# Gresham College Main logo

17 March 2015

**Cantor’s Infinities**

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**Slide: Title**

Thank you for coming to my lecture today. This is my last lecture of this academic year and the title is *Cantor’s infinities* and I am going to introduce you to some marvellous ideas about infinity that were developed by Georg Cantor.

It is sometimes said that because Cantor asked and then answered questions that had never been asked before that he created the first *truly* original mathematics since the Greeks. So a good topic to end this year’s lectures with – the first truly original mathematics since the Greeks!

**Slide: Georg Cantor and memorial**

Cantor was born in St Petersburg in 1845 and died towards the end of the First World War in 1918.

Let me give a little background to nineteenth century mathematics and how Cantor fitted in.

In the nineteenth century mathematics experienced a movement towards a progressively abstract style with an increased emphasis on putting itself on a sound and rigorous basis and on examining its foundations. This movement happened in the calculus, algebra and geometry.

In the 1820s the French mathematician Cauchy, the most prolific mathematician of the century, made a major advance in making the calculus rigorous by clarifying the concept of a limit. This idea of a limit is needed in the calculus, for example, where we have the ratio of two quantities and we want to see what happens to this ratio as both quantities move simultaneously towards zero so becoming infinitely small.

It was later in the century that the German mathematician, Weierstrass, gave a mathematically and logical solid definition of a limit and it is his definition of a limit that we use today and on which the calculus is founded.

However, as often happens, the resolution of one problem drew attention to another problem. It turned out that getting a sound definition of a limit necessitated a rigorous definition of the real numbers which in turn led to the study of infinite sets by Cantor.

This work arose from Cantor’s various investigations into the convergence of Fourier series and into whether if a function could be represented by a trigonometrical series was there only one such series?

This caused Cantor to start thinking about collections of numbers and the relation between the different kinds of numbers.

As a result Cantor created modern set theory. He established the importance of one-to one correspondences between sets and founded the theory of transfinite numbers, the mathematics of the size of sets of different sizes. He showed, in particular, that infinities can have different sizes.

As I have mentioned Cantor was born in St Petersburg and began his university studies at the Polytechnic in Zürich, moving after a year to the more prestigious Berlin University where he took his doctorate. In 1869 he became a lecturer at the University of Halle, being promoted to professor ten years later.

Although he had always hoped to obtain a position in Berlin, he remained in Halle for the rest of his life, teaching there for many years before succumbing to severe mental illness. He died in 1918 in Germany.

On the left of the slide is a bronze monument to Cantor in Halle-Neustadt. It is on one side of a cube commemorating four distinguished academics who worked there.

At the right of Cantor’s portrait the inscription reads;

Georg Cantor

mathematician

founder of set theory

1845 – 1918

Two other elements of the memorial across the centre are on the left one of his most famous formula and on the right a graphical presentation of Cantor's diagonal method. I will talk about both of these.

Finally at the bottom on the left, hard to see here, is a famous statement from Cantor

*The essence of mathematics lies in its freedom*

**Slide: Overview**

Let me now give you a brief overview of the lecture and introduce Cantor’s free and impressive thinking.

* Sets

Cantor introduced his theory of sets in a number of papers dating from 1874. For him, a set was:

*any collection into a whole of definite and separate objects of our intuition or of our thought.*

I will give some examples and introduce some notation.

* One-to-one correspondence

Next we will come to one of Cantor’s core ideas which is how to define when two sets have the same size. Two sets **A** and **B** are said to be *equivalent* — we can think of this as meaning that they have the same size — if we can match them up exactly; that is, there is a one-to-one correspondence between the elements of **A** and those of **B**. An example is the one-to-one correspondence between the set of digits of my left hand and the set of digits of my right hand. They pair up exactly and so there are as many digits on my left hand as on my right hand. So by using one-to-one correspondence I have shown that these two sets of digits have the same size and I have done it without counting the digits on each hand. In other words I have not needed to say that there are five digits on each hand and therefore they have the same size. There is a sense in which one-to-one correspondence is a more basic concept than counting.

