ causes confusion. If it is read as a process, then it asks the child to add two and three then do something with x and, if x is unknown, the operation cannot be done. There is the further difficulty of parsing the expression (breaking it down to do it in the appropriate sequence). If it is read in the usual order from left to right, it is 2+3 (which is 5) and then x, which gives $5x$. There is research (Macgregor & Stacey, 1993) which shows that when children are asked to write a simple sentence such as 'y equals the sum of 4 and x' into algebraic notation, many get it wrong, often writing $y=4x$, perhaps because the word 'and' between 4 and x is interpreted just by writing the symbols next to each other (because the child knows no mathematical symbol equivalent to 'and').

If the symbols are interpreted correctly then an equation such as
$3x+1=7$
is inherently easier than
$3x+1=2x+1$.

But this is not simply because the second is 'more complicated'. In essence, the first can be read as a *process* '3 times x plus 1 is 7', and this can be unravelled by seeing that if 'something plus 1' is 7, then that 'something', in this case '3 times x' equals 6, and this in turn shows x must be 2. The second equation is more difficult because it seems to have a process on either side and therefore may not make sense to children who regard '=' as 'makes'. One way of giving meaning is to consider each side as an expression, whose values are equal. The solution method taught in school involves 'doing the same thing to both sides', so that the two sides remain equal but are now different from what they were before — a most sophisticated and confusing idea for the less successful.

One 'solution' to these difficulties suitable for a wider range of children has involved linking them to the real world by interpreting an equation as a physical balance to explain 'adding the same thing to both sides'. I suspect this is a pseudo-scientific approach. Those who are insightful enough to give algebra an appropriate meaning do not need it and those who need it develop an approach which sometimes seems to the casual observer to be mathematics but in reality is likely to be something subtly different.

Thus traditional algebra proves to be cognitively difficult for the majority of students. Many educators attempt to improve the situation by giving it "real meaning'. But mathematicians think powerfully precisely because they use the links *within* mathematics and do not relate constantly to the real world. It may therefore be that curriculum designers who desire to make algebra widely understood are using methods that are broadly applicable but may be less suitable for the mathematically oriented who eventually need a more powerful abstract form of algebraic thinking.

Limiting processes

The idea of a limit also has notation that has connotations both as a limiting process and a limit concept. This has proved a significant obstacle to understanding the mathematical idea of a limit. For example infinite decimals are considered as 'going on forever' and 'never reaching' the 'final value'. Although students may regard 0.333... as a repeating decimal whose value is the fraction $^1/_3$ because of its familiarity, a decimal such as 0.4576121212... with the repeating pair 12 may not be seen as a fraction. For many years students are becoming familiar with approximate decimals and so find the precision of mathematical analysis and the formal limit process are foreign to their experience.

Thus the limit is a procept which behaves quite differently from the procepts of arithmetic which have built-in operations to give an answer in a finite number of steps. The limiting process is potentially infinite and causes difficulties because many students seem to believe that it never reaches its conclusion.

Changes in the nature of proof in elementary mathematics

Euclidean geometry requires more than the proof structure of Euclid to make it meaningful:

> The deductive geometry of Euclid from which a few things have been omitted cannot produce an elementary geometry. In order to be elementary, one will have to start from a world as perceived and already partially globally known by the children. The objective should be to analyze these phenomena and to establish a logical relationship. Only through an approach modified in this way can a geometry evolve that may be called elementary according to psychological principles. (van Hiele Geldof, 1984, p. 16)

The development of geometrical knowledge is very different from that of arithmetic and algebraic knowledge, involving teasing out the properties of geometric shapes, describing and refining their meaning to give verbal definitions of imagined 'perfect' platonic figures. The difficulty of the development of geometrical knowledge and proof has long been acknowledged.

The Mathematical Association was formed as the 'Association for the Improvement of Geometry Teaching' in 1871, yet the teaching of geometry has never proved satisfactory for a wide range of pupils. In the sixties there was an attempt to replace synthetic Euclidean geometry by transformation geometry. In terms of the theory presented here this is a significant move because the transformations are both processes of transformation and objects of a transformation group, and so they are procepts. The attempt was therefore to make geometry computation-oriented rather than proof-oriented. This however proved to be intellectually too demanding for the wide mass of pupils and it failed. So geometry as both a deductive and a computational science has largely been lost to elementary school to be replaced by broader concepts of 'space and shape'.

