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C O L L E G E

FIBONACCI'S FRACTIONAL FLOWERS

Mathematical Patterns in Seeds and Petals

A Lecture by

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We live in a universe of patterns. Human mind and culture have developed a formal system of thought for recognising, classifying, and exploiting those patterns: we call it mathematics. By using mathematics to organise and systematise our ideas about patterns, we have discovered a great secret: nature's patterns are not just there to be admired, they are vital clues to the rules that govern natural processes.

A very curious pattern indeed occurs in the petals of flowers. In nearly all flowers, the number of petals is one of the numbers that occur in the strange sequence 3, 5, 8, 13, 21, 34, 55, 89. For instance, lilies have three petals, buttercups have five, many delphiniums have eight, marigolds 13, asters 21, and most daisies have 34, 55, or 89. You don't find any other numbers anything like as often. Those numbers have a definite pattern, but one that takes a little digging out: each is obtained by adding the two previous ones together. For example 3+5 = 8, 5+8 = 13, and so on. The same numbers can be found in the spiral patterns of seeds in the head of a sunflower. This particular pattern was noticed many centuries ago, and has been widely studied ever since, but a really satisfactory explanation was not given until 1993.

Some features of the morphology of living creatures are genetic in origin, and some are a consequence of physics, chemistry, and the dynamics of growth. One way to tell the difference is that genetic influences can give pretty much anything you like, but physics, chemistry, and dynamics produce mathematical regularities.

The numbers that arise in plants — not just for petals but for all sorts of other features — display mathematical regularities. They form the beginning of the so-called Fibonacci series, in which each number is the sum of the two that precede it. Petals aren't the only places you find Fibonacci numbers, either. If you look at a giant sunflower you find a remarkable pattern of florets — tiny flowers that eventually become seeds — in its head. The florets are arranged in two intersecting families of spirals, one winding clockwise, the other counterclockwise. In some species the number of clockwise spirals is 34, and the number of counterclockwise spirals is 55. Both are Fibonacci numbers, occurring consecutively in the series. The precise numbers depend on the species of sunflower, but you often get 34 and 55, or 55 and 89, or even 89 and 144, the next Fibonacci number still. Pineapples have 8 rows of scales — the diamond-shaped markings — sloping to the left, and 13 sloping to the right.

Fibonacci invented his series in a problem about the growth of a population of rabbits, somewhere around 1202. It wasn't as realistic a model of rabbit population dynamics as the 'game of life' model that I've just discussed, but it was a very interesting piece of mathematics nevertheless because it was the first model of its kind, and because mathematicians find Fibonacci numbers fascinating and beautiful in their own right. The key question for this chapter is this: if genetics can choose to give a flower any number of petals it likes, or a pine cone any number of scales that it likes, why do we observe such a preponderance of Fibonacci numbers?

The answer, presumably, has to be that the numbers arise through some mechanism that is more mathematical than arbitrary genetic instructions. The most likely candidate is some kind of dynamic constraint on plant development, which naturally leads to Fibonacci numbers. Of course, appearances may be deceptive, it *could* be all in the genes. But if so, I'd like to know how the Fibonacci numbers got turned into DNA codes, and why it was those numbers. Maybe evolution started with the mathematical patterns that occurred naturally, and fine-tuned them by natural selection. I suspect a lot of that has happened, and it would explain why geneticists are convinced the patterns in living creatures are genetic and mathematicians keep insisting they are mathematical.

The arrangement of leaves, petals and the like in plants has a huge and distinguished literature. Early approaches are purely descriptive — they don't explain how the numbers relate to plant growth, they just sort out the geometry of the arrangements. The most dramatic insight yet comes from some very recent work of Stéphane Douady and Yves Couder, who devised a theory of the dynamics of plant growth and used computer models and laboratory experiments to show that it accounts for the Fibonacci pattern.

The basic idea is an old one. If you look at the tip of the shoot of a growing plant you can detect the bits and pieces from which all the main features of the plant — leaves, petals, sepals, florets, or whatever — develop. At the centre of the tip is a circular region of tissue with no special features, called the apex. Around the apex, one by one, tiny lumps form, called primordia. Each primordium migrates away from the apex — or more accurately the apex grows away from the lump, leaving it behind — and eventually the lump develops into a leaf, petal, or the like. Moreover, the general arrangement of those features is laid down right at the start, as the primordia form. So basically all you have to do is explain why you see spiral shapes and Fibonacci numbers in the primordia.

The first step is to appreciate that the spirals that are most apparent to the the human eye are not fundamental. The most important spiral is formed by considering the primordia in their order of appearance. Primordia that appear earlier migrate further, so you can deduce the order of appearance from the distance away from the apex. What you find is that successive primordia are spaced rather sparsely along a very tightly wound spiral, called the generative spiral. The human eye picks out the Fibonacci spirals because they are formed from primordia that appear near each other in space; but it is the sequence in time that really matters.