If the sets **A** and **B** are finite then they must have the same number of elements, but if the sets are infinite then things become much more interesting!

We will also use one-to-one correspondence to give a definition of what it means for a set to be infinite.

Another of Cantor’s core ideas was the way he viewed the infinite. Cantor accepted the actual infinite by viewing an infinite set as a mathematical object which could be thought of as a *totality* and could be studied.

* Countable

Now we will come to our first example of an infinite set, the natural or counting numbers, 1, 2, 3, 4, 5, 6, …. We say that any set which is equivalent to the set of the natural numbers, i.e. can be put in one-to-one correspondence with it, is *countable*.

I will show you some surprising examples of countable sets.

* Uncountable

But are all infinite sets countable? The answer is no! The set of the real numbers is not countable – it is a bigger infinity than the set of natural numbers. It is said to be uncountable and Cantor’s proof of this is, I believe, one of the most beautiful in mathematics. We can also use the result to tell us about the size of the set of irrational numbers and the size of the set of transcendental numbers. I’ll define these terms.

* Infinite number of infinite sets of different sizes

When we reach this point in the lecture we will have met two different sizes of infinities – corresponding to the size of the set of natural numbers and the size of the set of real numbers.

What I want to show you is the tool or technique Cantor created to obtain sets of larger infinities. If you hand me an infinite set I can use Cantor’s technique to construct a new set with size a larger infinity. I can then use the technique on the new set to get another set with a larger infinity and so on getting a hierarchy of an infinite number of infinite sets of increasing sizes.

* Continuum hypothesis

The continuum hypothesis can be simply stated as whether or not there is an infinite set whose size is greater than that of the natural numbers but smaller than that of the real numbers. The answer is truly amazing!

* Reception of Cantor’s ideas

I’ll finish by briefly considering the reception of Cantor’s ideas which not surprisingly generated a lot of controversy.

**Slide: sets**

Cantor defined a set as:

*any collection into a whole* ***M*** *of definite and separate objects m of our intuition or of our thought*

Broadly speaking a *set* is a collection of objects. In mathematics these objects are usually numbers or points in space or even other sets.

There are three ways to denote sets.

We could just list all the members of the set, usually within curly brackets.

Example: {1, 3, 4, 6, 8}

However most sets we consider are too large to do this – indeed some of them are infinite – and then we can use dots to show a list that is too long to write out.

Example: {1, 2, 3, …, 66} represents the whole numbers from 1 to 66.

Or

{2, 4, 6, 8, …} represents the set of all even positive whole numbers.

The third way to represent a set is probably the most important and defines the set using a property.

Example: {*x*: *x* is an even positive integer} which we read as

*The set of x such that x is an even positive integer*

{*x*: *x* is a prime number less than a million} which is

*The set of x such that x is a prime number less than a million*

**Slide: One-to-one correspondence**

Cantor’s investigations led him to defining a means of deciding if two sets have the same size. He came up with the idea of one-to-one correspondence which is a pairing up of the elements of the two sets so that no element of either set is left unpaired.

Cantor put one-to-one correspondence like this:

*Two sets* ***M*** *and* ***N*** *are equivalent … if it is possible to put them, by some law, in such a relation to one another that to every element of each one of them corresponds one and only one element of the other.*

If **M** and **N** are *equivalent* we often say that they have they have the *same* *cardinality* or the *same power.*

As I have already mentioned the idea of one-to-one correspondence seems to be more basic than counting because using it we can decide if two sets are the same size without counting their number of elements. Indeed if **M** and **N** are finite sets then they can be put into one-to-one correspondence exactly when they have the same number of elements.

But what about the case when **M** and **N** are infinite?

Now Cantor was moving into unmapped and exciting new territory.

Let us see some of the consequences of Cantor’s two key premises – first that we can think of infinite sets as a totality and that this is a valid mathematical concept. Second, that the equal cardinality of sets can be established using one-to-one correspondence.

