GRESHAM COLLEGE

Founded 1597

Butterflies, Chaos and Fractals

Raymond Flood

Gresham Professor of Geometry

1 pm on Tuesdays at the Museum of London

Butterflies, Chaos and Fractals

Tuesday 17 September 2013

Public Key Cryptography: Secrecy in Public

Tuesday 22 October 2013

Symmetries and Groups

Tuesday 19 November 2013

Surfaces and Topology

Tuesday 21 January 2014

Probability and its Limits

Tuesday 18 February 2014

Modelling the Spread of Infectious Diseases

Tuesday 18 March 2014

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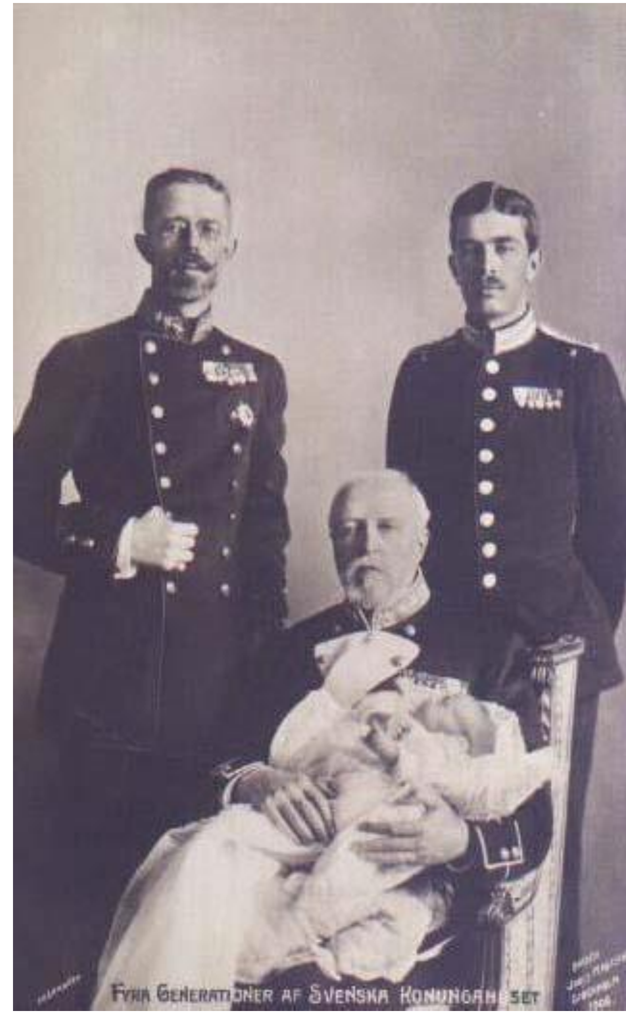
Overview

- Is the solar system stable?
- Dynamical systems
- Logistic equation
- Deterministic chaos
- Sensitivity to initial conditions
- Predictability horizon
- Lorenz attractor
- Mandelbrot set
- Fractals
- Predictability in real physical systems

Henri Poincare
1854–1912

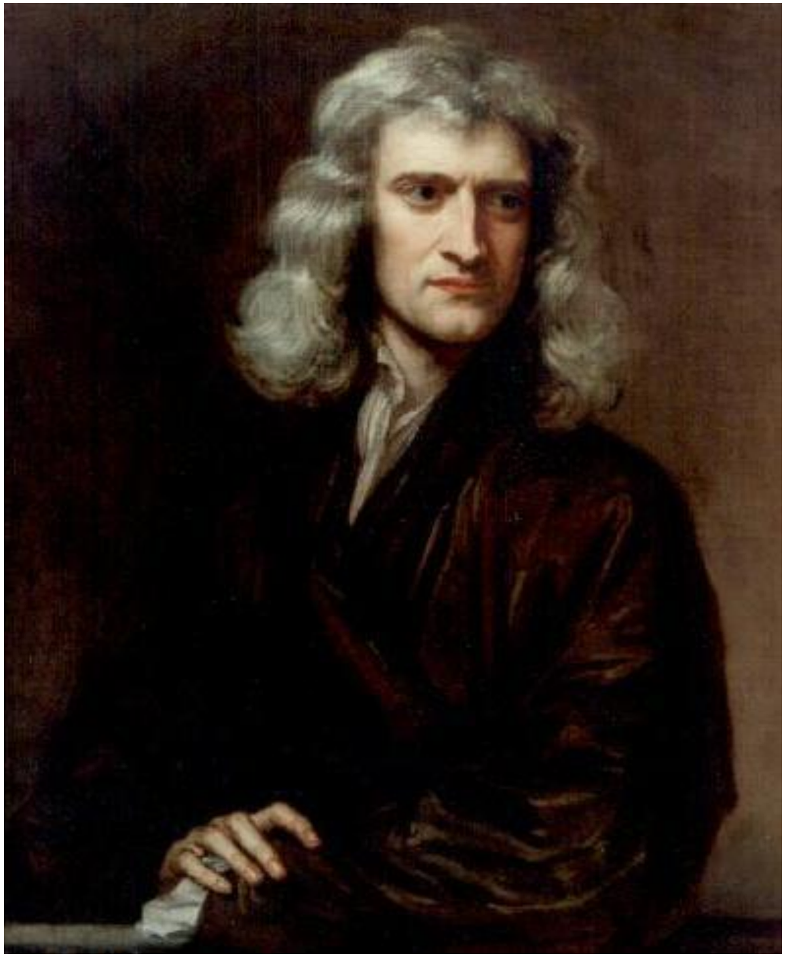
Is the Solar System Stable?