Again there is a loss for mathematically oriented students who might find delight in seeing how proof is constructed in a verbal way based on visual representations of geometric figures.

Reflections

The result of this analysis is that the mathematics of arithmetic, algebra and calculus uses symbols both as processes and concepts and the mathematically oriented student develops flexible ways of using them, both as compact symbols that can be considered as mental objects in the limited focus of attention and as ongoing mathematical processes to be able to obtain answers to problems. There is evidence to support the hypothesis that the less successful student tends to cling more to the security of known procedures to 'get answers' but that these are less suitable for thinking about than flexible symbols which can also be considered as mathematical objects to be compared, related and operated upon.

The idea that we all go through the same developmental stages but perhaps at a different pace is comforting to curriculum developers who may therefore decide to design such a curriculum in successive levels for children to study. It is a straw for politicians to grasp, for example in the legally enforced National Curriculum in England, for it suggests that progress may be measured by the level attained. According to this theory, children's progress can be monitored by assessing progress through specified stages and the changes will give a measure of the quality of learning and teaching.

But it is a naive and damaging assumption. It is not only the things that children can *do* that measures progress, but *how* they do them, and whether their methods are of a kind that can be built on in subsequent development. This discussion has shown that there is a broad spectrum of performance in which those who are successful develop a flexible way of handling symbols so that they may be flexibly manipulated to derive new facts from known ones. Those at the other end of the spectrum learn fewer facts, are often unable to derive new facts from old and in arithmetic fall into inflexible counting procedures related to physical representations that tend not to generalise to problems involving larger numbers. Short-term success might be bought for a time by their learning routine procedures and attempting to rote-learn facts but this may only store up problems for a later stage.

A curriculum built up on evaluating what children can do, in which it is possible to succeed at a given level by radically different thinking processes, can lay foundations for eventual failure for those who do not develop methods that will lead on to later developments and may limit those whose cognitive structure develops in a way which is suitable for more powerful thinking.

Democracy in education does not therefore mean giving every child the same sequences of learning, but at different paces. It means giving each child the education that best suits the child's individual needs appropriate for his or her growing cognitive structure. And this in turn may very well mean that many children need help with the physical meanings and relationships with real world referents, but those who are succeeding in a flexible manner may need a more reflective learning environment, less dependent on physical referents, that encourages growth of more powerful conceptual relationships.

In the debate between university mathematicians and schoolteachers there may be a situation in which each is focusing on different forms of mathematical need, the schoolteacher for the need of a wide spectrum of children and the mathematician on that part of the spectrum which may move on to study university mathematics.

The perceived difficulties formulated at the beginning of this paper are all consonant with the possibility that the more mathematically oriented children in school are not getting the kind of curriculum for which they are capable, in the name of producing a curriculum ladder for all. It is a curriculum ladder which demands too much of the less successful so that they reach various plateaux where their cognitive structure is no longer able to cope with the increasing complexity, yet fails to support the mathematically able who need a more powerful approach to build the long-term development needed in professional mathematics.

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Summary Of The Discussions

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After each talk there was a discussion of twenty minutes, open to all participants. Rather than transcribe what was said (interesting though that would be) here I shall single out a number of issues which came to light and which I think are worth stressing. The first is of course the fact that there are a number of groups here, with differing outlook and interests. There are the teachers, represented on the panel by Chris Belsom, and in the audience by a number of teachers, including some heads of mathematics, from schools in the state and independent sectors. Next there are the end-users, most vociferous among them the university mathematicians, though there were contributions from university Engineering Departments, the Institute of Actuaries and the London Business School. Finally, there are the mathematics educationalists, of whom a somewhat heterodox representative, David Tall, was on the panel. What I found striking, was the extent to which members of the three groups were so clearly identifiable in the discussion.

Is there a problem?

There was little consensus as to the nature and extent of the problem, or, (if there is a problem) its causes and what to do about it. Several teachers (including Chris Belsom in his talk) made the point that the teaching profession is already under a great deal of strain, and criticism of school mathematics from the universities is more likely to demoralise teachers and pupils than help to raise standards; it is liable to be seen as an attack, and not only on the mathematical abilities of school leavers but on those of their teachers. Teachers also questioned the extent to which a decline has really occurred; here is Michael Davis of Westminster School:

> I think it is too easy to look back at a golden age. I am just old enough to have done lots of Euclidean geometry and to have worked my way through Durell. I was at a grammar school in a class where most people got A's in O-level mathematics, and I am quite sure that though they were well able to do the questions in the exam, very few of them had any idea of what they were doing. What they were doing was producing proofs, not understanding what proofs are. There is a difference between the two, and teaching one is much easier than teaching the other, so I think you're right to say that technical things are important, and we're very keen on teaching technical fluency, I think it's well worth spending time teaching and practising, but it isn't the whole story; even technical fluency in quite difficult things like constructing proofs in Euclidean geometry. The attempt to go further and get children to understand what they're doing is extremely important, even though it's extremely difficult and in many cases isn't successful.