As we will see these two premises gave strange results.

**Slide: Countable**

Let

**N** = {1, 2, 3, …} the set of natural numbers

**E** = {2, 4, 6, …} the set of even natural numbers

Then by Cantor’s definition **N** and **E** can be easily seen to have the same cardinality since we can set up a one-to-one correspondence between their members.

**N**: 1 2 3 4 5 … n …

**E**: 2 4 6 8 10 … 2n …

This is a one-to-one correspondence between the natural numbers and the even natural numbers (we pair each natural number with its double) so according to Cantor these two infinite sets are of the same size. This might seem paradoxical since it seems that there should be half as many even numbers as whole numbers. But it is consistent with Cantor’s premises that we can regard the set of natural numbers as a totality and the beautifully simple definition of equal cardinality using one-to-one correspondence.

**Slide: Definition of infinity**

Indeed we will in a sense turn this paradox on its head and *define* an infinite set as one that can be put into one-to-one correspondence with a proper subset of itself. A *proper* subset does not contain all the elements of the set.

This example shows that **N** and **E** have the same cardinality. But we can show that **N** and the set of all integers have the same cardinality.

**Slide: Integers are countable**

Likewise if

**Z** = {… -3, -2, -1, 0, 1, 2, 3, …} the set of all integers

We can show that Z has the same cardinality as **N**. A one-to-one correspondence showing this is given as:

**N**: 1 2 3 4 5 6 7 8 9 …

**Z**: 0 1 -1 2 -2 3 -3 4 -4 …

This is a one-to-one correspondence but note that it does not preserve the natural order of the integers.

Cantor now made the definition:

**Slide: countably infinite or denumerable.**

Any set that can be into one-to-one correspondence with **N** is called *countably infinite* or *denumerable*.

So **E** and **Z** are countably infinite or countable, for short.

He chose a symbol to denote the size of a countable set. This symbol he chose was ℵ0 (read as aleph-nought or aleph-null) after the first letter of the Hebrew alphabet.

So Cantor’s study of infinite sets had led him to a new number indeed a new kind of number called a *transfinite cardinal*.

**N** is like a benchmark for these new numbers and we have shown

Cardinality of **E** = cardinality of **Z** = cardinality of **N =** ℵ0

To exploresome properties of aleph-null we will use an idea reportedly introduced by the German mathematician David Hilbert in a lecture in 1924. It is sometimes called Hilbert’s Grand Hotel.

**Slide: Hilbert’s Grand Hotel and one new arrival**

This hotel has a countably infinite number of rooms and it is full. Every room is occupied as we can see. But somebody arrives looking for a room and after a moment’s thought the manager, who as it happens is called Cantor, comes up with a solution.

**Slide: Hilbert’s Grand Hotel and one new arrival - solution**

He moves everybody up a room – the person in room 1 moves to room 2, the person in room 2 moves to room 3 and so on. Cantor then puts the new arrival in the now empty room 1.

Done!

And we note that adding a new member to a countable set still leaves it countable. Or 1 + ℵ0 = ℵ0

**Slide: Hilbert’s Grand Hotel and a finite number of new arrivals**

Next a bus arrives with 66 people looking for rooms. Cantor thinks easy – if I can solve it for one I can solve the problem for 66 new arrivals.

**Slide: Hilbert’s Grand Hotel and a finite number of new arrivals -solution**

* everybody moves up 66 rooms. So if they are in room n they move to room n + 66
* New arrival put in room 1
* Done!
* 66 + ℵ0 = ℵ0

Cantor of course realises that if he can solve the problem for 66 he can solve it for any finite number.

And we note that adding a finite number of members to a countable set still leaves it countable.

Cantor is just settling down for a well-earned coffee when another bus arrives but this time it has on board an infinite number, a countably infinite number of people!