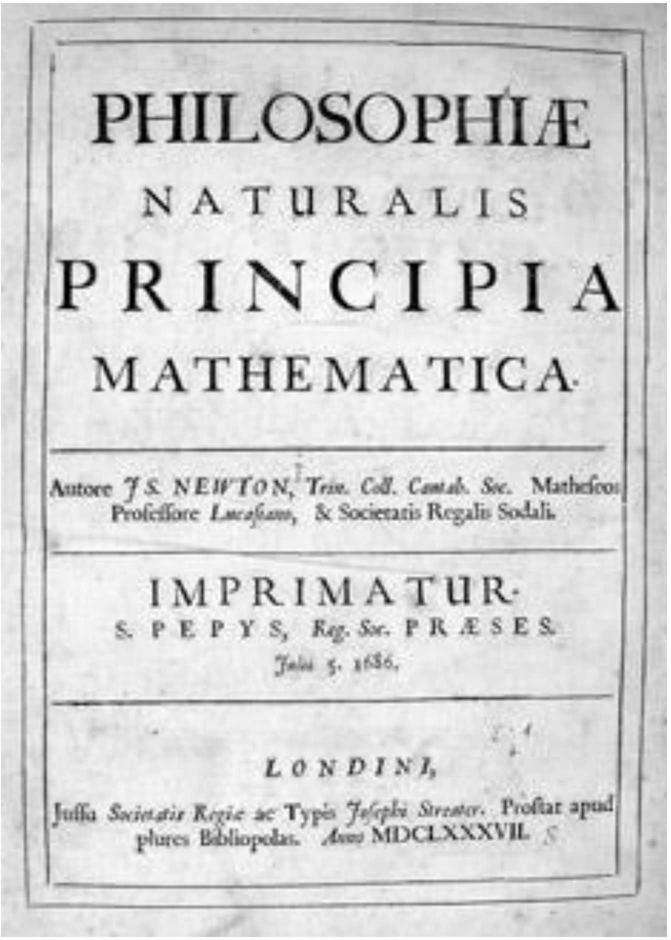
Given a system of arbitrarily many mass points that attract each according to Newton's law, under the assumption that no two points ever collide, try to find a representation of the coordinates of each point as a series in a variable that is some known function of time and for all of whose values the series converges uniformly



King Oscar II, his son Gustav, grandson Gustav-Adolf and great-grandson Prince Gustav-Adolf

Motion of two bodies under gravitational attraction





PHILOSOPHIÆ

NATURALIS

PRINCIPIA

MATHEMATICA.

Autore J S. NEWTON, Trin. Coll. Cantab. Soc. Matheseos Professore Lucasiano, & Societatis Regalis Sodali.

IMPRIMATUR.

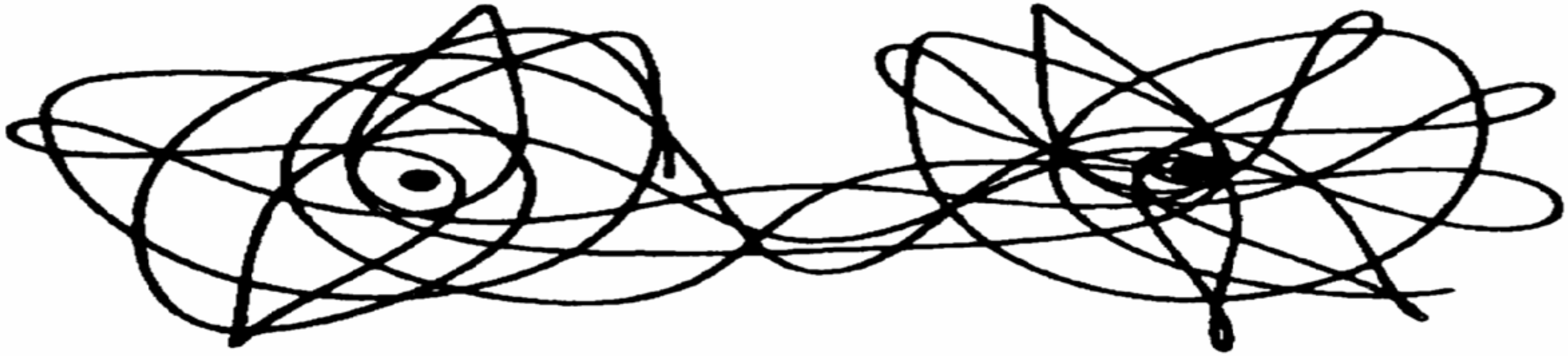
S. PEPYS, Reg. Soc. PRÆSES.