This is also the view of some of the mathematics educationalists: what is needed is that the students should have a more reflective, more critical understanding of the procedures which we teach them. Here is Barbara Jaworski, of the Department of Educational Studies, Oxford University:

> ...if right at the beginning, when schemes for learning are at hand, we are encouraging the

> students to be aware of the processes, so that when they have developed a number of ways of adding we can actually have a discussion with them about which are the most effective ways of doing it and why, and encouraging the students to be critical thinkers at the level of doing simple addition, then that critical thinking is likely to carry forward into other aspects of their doing mathematics at different levels. Now this is not to say that all students will be able to think critically about analysis, but it's likely that when a student has come from an education that encouraged them to think critically at all levels,... then when they get to do analysis at university level they might be able to cope with it more readily...

and Leone Burton of the School of Education, Birmingham University

> I don't think we can present 'the best way' [to perform a mathematical operation] and expect that anyone will understand why it is the best way. What we have to do is put them in a position where they can begin to make an analysis of why their way is good for some things but less good for others, and I think that very early on in mathematics education we lose that critique and we start to impose a view of mathematics, and at the same time we start to see young people losing interest.

It is through "humanising" the teaching of mathematics, rather than through insisting on the learning of more difficult technical procedures, that our students will learn more. This view is met with a certain amount of scepticism from university mathematicians and engineers. Here is Ian Stewart, in the discussion after David Tall's talk:

> What is your criterion for success in teaching analysis, for example? It is not that the process of learning it is painless and easy, and the student at each stage knows what they are doing. It is that at some point later on they understand it and how to use it. Warwick's mathematics students, by their final year, understand an awful lot of analysis well, and I would say that we do teach it effectively, but we do not teach it in a painless way. Now if you start thinking about other areas you begin to see why. If you went out to the ballet schools and suggested to them that learning to do ballet at the highest levels should be a painless process, and that if the students feel at all uncomfortable then the teacher has got something wrong, they'd laugh at you!

Tony Gardiner, of the School of Mathematics, Birmingham University (one of the authors of the LMS report) responded to Barbara Jaworski's call for a more reflective, more critical approach to mathematics in schools thus:

> [Barbara] said that if the student doesn't understand what is the problem in analysis, how can they get their mind round the processes? Well if you think about the infant learning to read, what you find is that millions and billions of human beings get their mind round the processes without ever addressing the problem ...

That is, in his view it is not always necessary for students to understand the rationale behind what they are being taught, and moreover some skills are needed so urgently that it is not possible to wait until students are able to understand their purpose.

Steven Syklos (Department of Applied Mathematics, University of Cambridge), believed that the current deficiencies are not confined to mathematics:

> My colleagues in history say that even though their students are now good at looking up sources, they lack the technical facility to write an essay, and my colleagues in modern languages say that even though the students can look at a piece of text, they don't have the technical facility to cope with writing fluently and properly. I just wonder if we're not looking at part of a wider problem. Our students now are not grasping analysis even after an extra six months because they have not been doing anything technically difficult since the age of five.

Besides university mathematicians and engineers, two other groups raised their concern about standards. Liz Goodwin, of the Institute of Actuaries, in a written submission, said

> The actuarial profession needs people who can analyse, communicate and use judgement. Our ground roots are in probability and mathematics. We are concerned that our students today find it difficult to tackle more open-ended problems. They can do single, one-stage problems, but for example they cannot use lateral thinking. We are vitally interested in the issues debated today, and urge that in any review the views of employers of mathematicians, and of professional bodies such as the Institute of Actuaries, should be taken into account. This debate does not concern only schools and universities.

Peter Moore, of the London Business School, complained about the poor communications skills of today's mathematics graduates. In his experience many of the people who apply to the London Business School are either illiterate or innumerate. The division between the two cultures is still very much in evidence, and is the cause of serious difficulties for British business.