**Slide: Hilbert’s Grand Hotel and an infinite number of new arrivals**

No problem – Cantor remembers an earlier slide in this lecture which showed the one-to-one correspondence:

**Slide: Hilbert’s Grand Hotel and an infinite number of new arrivals - solution**

**N**: 1 2 3 4 5 … n …

**E**: 2 4 6 8 10 … 2n …

between the set of natural numbers and the set of even natural numbers.

So Cantor asks for everybody to move to the room with number twice that of their current room. This leaves all the odd numbered rooms free and he uses them to accommodate the infinite number of people on the bus.

Done!

And we note that adding a countable set to a countable set still leaves it countable.

ℵ0 + ℵ0 = ℵ0

But the most challenging case is yet to come!

A countably infinite number of buses each with countably infinite passengers!

**Slide: Countably infinite number of buses each with countably infinite passengers.**

On the bottom right we see three of the infinite number of buses that arrive at the hotel. Cantor has an idea which can be thought of as rebuilding the hotel in a horizontal pyramid as shown here.

**Slide: Hilbert’s Grand Hotel in a horizontal pyramid**

You can see how the rooms are numbered – the first row only has room number one the second the next two rooms and so on for ever.

Cantor moves the existing occupants into the column formed by the rightmost rooms as shown here – shown in yellow. Their new room numbers are sometimes known as the triangular numbers.

**Slide: Hilbert’s Grand Hotel in a horizontal pyramid** **with right column**

The crucial thing to note is that when this is done the empty rooms form another pyramid. So move the members of the first bus into the right hand column of these remaining empty rooms as shown here in red.

**Slide: Hilbert’s Grand Hotel in a horizontal pyramid** **with first bus**

Now the remaining empty rooms again form a pyramid so move the members of the second bus into the right hand column of these remaining empty rooms as shown here in blue.

And so and so on!

So Cantor can allocate a room to each person on the countably infinite number of buses each with countably infinite passengers.

As an aside, Cantor does **not** have to rebuild the hotel he can use the plan of it as a pyramid to allocate all the members of the infinite number of buses to a room.

More importantly we can write what he has done as:

**Slide: ℵ0 times ℵ0 = ℵ0**

ℵ0 times ℵ0 = ℵ0

We have been using Hilbert’s hotel to introduce some of the strange and fascinating arithmetic of these infinite cardinals.

**Slide: But you might be saying to yourself**

But you might be saying to yourself:

*I see what might be going on – we can do this because these infinite sets are discrete, have gaps, and this is what allows the method to work because we can somehow interleave them and this is why we always end up with ℵ0.*

**Slide: Afraid not!**

Afraid not!

**Slide: Afraid not! Rationals**

Let us try to think of an infinite set that does not have gaps between its elements the way the integers have.

An example is the set of fractions or rationals. A rational number or fraction is any integer divided by any nonzero integer, for example, 5/4, 87/32, -567/981.

The rationals don’t have gaps in the sense that between any two rationals there is another rational. To see this choose any two rationals and form their average. This average is also a rational and lies between the two you first thought of. For example the average of ½ and ¼ is $^{3}/\_{8}$ which lies between ½ and 1/4 .

**Slide: The positive rationals are countable**

I’m going to show that the set of positive rationals or fractions is countable.

We first list all the positive fractions, as shown: the first row lists the integers, the second row lists the ‘halves’, and so on.

We then ‘snake around’ the diagonals of this array of numbers, deleting any numbers that we have seen before: this gives the list

1, 2, $ ^{1}/\_{2}$, $^{1}/\_{3}$, 3, 4, 1$^{1}/\_{2}$, $^{2}/\_{3}$, $^{1}/\_{4}$, $^{1}/\_{5}$, 5, 6, 2$^{1}/\_{2}$, …

This list contains all the positive fractions in other words provides a one-to-one correspondence between the positive rationals and the natural numbers.

To list all the fractions (positive, negative and zero) in order, we then alternate + and - , as we did with the integers.

So the set of all fractions is countable.

So, are all infinite sets countable?

No.