The essential quantitative feature is the *angle* between *successive* primordia, which are pretty much equal. Their common value is called the *divergence angle*. In other words, the primordia are equally spaced — angularly — along the generative spiral. Moreover, the divergence angle is usually very close to 137.5°, a fact first emphasised by the crystallographer Auguste Bravais and his brother Louis. To see why that number is significant, take two consecutive numbers in the Fibonacci series, for example 34 and 55. Now form the corresponding fraction 34/55 and multiply by 360° to get 222.5°. Since this is more than 180°, we should measure it in the opposite direction round the circle, or equivalently subtract it from 360°. The result is now 137.5°, the value observed by the Bravais brothers.

The ratio of consecutive Fibonacci numbers gets closer and closer to the number 0.618034. For instance, 34/55 = 0.6182 which is already quite close. The limiting value is exactly $(\sqrt{5}-1)/2$, the so-called 'golden number', often denoted by the Greek letter phi (φ). Nature has left a clue for mathematical detectives: the angle between successive primordia is the 'golden angle' of $360(1-\varphi)^{\circ} = 137.5^{\circ}$. In 1907 G.Van Iterson followed up this clue, and worked out what happens when you plot successive points on a tightly wound spiral separated by angles of 137.5°. Because of the way neighbouring points align, the human eye picks out two families of interpenetrating spirals — one winding clockwise and the other counterclockwise. And because of the relation between Fibonacci numbers and the golden number, the numbers of spirals in the two families are consecutive Fibonacci numbers. *Which* Fibonacci numbers depends on the tightness of the spiral. How does that explain the numbers of petals? Basically, you get one petal at the outer edge of each spiral in just one of the families.

At any rate, it all boils down to explaining why successive primordia are separated by the golden angle: then everything else follows.

Douady and Couder found a dynamic explanation for the golden angle. They built their ideas upon an important insight of H.Vogel, dating from 1979. He performed numerical experiments which strongly suggest that if successive primordia are placed along the generative spiral using the golden angle then they pack together most efficiently. For instance, suppose that instead of the golden angle you try a divergence angle of 90°, which divides 360° exactly. Then successive primordia are arranged along four radial lines forming a cross. In fact, if you use a divergence angle that is a rational multiple of 360° then you always get a system of radial lines. So there are gaps between the lines and the promordia don't pack efficiently. Conclusion: to fill the space efficiently you need a divergence angle that is an *irrational* multiple of 360° — a multiple by a number that is not an exact fraction.

Which irrational number? Numbers are either irrational or not, but — like equality in George Orwell's *Animal Farm* — some are more irrational than others. Number theorists have long known that the most irrational number is the golden number. It is 'badly approximable' by rational numbers, and if you quantify how badly, it's the worst of them all. Which, turing the argument on its head, means that a golden divergence angle should pack the primordia most closely. Vogel's computer experiments confirm this expectation, but do not prove it in full logical rigour.

The main new thing that Douady and Couder did was to obtain the golden angle as a *consequence* of simple dynamics, rather than postulate it directly on grounds of efficient packing. They assumed that successive elements of some kind — representing primordia — form at equally spaced intervals of time somewhere on the rim of a small circle, representing the apex; and that these elements then migrate radially at some specified initial velocity. In addition, they assume that the elements repel each other — like equal electric charges or magnets with the same polarity. This ensures that the radial motion keeps going, and that each new element appears as far as possible from its immediate predecessors. It's a good bet that such a system will satisfy Vogel's criterion of efficient packing, so you'd expect the golden angle to show up of its own accord. And so it does.

Douady and Couder performed an experiment — not with plants, but using a circular dish full of silicone oil placed in a vertical magnetic field. They let tiny drops of magnetic fluid fall at regular intervals of time into the centre of the dish. The drops were polarised by the magnetic field, and repelled each other. They were given a boost in the radial direction by making the magnetic field stronger at the edge of the dish than it is in the middle. The patterns that appeared depended on how big the intervals between drops were; but a very prevalent pattern was one in which successive drops lay on a spiral with divergence angle very close to the golden angle, giving a sunflower seed pattern of interlaced spirals. Douady and Couder also carried out computer calculations, with very similar results. By both methods they found that the divergence angle depends upon the interval between drops according to a complicated branching pattern of wiggly curves. Each section of a curve between successive wiggles corresponds to a particular pair of numbers of spirals. The main branch is very close to a divergence angle of 137.5°, and along it you find all possible pairs of consecutive Fibonacci numbers, one after the other in numerical sequence. The gaps between branches represent 'bifurcations' where the dynamics undergoes significant changes. The final step, making their analysis fully rigorous, was made by M.Kunz of the University of Lausanne.

I'm not suggesting that botany is *quite* as perfectly mathematical as this model. In particular in many plants the rate of appearance of primordia can speed up or slow down, and changes in morphology — whether a given primordium becomes a leaf or a petal, say — often accompany such variations. So maybe what the genes do is affect the timing of the appearance of the primordia. But plants don't need their genes to tell them how to space their primordia out: that's done by the dynamics. It's a partnership of physics and genetics, and you need both to understand what's happening.

Further Reading

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