Several people remarked on a lack of serious statistical evidence of the nature and extent of the problem, which makes all the more pointed the remark by Kath Hart, of the Shell Centre for Mathematics Education, Nottingham University, that this year the Education Secretary Gillian Shephard sent back £11 million unspent which has been earmarked for research into the National Curriculum and its effects.

Is the National Curriculum to blame?
Several possible causes for the decline came up in the discussion. First was the weakening of the syllabus and of the demands of the examination boards, in both pre- and post-16 mathematics. The university mathematicians who spoke laid particular emphasis on the (supposed) removal of proof – for them this was clearly an important factor in the decline. Among the teachers present, the issue was less clear. Michael Davis (quoted above) doubted whether the proof that was previously taught had really been understood, except by a very few. Terry Heard (City of London School) said that proof was still present, there in the National Curriculum at level 8, but Roger Green (Head of Mathematics at Latymer Upper School) commented that

> proof turns up only in course-work. It doesn't need to be examined, so most teachers see no need to teach it. Moreover, since most course-work is not done by the students themselves anyway, the weakness of students in proof is not surprising.

Perhaps the National Curriculum itself cannot be blamed (or praised), since its formulation leaves room for widely differing interpretations[1]. Echoing points made earlier by Peter Saunders in his talk, and by Roger Green, Michael Jennings, (Head of Mathematics at the Royal Grammar School, Guildford), laid the blame for students' weakness in proof on the examining boards:

[1] See Appendix 1

> ...the problem is that examination questions are now overly structured. Teachers will teach to exams, and if you set structured questions in exams then the teachers will teach towards them, and will not ask the students to make the progression ... of putting two things together. I think the main problem is at GCSE and A-level. I also set A-level questions, and if I set a questions where there is not enough structure, I get shouted down and told that it is too difficult. The only way you will change teachers' attitudes is to change the examinations, because teachers do teach to examinations.

Funding
A second factor in the current difficulties is the generally depressing issue of financial constraints. This lurked in the background in the discussion but was not dealt with at length.

Christine Crisp, Head of Mathematics at Shrewsbury Sixth Form College, reported that Further Education Funding Council policies discourage colleges from allowing students to take anything more than three A-levels. Each of the first three A-levels attracts 56 units of funding per student, whereas, for example, an extra AS level attracts only 10 units of funding per student, instead of the 28 one would expect for a course equivalent to half an A-level. Since for obvious reasons AS-level mathematics is usually taught only to students who are doing three A-levels, colleges have a financial incentive not to teach it, except in groups 2.8 times the size of their A-level groups. In response to Christine Crisp's complaint, Geoffrey Howson said:

> There is a remark in the LMS report about this problem ... The chairman and head of finances of the FEFC both descended on me and said this is nonsense. ... It would help if the people here communicated these problems to the FEFC, because they might take your word for it better than mine, because I'm only saying things second hand.

Calculators and computers
A third factor often blamed for a decline in manipulative skills is the use of calculators. Chris Belsom's rather tentative support of this view attracted a vigorous rebuttal from Leone Burton:

> Chris said that he believes that calculators have had a detrimental effect on primary school mathematics performance. I don't know where that belief comes from, but I do know that there is a small but growing amount of research that is being done, some of it by people in this room, which does not bear it out, and I find very worrying that, despite this research, that belief is quite widespread. He also said that calculators have profound implications for teaching. They have even more profound implications for learning ... Our classrooms have in them people who have many different ways of understanding mathematics, but we don't pay attention to the heterogeneity of the learners. Neither do we pay attention to the heterogeneity in the comprehension of mathematics.
>
> The calculator is a very good example of how we fail to do so. Because as adults we feel that calculators are best for dealing with big numbers and for checking, we tend to bring these feelings into the classroom. Some of the worst use of calculators that you can see is around that kind of use. Whereas some of the best use of calculators and computers in the classroom involves using them as devices to explore one's own understanding of mathematics. If only we could acknowledge the research facts which are becoming more profound in this area, I would feel more comfortable with the kind of engagement we're having during the day.

The pizza on the hypotenuse
Peter Saunders, in his talk, placed some of the blame for the decline on the insistence on teaching

mathematics 'in context'. Rosamund Sutherland, of the Department of Education of Bristol University, and the author of a report on school mathematics commissioned by the Engineering Council, felt that he had overstated the need to develop mathematics as an abstract discipline:

> ... where you got too carried away was with the idea that mathematics is a universal generalising tool which will allow you to see the abstract mathematics in everything around you ... There is another problem there, and I think that in some ways schools have been trying to deal with that other problem, and that's where things have got muddled up.

