Mathematics:

The next generation

Peter J. Cameron

LMS–Gresham College Lecture
14 May 2013

A mathematics lecture

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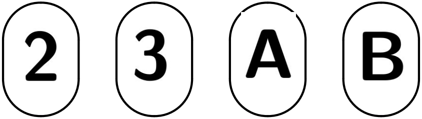
> The reader should expect to make use of pen and paper in many places; mathematics is not a spectator sport!
>
> Julian Havel, *Gamma: Exploring Euler's Constant*

A problem

Each card in a pack has a number on one side and a letter on the other. Four cards are placed on the table:



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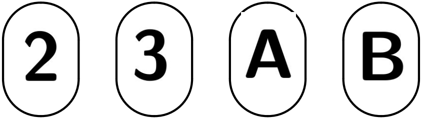

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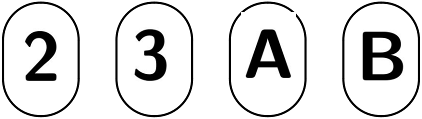

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> *A card which has an even number on one side has a vowel on the other.*

You are allowed to turn over two cards. Which cards should you turn?

- 2 and A;
- 2 and B;
- a different pair;
- it's not possible.

Mathematics is important!

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- Wiles' proof of Fermat's last theorem;
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We cannot tell which bit of mathematics will be important next; so it is vital that we produce able and enthusiastic mathematicians!

A good career

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From the Jobs Rated website (2009 data), out of 200 professions surveyed:

1. Mathematician			
Applies mathematical theories and formulas to teach or solve problems in a business, educational, or industrial climate.			
Overall Ranking: 1	Overall Score: 104	Work Environment: 89.720	
Physical Demands: 3.97	Stress: 24.670	Income: $94,160	Hours Per Week: 45

Why major in mathematics?

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One could say that mathematics is simply an exciting, fascinating, utterly satisfying, and rapidly expanding discipline. Many people are drawn to the level of challenge it presents, the creativity it requires, and the clarity it affords in knowing when you are right.

If you like solving puzzles and hunting for patterns and hidden structures – if you enjoy logical analysis, deduction, and investigating the unknown – if you want to understand the connections between seemingly widely different areas of science and technology, and how mathematics can be used to explain and control natural phenomena – then being a math major might be a good choice for you.

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[Emphases mine]

But how do we get from here . . .

OCR Specimen Exam Paper
Sketch the graph of $y = \cos x^\circ$, for values of x from 0 to 360.
Sketch, on the same diagram, the graph of $y = \cos(x - 60)^\circ$.
Use your diagram to solve the equation

$$\cos x^\circ = \cos(x - 60^\circ)$$

for values of x between 0 and 360. Indicate clearly on your diagram how the solutions relate to the graphs.
State how many values of x satisfying the equation

$$\cos(10x)^\circ = \cos(10x - 60)^\circ$$

lie beween 0 and 360. (You should explain your reasoning briefly, but no further detailed working or sketching is necessary.)

. . . to here?

Annals of Mathematics, **142** (1995), 443–551

Modular elliptic curves and Fermat's Last Theorem

By Andrew Wiles*

For Nada, Clare, Kate and Olivia

Cubum autem in duos cubos, aut quadratoquadratum in duos quadratoquadratos, et generaliter nullam in infinitum ultra quadratum potestatem in duos ejusdem nominis fas est dividere: cujus rei demonstrationem mirabilem sane detexi. Hanc marginis exiguitas non caperet.

Pierre de Fermat

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We are more interested in helping the students to *understand* what they are doing than to cram stuff into short-term memory for the exams.

On the other hand, there is so much mathematics to know; there are voices saying “You can’t call yourself a maths graduate unless you know about Lebesgue measure/finite simple groups/the Index Theorem/sheaves/…”.

Our response

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The hope was that it would teach mathematical thinking and understanding, and give the students good habits, which would transfer to their other modules.
I was given the job of producing and delivering this module. I would like to thank Thomas Prellberg, whose vision led to this, and whose unwavering support was crucial to its success.