There is an infinite set which is uncountable - the real numbers and Cantor gave a marvellous proof of this result. In fact we will prove that the set of real numbers in the interval from 0 up to 1 is not countable.

**Slide: The reals**

Our approach uses *proof by contradiction*. We start by assuming the natural numbers and the set of real numbers in the interval from 0 up to 1 can be matched up in a one-to-one fashion – that is just what countable means – and then obtain a contradiction so that our original assumption is false.

Suppose this matching up is as below where the left hand column contains all the natural number and the right hand column all the real numbers in the interval from 0 up to 1 and the symbol ↔ just means matched up with so 1 is matched up with *x1* = 0.256173…

1 ↔ *x1* = 0.256173…

2 ↔ *x2* = 0.654321…

3 ↔ *x3* = 0.876241…

4 ↔ *x4* = 0.60000…

5 ↔ *x5* = 0.67678…

6 ↔ *x6* = 0.38751…

. . . .

. . . .

*n* ↔ *xn* = 0.a1a2a3a4a5 …an …

. . . .

. . . .

. . . .

Now remember that by assumption this list contains all the real numbers in the interval from 0 up to 1. We are however going to construct a number that is not in the list!

**Slide: construct the number *b***

Now construct the number which I will call *b* as follows:

Its first decimal place will be chosen different from the first decimal place of *x1*. The first decimal place of *x1* is 2 so I will choose 4 as the first decimal place of *b*.

So *b* will be different from *a1*.

Its second decimal place will be chosen different from the second decimal place of *x2*. The second decimal place of *x2* is 5 so I will choose 7 as the second decimal place of *b*.

So *b* will be different from *a2*.

Its third decimal place will be chosen different from the third decimal place of *x3*. The third decimal place of *x3* is 6 so I will choose 8 as the third decimal place of *b*.

So *b* will be different from *a3*.

Its fourth decimal place will be chosen different from the fourth decimal place of *x4*. The fourth decimal place of *x4* is 0 so I will choose 3 as the fourth decimal place of *b*.

So *b* will be different from *a4*.

Its fifth decimal place will be chosen different from the fifth decimal place of *x5*. The fifth decimal place of *x5* is 1 so I will choose 7 as the fifth decimal place of *b*.

So *b* will be different from *a5*.

And so on

Its *n*th decimal place will be chosen different from the *n*th decimal place of *xn*. The *n*th decimal place of *xn* is *an* so choose a number different from *an* as the *n*th decimal place of *b*.

So *b* will be different from *an*.

Now *b* = 0.*b1b2b3b4b5* … = 0.47837… is a clearly a number in the interval from 0 to 1 and so should appear in the list but by our construction it does not because it is different from every number in the list!

**Slide: *b* is not on the list**

This tells us that our original assumption that there is a one-to-one correspondence between the natural numbers and all the real numbers in the interval between 0 and 1 was wrong. There is no one-to-one correspondence between the natural numbers and all the real numbers in the interval between 0 and 1.

In other words the set of real numbers in the interval between 0 and 1 is not countable.

**Slide: cardinality of the reals is the same as that of the interval of the reals between 0 and 1**

We can show that the cardinality of the reals is the same as that of the interval of the reals between 0 and 1.

A one-to-one correspondence between these two sets is given, for example by the function

*y =* $\frac{2x-1}{x- x^{2}}$

This matches up every real number with a number between 0 and 1.

The cardinality of the reals is often denoted by *c* for the continuum of real numbers.

**Slide: Decimal expansion of a rational terminates or repeats**

Rationals can also be thought of as the collection of decimals which terminate or repeat e.g.

5/4 = 1.25000000 …

17/7 = 2.428571428571428571 …

-133/990 = - 0.134343434…

The decimal expansion of a fraction must terminate or repeat. This is because when you divide the bottom integer into the top one there are only a limited number of remainders you can get and so eventually you will get a remainder you’ve obtained before and then the results of the division will repeat itself e.g.