Ian Stewart commented:

> The problem is if you go too far in embedding mathematics into the real world. The point comes where the students start thinking about the problem in terms of the model instead of in terms of the mathematics they derive from the model. So when they're thinking about a problem involving cutting a pizza into fractions, they're not thinking about fractions but about pizza, and maybe you'll burn your fingers when you cut it. I think Peter is right, that the emphasis on context has gone much too far.

What is to be done?
Beyond the suggestion, implicit in many of the comments made, to undo the changes that have contributed to the decline, two further approaches to this question were raised in discussion. The first is to consider the kind of political or institutional action that might lead to the necessary changes in school mathematics. This was touched on by Hugh Burkhardt, of the Shell Centre for Mathematical Education of the University of Nottingham:

> In the US the National Academy of Science has created the Mathematical Science Education Board which it has structured to have representatives of all of the major constituencies, including professional mathematicians and scientists, politics and commerce, and heavily the teaching profession and those who are in mathematics education. What it has sought to do is to maintain a review of the situation in schools and in undergraduate work, on behalf of the mathematical community.... I do believe that such a body could be constituted here and could make a very great contribution.

The second was to consider ways in which universities might modify their teaching to take account of changing circumstances. In the discussion following Chris Belsom's talk, Tony Gardiner made the following suggestion:

> The LMS report did try to raise the idea that university mathematics departments should be encouraged to lay on mathematics courses ... that encouraged you to think in a more reflective way about what you've learned, rather than dash on to something new; courses which might encourage those with an inclination possibly to go into teaching to think about the interest of the subject and of communicating it to others.
>
> There appears to be no mechanism for encouraging mathematics departments to do this other than their own altruism. ... Perhaps someone present - and I don't look in any particular direction - has some idea how to dangle carrots, in the sense of earmarked funding to mathematics departments, to make them think about this.

Hugh Burkhardt and Christine Shiu (Centre for Mathematics Education, University of Bristol) concurred, and reported with approval on courses of this kind already in existence at Nottingham and Southampton respectively.

Teachers, mathematics educationalists and university mathematicians at it again
Overall, one of the most striking features of the seminar was the polarisation between teachers, mathematics educationalists and university mathematicians. This situation is typified by one of the last comments made at the conference. Tony Gardiner, a university mathematician, acknowledged that mathematics educationalists had a useful role to play. However, he suggested that some of their work was not very helpful to teachers:

> I suspect that those in this room would work more strongly together in tackling this problem if we were able to recognise a simple distinction - between concentrating on teaching and concentrating on learning. If you look at learning, it's infinitely complex and intriguing, and you can spend a whole lifetime trying to understand the complex picture of the brain. If you're a teacher, on the other hand, you don't have the space to indulge yourself in what is going on in little Bernadette's head or little Herminia's head, you want them to make progress. So we have here a science of how people learn, and a craft of teaching. The problem is that we often confuse these two. ... We need people who are interested in the complexities of how people learn, who will then reformulate them in words which help the craftsmen do their job better. The craft is different from the science, and we have to tease them apart. And until we do that, the advice that filters through into the National Curriculum, into advice to teachers and so on, will be thoroughly misleading, because a lot of it stems from the standpoint of the 'infinite complexity of the learning process' whereas what teachers need are simple things that work. What is called pedagogy in many countries, but which we don't have in this country.

As a university mathematician, my inclination is still to endorse this view. I don't know whether to the reader this report will serve merely to reinforce entrenched views. I hope it will at least alert them to the very real needs of the 'other sides'.

Concluding Comments

David Mond

This is not the place to attempt to synthesise, reconcile or second-guess the views offered in the main talks; and I am certainly not qualified to undertake such a task.

Instead, I offer four comments arising from the meeting itself, and from the task of editing the discussion. I make them from the viewpoint of a university mathematician; I can lay claim to no other.

The first is that a significant component of this problem is a lack of communication between schools and universities. As I see it, neither side shows much awareness of the other's aims or requirements. This state of affairs is nothing new; what has converted it into a problem are the changes in secondary education and in the demographics of higher education. Alarmed by the lower standards of many of their students, university mathematicians have discovered that they have no representation on the bodies responsible for these changes, nor indeed much idea of what these bodies are, nor who belongs to them. Whatever the nature of the problem, some kind of forum must be found where universities can give their views on the kind of mathematical preparation they feel A-level should provide; presumably in the process they would also learn something of the views of the other interested parties. The creation of a National Mathematical Council, along the lines suggested by Hugh Burkhardt in the discussion, is long overdue.