The specification (extract)

- It should be a first-semester module, compulsory for mathematics students (including those on joint programmes), designed to introduce them to rigorous mathematical thinking and fundamental objects such as sets, functions, and numbers.

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I was nervous of putting "rigorous mathematical thinking" into a ghetto, hence the second clause.

What was in the course?

The ten chapters were as follows:

- Introduction
- Sets
- Infinity
- Functions and relations
- Natural numbers
- Integers and rational numbers
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- Proofs
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The intention was to keep the content minimal, to allow time for discussion of how to do mathematics.

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I collected interesting examples and problems from many colleagues and friends. Special thanks to Chris Budd!

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So I provided the students with a table of the Greek alphabet with the names of the letters written out, and for every new kind of formula, notes about how to read it.
There was also advice to students on how to write blackboard bold characters $\mathbb{N}$, $\mathbb{Z}$, etc., by hand in their notes (and why we use the particular letters we do), the difference between a/b and $a \mid b$, and so on.

Tutorials

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The tutorials were very successful (as I will discuss later).

What's it all about?

A new module has to have learning outcomes, key objectives, and all that (or whatever the jargon is now), and this one did. I was pleased to be able to get a quote from T. S. Eliot into the course description: "precise but not pedantic".

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The module has three main aims:

1. to introduce the basic objects of mathematics (numbers, sets and functions), and their properties;
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Of course the third is the most important!

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1. Question everything. [Don’t take anyone else’s word for it, or as the Royal Society has it, *Nullius in Verba*. It’s true for all scientists, but far more so for mathematicians.]
2. Write in sentences. [One of the commonest mistakes students make is to write a chain of formulae connected with $=$ or $\Leftrightarrow$ signs and expect to get full marks.]

Entering a mathematics department

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My aim was to build on this, not to tear everything up and start again with abstractions. This decision had several consequences.

Sets and functions

Philosophers have not found it easy to sort out sets . . .

D. M. Armstrong, *A Combinatorial Theory of Possibility*

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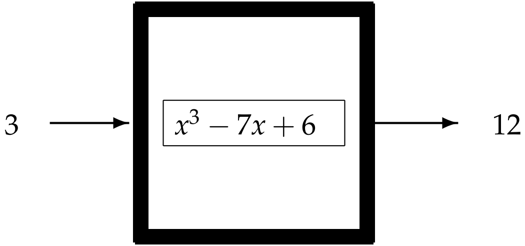

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> *You can, in principle, start at* 1 *and count up to any natural number.*

When my children were in primary school, there was a vogue for a rhyme that went:

> *One, two*
> *Missed a few,*
> *Ninety-nine,*
> *A hundred.*

Or, as a mathematician would say, $1, 2, \ldots, 99, 100$.

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The naive way to do this is to say: an integer is either 0 or a natural number with $+$ or $-$ in front of it. The trouble with this definition is that, to define addition of integers, you have to consider thirteen separate cases!
The mathematician's definition of an integer is an equivalence class of ordered pairs of natural numbers. The pair (a, b) represents the integer $b - a$. But we don't think of integers this way …

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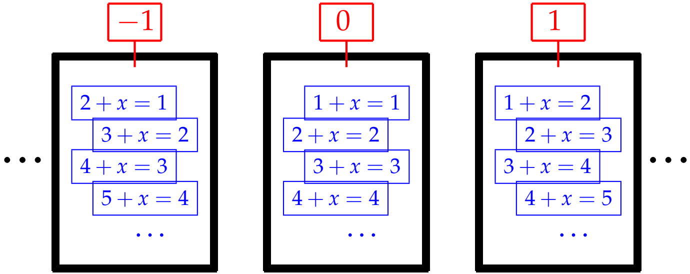

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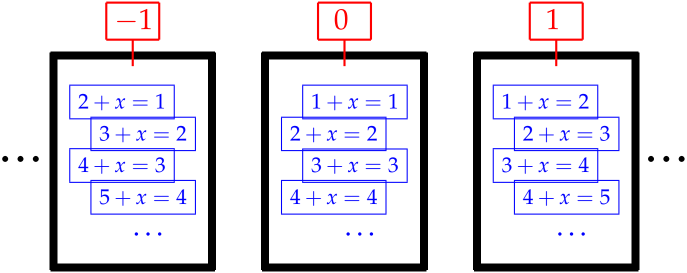

Now the equation $b + x = a$ is a way of giving meaning to the ordered pair (a, b), and we can define addition and multiplication without cases.