1/7 starts with

0.1 remainder 3 then

0.14 remainder 2 then

0.142 remainder 6 then

0.1428 remainder 4 then

0.14285 remainder 5 then

0.142857 remainder 1 which we have had **before** at the start so process **repeats**

1/7 **=** 0.**142857**142857**142857 …**

So a fraction gives a decimal which terminates (if remainder 0 is obtained) or repeats.

The converse is also true – a repeating decimal is a fraction e.g.

Consider *x* = 0.123123123123 …

This has a repeating block of length 3. Multiply by 103 to get

1000 *x* = 123.123123123 … Subtract *x*

 *x* = 0.123123123 …

 999*x* = 123

 *x =* 123/999 = 41/333

The *irrationals* are those real numbers which are *not* rational

So their decimal expansions do not terminate or repeat.

**Slide: Table without algebraic numbers**

Let me summarise in table form some of the results we have obtained.

|  |  |  |
| --- | --- | --- |
| Set | Description | Cardinality |
| Natural numbers | 1, 2, 3, 4, 5, … | ℵ0 |
| Integers | …, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, … | ℵ0 |
| Rational numbers or fractions | All the decimals which terminate or repeat | ℵ0 |
| Irrational numbers | All the decimals which do not terminate or repeat | *c* |
| Real numbers | All decimals | *c* |



The reason the irrationals are not countable is because if they were we could think of them as being on a bus arriving at Hilbert’s hotel which was occupied by the rationals. We have seen that we could fit them all in showing that their union is countable. But their union is all the reals which *is* uncountable so the irrationals must be uncountable.

So, in a sense, there are many more irrationals than rationals.

It is possible to show that the irrationals have indeed cardinality the same as the reals.

Now to show you an amazing result due to Cantor.

**Slide: Table with algebraic numbers**

|  |  |  |
| --- | --- | --- |
| Set | Description | Cardinality |
| Real numbers | All decimals | *c* |
| Algebraic numbers | All solutions of polynomial equations with integer coefficients. All rationals are algebraic as well as many irrationals $ e.g. √2.$ | **ℵ0** |
| Transcendental numbers | All reals which are not algebraic numbers e.g. *π* | ***c*** |

****

There is a set of numbers called the *algebraic numbers* which is the set of all numbers which are solutions of polynomial equations with integer coefficients.

All rationals are algebraic for example ¼ is the solution of 4*x* = 1, as well as many irrationals $ e.g. \sqrt{2}$ which is the solution of *x2* = 2.

But the algebraic numbers are still countable - it is like Hilbert’s hotel where a countable infinite number of buses each with a finite number of people arrive. They can all be accommodated!

Any number which is not algebraic is called *transcendental*.

In the same way as we did with the rationals and irrationals we deduce that the set of transcendentals is uncountable.

So they are much more numerous than the algebraic numbers. The amazing thing is that Cantor proved this in 1874 and by that time only a very few numbers had been proven to be transcendental and here was Cantor proving that most numbers were transcendental! It was only in 1882 that *π* was shown to be transcendental.

Cantor now has two types of infinity, two cardinalities, that of the set of natural numbers and that of the set of reals. Are there other cardinalities?

He looked to see if the points of the plane are more abundant than the points of the real line and showed that they were not!

As he wrote

*I see it but I do not believe it!*

He had to look elsewhere and the result is one of his most powerful. First I need to introduce a concept.

**Slide: Power set of a set**

Given a set A, the *power set* of A, denoted by P[A], is the set of all subsets of A.

A = {a, b, c}

Then A has eight = 23 subsets and the power set of A is the set containing these eight subsets.

P[A] = { { }, {a}, {b}, {c}, {a, b}, {a, c}, {b, c}, {a, b, c} }

{ } is the empty set.

The power set is itself a set.

If a set has *n* elements it has 2*n* subsets.

**Slide: No set can be placed in one-to-one correspondence with its power set**

The proof is again by contradiction. We assume a set, A, can be placed in one-to-one correspondence with its power set and obtain a contradiction.