The same communication gap used to exist in the United States; it was bridged some ten years ago with the creation of the Mathematical Sciences Education Board in Washington DC. This body was largely self-elected, but had the backing of most major educational organisations, and took immediate steps to enlist the cooperation of other bodies with an interest in the mathematical education of the nation's children, and to appoint representatives from those bodies. Over the last decade the MSEB has initiated a wide range of changes to school and university curricula and teaching methods in the United States, developed pilot projects, and helped to turn them into successful nation-wide reforms. Interest in mathematics and science among American schoolchildren, for example, is now on the increase after years of decline.

Another avenue of communication between schools and universities was advocated by Chris Belsom: each university mathematics department should have a 'first year co-ordinator' whose job it was to find out what, exactly, incoming students actually could be assumed to know, and where the university courses should begin. The first year co-ordinator might profit by liaising with selected local A-level mathematics teachers.

The second comment concerns something that was skated round, but not addressed directly, in the discussion. In response to the lower level of knowledge of university entrants, most English and Welsh universities are lowering the level of the beginning of their mathematics courses. It is inevitable that the level of their final degree will also be lowered. Whether this is a problem for the nation is a moot point. A mathematics degree is not a professional training, and after leaving university the great majority of graduates use only a tiny fraction of what they have learned at university. However, to assume that therefore it doesn't really matter if mathematics students no longer take Topology 3 or Functional Analysis 4, is implicitly to call into question the belief that underpins all of higher education, that excellence, in whatever sphere of intellectual endeavour, is a social good. Universities often advertise a mathematics degree as a good preparation, by virtue of its training in rigorous abstract thought, for a wide range of future careers. If this is so, then might it not be that by reducing the amount of mathematics that students learn, the depth to which they penetrate, the intricacy of the body of knowledge they acquire, the value of this training will be diminished? Because of the indirectness of the contribution made to later professional life by a training in mathematics, it is harder to assess the consequences of dropping Topology 3 than, say, of dropping Cardio-Vascular Disease 3 from a medical training. But this does not mean that such a change is not a loss.

On the other hand, this is not to deny the value of the reappraisal of the structure of mathematics degrees currently taking place in many universities, prompted at least in part by the changes in A-level. Obviously, there is no point insisting that students should do what they manifestly cannot.

The third comment is that university mathematics degrees must find ways to encourage more graduates to go into teaching. In departments which tenaciously cling to former standards, it is highly likely that all but the very best students will end their studies with the feeling that mathematics is hard, confusing and not much fun. Graduates with this view of mathematics are not ideally prepared for teaching, and besides will only consider it as a last resort. Tony Gardiner's suggestion that mathematics degrees should contain a course or courses in which the students reflect on and consolidate the knowledge they have acquired, rather than striving for ever more, deserves the strongest possible support. Ironically, with this suggestion Tony is supporting precisely the view advanced earlier in the discussion by Barbara Jaworski and Leone Burton, although in a rather different context.

I'd like to raise a final strategic question: why did so few of the mathematics educationalists made any reference to the issues as laid out by Ian Stewart in his opening talk? Instead of discussing a temporary state of affairs perhaps owing more to recent government legislation, the structure of the National Curriculum or to current funding practices than to the psychology of learning, they concentrated on the latter. Why is there this focus on the eternal verities of the learning process? Why has there been so little research on the effects of the National Curriculum, and of the other changes in secondary education introduced over the last ten or fifteen years (such as the attempt, described by Geoffrey Howson, to 'increase yield')? Is it because ultimately the pressure on mathematics educationalists is to write and publish (preferably in international journals) on the psychology of learning, rather than on the less transcendent realities of the current British educational process?

Appendix 1

In this appendix, I have selected short passages from the National Curriculum which, I hope, will help to illuminate some of the issues raised in the discussion. Both are from the version of the National Curriculum superseded by the 1995 revised version; however, it is this one whose content and possible effects were under discussion at the meeting.

Proof in the National Curriculum

Here are levels 8, 9 and 10 of Attainment Target 1: Using and Applying Mathematics.