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It's easy to prove that the set of real numbers is uncountable, that every positive real number has a real square root, and so on.

There are some difficulties, which I simply skipped over. For example, consider

$$x = 0.386732054789\ldots + 0.613467945210\ldots$$

Is x greater than, equal to or less than 1?

Reasoning and logic

> Reasoning and logic are to each other as health is to medicine, or — better — as conduct is to morality. Reasoning refers to a gamut of natural thought processes in the everyday world. Logic is how we ought to think if objective truth is our goal — and the everyday world is very little concerned with objective truth. Logic is the science of the justification of conclusions we have reached by natural reasoning. My point here is that, for such natural reasoning to occur, consciousness is not necessary. The very reason we need logic at all is because most reasoning is not conscious at all.
>
> Julian Jaynes, *The Origin of Consciousness in the Breakdown of the Bicameral Mind*

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Suppose I say to you,

> *If it’s fine tomorrow, I’ll take you to the Zoo.*

The only situation where I have lied is if it is fine and I don’t take you to the Zoo. If it rains, my statement is not wrong, no matter what we do.

A problem revisited

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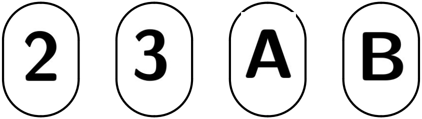

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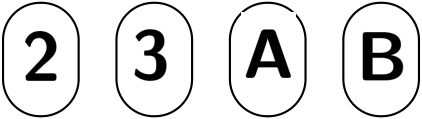

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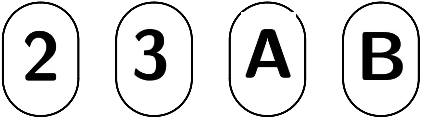

You have to test the following hypothesis:

> *A card which has an even number on one side has a vowel on the other.*

The hypothesis is false only if there is a card with an even number on one side and a consonant on the other. So we have to check cards 2 and B.

Proof by contradiction

> The proof [of the existence of an infinity of prime numbers] is by *reductio ad absurdum*, and *reductio ad absurdum*, which Euclid loved so much, is one of a mathematician's favourite weapons. It is a far finer gambit than any chess gambit: a chess player may offer the sacrifice of a pawn or even a piece, but a mathematician offers *the game*.
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>
> G. H. Hardy, *A Mathematician's Apology*

If the assumption that A is false leads us to an impossible or nonsensical situation, then we know that A must be true.

Euclid's proof

Theorem

There are infinitely many prime numbers.

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A proof is a convincing argument

When I gave this proof, I told the students that a proof is meant to be a convincing argument; if they were not convinced, I had not done my job, and they should ask questions. Many of them took me up on this, and continued to do so throughout the course.

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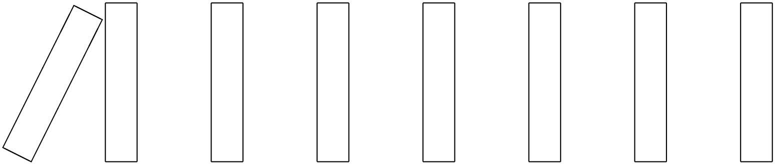
There is a weak spot in the proof. Why does N have to have a prime divisor at all? The fact that every natural number greater than 1 has a prime divisor needs to be proved, and the proof of this depends on the most important property of the natural numbers, Induction.