Elements of **A** Elements of P[**A**]

 (i.e. subsets of **A**)

 *a {c, d}*

 *b {e}*

 *c {b, c, d, e}*

 *d { }*

 *e* **A**

 *f {a, c, e, g, …}*

 *g {b, k, m, …}*

 **. .**

 **. .**

 **. .**

Now we define the set **B** as the set of each and every element of the original set **A** that is not a member of the subset with which it is matched.

**B** = {*a, b, d, f, g, …}*

Now **B** is just a subset of **A** so must appear somewhere in the right-hand column and so is matched with some element of **A** say *z*.

Now for the fatal question!

Is *z* an element of **B**?

 **. .**

 **. .**

 **. .**

 *z* **B**

**. .**

 **. .**

 **. .**

Case 1: Suppose z is an element of **B**

Then *z* satisfies the defining property of B which is that it consists of elements which do not belong to their matching subset so *z* does not belong to **B**!

Case 2: Suppose z is not an element of **B**

Then *z* satisfies the defining property of B which is that it consists of elements which do not belong to their matching subset so *z* does belong to B!

In each case we have a contradiction so the only conclusion is that are our original assumption that the set **A** could be put in one-to-one correspondence with its power set is wrong.

**Slide: Infinity of infinities**

The Reals have smaller cardinality than the power set of the reals,

which is smaller than the power set of the power set of the reals,

which is smaller than the power set of the power set of the power set of the reals,

which is smaller than the power set of the power set of the power set of the reals

etc!

**Slide:** $ c =2^{ℵ\_{0}}$

Indeed we can show that the reals have the cardinality of the power set of the natural numbers which is often written as above and this is our last example of transfinite arithmetic and appears on Cantor’s monument.

**Slide: Continuum hypothesis**

The continuum hypothesis was a problem that occupied Cantor for years and he was never able to solve it, sometimes writing to colleagues that he had proved it true then shortly afterwards writing that he had proved it false! His difficulties were not surprising as we shall see.

The Continuum hypothesis states:

There is no transfinite cardinal falling ***strictly*** between ℵ0 and *c*

Any infinite subset of the real numbers is ***either*** countable, i.e. can be put in one-to-one correspondence with **N,** ***or*** can be put in one-to-one correspondence with the reals

Work of Gödel (1940) and of Cohen (1963) together implied that the continuum hypothesis was independent of the other axioms of set theory.

In 1940 Gödel showed that the continuum is consistent with the other axioms of set theory so cannot be disproved.

In 1963 Cohen showed that beginning with the axioms of set theory we could not prove the continuum hypothesis either. Together these results showed that the continuum hypothesis was independent of the other axioms of set theory.

**Slide: Reception of Cantor’s work**

Cantor’s work was very controversial because it raised really difficult questions and many powerful and established mathematical figures were critical of what he had achieved. But by the end of his career he had achieved recognition and prizes but there were problems with his set theory which he could not resolve, for example Russell’s paradox involving the set of all sets that were not members of themselves. Is it a member of itself or not?

Cantor’s approach was to regard such collections as *being too large* and not really sets at all or as he put it *inconsistent aggregates*. Other mathematicians began to axiomatise set theory to try to exclude the possibility of contradictions.

But this new set of questions that arose form Cantor’s work should not take away from his magnificent achievement of enabling us all to deal with the concept of the infinite in a careful, thoughtful and consistent manner.

Let me leave the final words to Cantor:

**Slide: Cantor quote**

My theory stands as firm as a rock; every arrow directed against it will return quickly to its archer. How do I know this? Because I have studied it from all sides for many years; because I have examined all objections which have ever been made against the infinite numbers; and above all because I have followed its roots, so to speak, to the first infallible cause of all created things.

 © Professor Raymond Flood, March 2015

Pleased to tell you that I’ve been appointed for a fourth and final year – let me tell you the dates for next year.

**Slide: Academic year 2015 –16: dates for your diary**

20 October 2015

17 November 2015

19 January 2016

16 February 2016

15 March 2016