LEVEL 8

• devising and extending a mathematical task, making a detailed plan of the work; working methodically; checking information; considering whether results are of the right order.	a) Give logical accounts of work with reasons for choices made.	*Decide where to put a telephone box in the locality, giving reasons for their decisions.* *Consider the effects in various countries of the melting of polar ice-caps and consequent rise in sea-levels. Compare areas of land which would be covered by a rise of 50m or 100m, eg Britain and Bangladesh.* *Explain why they have chosen to limit their investigations of packaging shapes to those that are possible without having to use glue.*
• making statements of conjecture using 'if ... then ...'; defining, reasoning, proving and disproving, using counter-examples. • construct an extended chain or argument using 'if ... then ...' appropriately.	b) Understand the role of counter-examples in disproving generalisations or hypotheses.	*Having made the conjecture that the median is always smaller than the mean, try successfully to find an example in which this is not true.* *Having tried out several examples in order to investigate which is larger, 2n or n−2, and found 2n to be always greater, search for a counter-example to demonstrate that this is not always true.*

LEVEL 9

• designing, planning and carrying through a mathematical task to a successful conclusion. • stating whether a conjecture is true, false or not proven; defining and reasoning; proving and disproving; using symbolisation; recognising and using necessary and sufficient conditions.	a) Co-ordinate a number of features or variables in solving problems.	*Given the task of conducting a survey on what local people feel about a new planning proposal, using a sample of only 100 people, decide on the best way of selecting a representative sample.* *Examine the relationship between the number of grid-points on the perimeter and the number of interior grid-points for different shapes.*
	b) Justify their solutions to problems involving a number of features or variables.	*Justify their solutions to the problem of what size, and how many, chairs a cafe owner should order to equip a new cafe of specified shape, by referring to: the costs; the number of people likely to come in groups; the profit; and the undesirability of customers feeling cramped, or having to get up to let others past.* *When reporting on the best way to stock and transport pipes, explain solutions in terms of costing, volume, surface area and stability.*

LEVEL 10

• designing, planning and carrying through a mathematical task to a successful conclusion; presenting alternative solutions and justifying the selected route. • exploring, developing and using constructively an area of mathematics that is new to them.	a) Explore independently a new area of mathematics.	*Use books from the library in order to complete a project on matrices and transformations to present to the class.* *Write a booklet for other pupils about the mathematics involved in bell-ringing, in particular how group theory can be applied to change-ringing.* *Explore conditional probability.* *Explore relationships between trigonometric functions.*
• giving definitions which are sufficient and minimal. • using symbolisation with confidence; constructing a proof, including proof by contradiction.	b) Handle abstract concepts of proof and definition.	*Follow from a book Euclid's proof by contradiction that $\sqrt{2}$ is irrational, write out a similar proof for $\sqrt{3}$ and find why it breaks down for $\sqrt{4}$.* *Prove that the first differences in the sequence of square numbers generate the sequence of odd numbers.* *Find their own proof that the angle in a semi-circle is a right angle, and its converse, stating what prior results have been assumed.*

The National Curriculum and 'mathematics in context'

Here are two quotes from the National Curriculum; they are selected in order to show its somewhat inscrutable nature. First, in *Introduction to Mathematics in the National Curriculum, 2.4 (June 1989)* we find a classic, almost parodied statement of the 'pizza on the hypotenuse' approach to teaching mathematics:

> Learning skills, such as adding two numbers, calculating the area of a triangle or solving an equation, form a large part of pupils' work in school mathematics. Important though they are, such skills are only a means to an end, and should be taught and learned in a context that provides purpose and meaning. It is through, for example, handling money when shopping, analysing the results of a survey in geography or measuring fabric for a garment in design, that the importance of such skills is seen.

But four pages further on is a statement with which few mathematicians would quarrel: in *Classroom approaches to using and applying mathematics, 2.3 (June 1989)* we find

> Mathematics provides a way of viewing and making sense of the real world. It is also a means of creating new imaginative worlds to explore. An approach to mathematics which includes just those aspects which relate to knowledge, skills and understanding and their application to problems in the 'real' world, is deficient. It fails to provide pupils with insights into the unique character of mathematics, the opportunity it gives for intellectual excitement and an appreciation of the essential creativity of mathematics. Moreover, this aspect of mathematics which encourages pupils to explore and explain the structure, patterns and relationships within mathematics is an important factor in enabling them to recognise and utilise the power of mathematics in solving problems and to develop their own mathematical thinking.

Each of these two passages, quoted without the other, would have served in the discussion as convincing evidence for one position or another.