Induction

Suppose we have a long (potentially infinite) line of dominoes:



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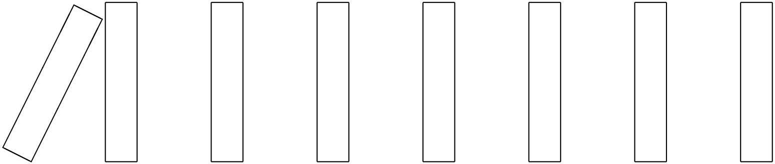

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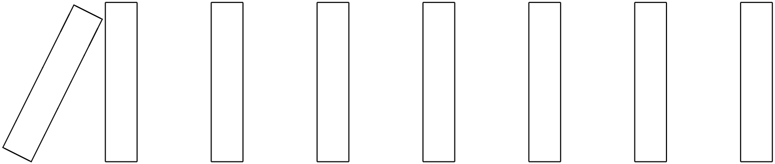

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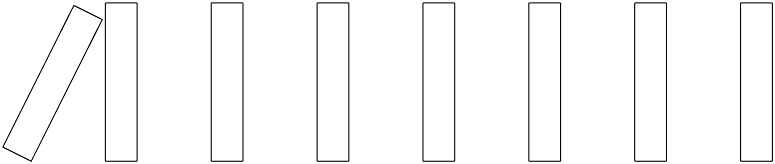

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This is the Principle of Induction.

More formally

Suppose that $P(n)$ is some statement about the natural number n. Suppose that

- *$P(1)$ is true;*
- *$P(n)$ implies $P(n+1)$, for any natural number n.*

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But compare the two statements, on the last slide and this one. The first is to be *understood*, the second to be learned and trotted out in an exam.

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The statement that every natural number greater than 1 has a prime factor (which we needed for Euclid's proof) can be proved by induction. But I won't inflict the proof on you now. (It is an "exercise for the reader".)

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This is a proof by induction. Let $P(n)$ be the statement:

> *In any set of n horses, all the horses have the same colour.*

We prove $P(n)$ by induction.

Proof.

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It follows that all of $H_1, \ldots, H_{n+1}$ have the same colour, and so the inductive step is complete, and with it the proof. □

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At the start of the semester, two of my colleagues had a bet, one claiming that the students would stop turning up at tutorials by the mid-term.
I am happy to report that the cynic lost the bet! Tutors were reporting full attendance, or emailed apologies in advance, at the end of the semester.

Tutors' comments

Typical comment:

Good engagement with the group in the tutorial. But reluctance to get involved - "can't see how to write it down for case "n" when they argued it perfectly for "n=4" re. no two people have the same number of friends. Have asked them to hand in the solutions we discussed for me to look at - hope this doesn't break too many rules? I had 3 of the 6 stand up and write on the board which wasn't bad. Generally nice problems to talk around in Ex 1 and 2. I enjoyed the tutorial.

Student comments: positive

- Different view of maths than I had previously encountered.
- Professor Cameron makes the lectures interesting by adding some history/philosophical thoughts in.
- Tutorials are very helpful, small classes enable a lot of learning to be done. Online notes are useful.
- Intervals at the appropriate times for questions.
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There were many positive comments about the small group tutorials.

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Students asked for notes to be put online in advance of the lectures. Rightly or wrongly, I do not do this. I believe that taking notes is an important part of learning and helps get the material into the students' brains.

An email

Dear Professor Cameron,
I'd just like to say a HUGE thank you! You made the transition from A-level to University much easier for me, you taught me to look at mathematics in a different way and I feel I am now able to approach this course in the way I should after learning Mathematical Structures. Every single lecture you gave was so intriguing and I didn't realise how much I had learnt from them until the mid-term exam.

Approaching the Midterms, Mathematical Structures was the one I was most worried about as it required a different approach compared to Calculus, Probability and Mathematical Computing, but after reading through my notes I realised that I actually knew most of the material and ended up getting 97/100 (which was my highest mark!).

I would just like to say thank you one more time, you have made me very confident in knowing that I can achieve my best at Queen Mary and in Mathematics as a whole and I think I can speak for everyone on the course when I say I hope you were able to stay and teach all of our modules! Hope you have a lovely time in Portugal and a great Christmas!

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So the course was a success!