APPENDIX 2

How are changes made to syllabuses?

The following is a note from Chris Belsom in response to an enquiry from the rapporteur.

Examination boards have their own A-level syllabus, sometimes with one or two variants. All such syllabuses have been approved by SCAA (Schools Curriculum and Assessment Authority). The SCAA defines a *common core* of material which every A-level mathematics syllabus must contain; the remainder of the syllabus is decided by the examination board itself. Most, if not all, examination boards have representatives from schools and universities on each of their subject panels.

Changes to syllabus content or philosophy could be proposed to an examination board. Its committees would discuss the proposal. If it wished to modify its syllabus as a consequence, then it would make an appropriate proposal to SCAA, probably having first consulted its centres on the desirability of such a change. If the change is approved, then schools would be notified, and proper time would be allowed for its assimilation into the examination programme.

On the other hand, suggestions for modifications to the common core would presumably be best made directly to the SCAA.

An example of a more major curriculum change is provided by the development of 16-19 Mathematics by the School Mathematics Project (SMP). The following is a potted history of this development.

Concerned about provision in post-16 mathematics, SMP, under the chairmanship of Geoffrey Howson, launched a major national conference to discuss the issues. Interested teachers from schools and some (rather fewer) universities attended. Sufficient concern was expressed for the launch of a major development, with the aim of providing a new course in mathematics at A-level. Following national advertising, SMP appointed a full- time project director (Stan Dolan). Support services were set up, and funding was agreed. SMP then called a second conference to discuss issues raised and to launch the project. Working groups were established to write the material; most of the writers were practising teachers, though some were from the universities. An advisory group was appointed to work on general issues, putting the syllabus together, deciding on appropriate assessment procedures, etc. Various consultative meetings were held, with employers from industry and with universities, for example. SMP began negotiations with an examination board which would ultimately adopt the course and support the proposal to SCAA. In the case of 16-19 mathematics, the Oxford and Cambridge Examinations Board adopted the course for the first three years, and the Northern Examinations and Assessment Board (NEAB) thereafter. A pilot scheme was proposed and approved by SCAA for trials in 30 centres. For the next three years, materials and ideas were tested extensively in 30 schools and colleges across the country. Further development took place as a consequence of the trials, and after more rounds of testing, the finished material was published by Cambridge University Press. The full scheme was proposed to SCAA, and the examination board attended various meetings at SCAA which led to approval of the course and its assessment procedures. 16-19 Mathematics is now available nationally through the NEAB. Currently about 10,000-12,000 students take the course each year.

Further Reading

The changing mathematical background of undergraduate engineers, a review of the issues, by Dr. Rosamund Sutherland and Mr. Stefano Pozzi, The Engineering Council, available free of charge on receipt of an 85p stamped self-addressed A4 envelope, from the Engineering Council, 10 Maltravers St., London WC2R 3ER

Tackling the Mathematics Problem, London Mathematical Society, Institute of Mathematics and its Applications, Royal Statistical Society, October 1995, available free of charge on receipt of a 57p stamped self-addressed A4 envelope, from the London Mathematical Society, Burlington House, Piccadilly, London W1V 0NL

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Mr Howard Truelove	Mercers' Company
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GRESHAM

COLLEGE

Gresham College was established in 1597 under the Will of the Elizabethan financier Sir Thomas Gresham, who nominated the Corporation of the City of London and the Worshipful Company of Mercers to be his Trustees. They manage the Estate through the Joint Grand Gresham Committee. The College has been maintained in various forms since the foundation. The one continuing activity (excepting the period 1939-1945) has been the annual appointment of seven distinguished academics 'sufficiently learned to reade the lectures of divyntye, astronomy, musicke, and geometry' (appointed by the Corporation), 'meete to reade the lectures of lawe, phissicke, and rhethoricke', (appointed by the Mercers' Company). From the 16th century the Gresham Professors have given free public lectures in the City. A Mercers' School Memorial Chair of Commerce has been added to the seven 'ancient' Chairs.

The College was formally reconstituted as an independent foundation in 1984. The Governing Body, with nominations from the City Corporation, the Mercers' Company, the Gresham Professors and the City University, reports to the Joint Grand Gresham Committee. Its objectives are to sponsor innovative research and to supplement and complement existing facilities in higher education. It does not award degrees and diplomas, rather it is an active collaborator with institutions of higher education, learned societies and professional bodies.

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