Julii 5. 1686.

LONDINI,

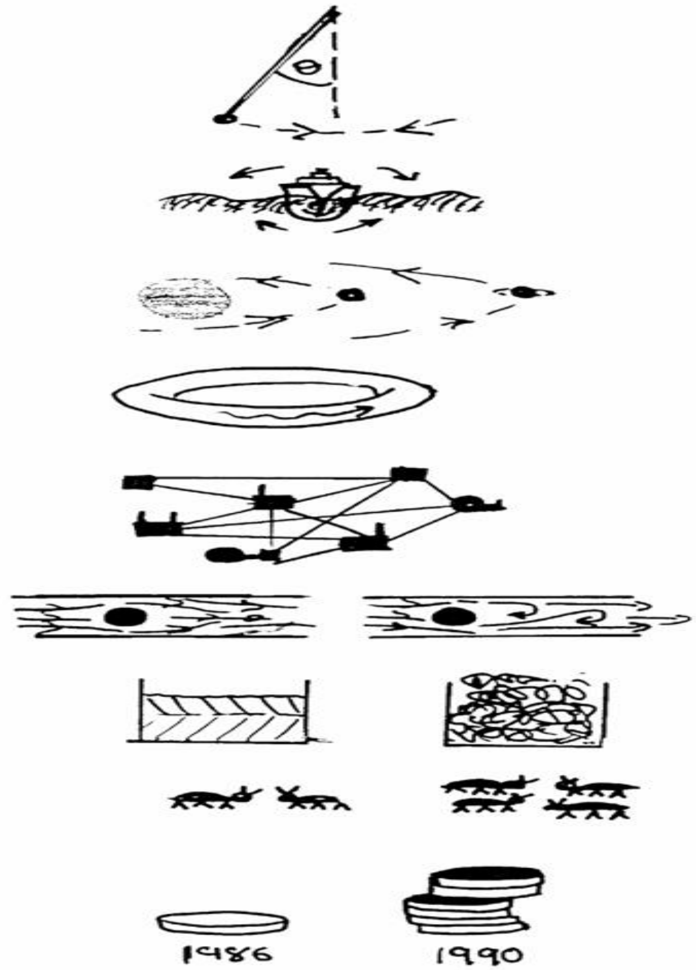
Jussu Societatis Regiæ ac Typis Josephi Streater. Prostat apud plures Bibliopolas. Anno MDCLXXXVII.

The complexities of three-body motion: here is a typical trajectories of a dust particle as it orbits two fixed planets of equal mass.



Examples of Dynamical systems

- Swinging pendulum
- Ship at sea
- Solar system
- Particle accelerator
- Power networks
- Fluid dynamics
- Chemical reactions
- Population dynamics
- Stockmarkets



Drawings: Robert Lambourne, Open University

Discrete Dynamical systems

A **discrete dynamical system** is one that evolves in jumps.

Example: the system could be the amount of money in a savings account at the start of each year and the underlying dynamic is to add the interest once a year

This could be modelled by a difference equation and written

$$S(n + 1) = S(n) + 0.1 \times S(n)$$

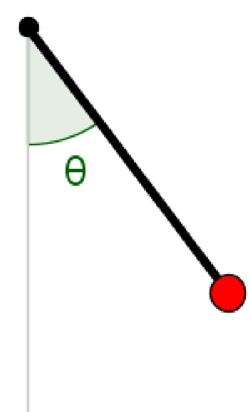
$S(n)$ means the amount of money in the account in year n. The number 0.1 is the interest rate.

Continuous Dynamical systems

This is where the state of the system varies continuously with time and is usually given by differential equations.

For example: for a swinging pendulum the angle of inclination, θ, of the angle of the string supporting the pendulum bob from the vertical is given by:

$$\frac{d^2\theta}{dt^2} + \frac{g}{\ell}\sin\theta = 0$$



Our dynamical systems are ***deterministic***

- for our savings account example, if we know the *exact* sum of money put into the bank at year 1 then this determines how much is in the account in all subsequent years.
- For the pendulum if we know *exactly* the angle at which we start of the motion then this determines the value of θ at all subsequent times.

Pierre Simon Laplace 1749 – 1827

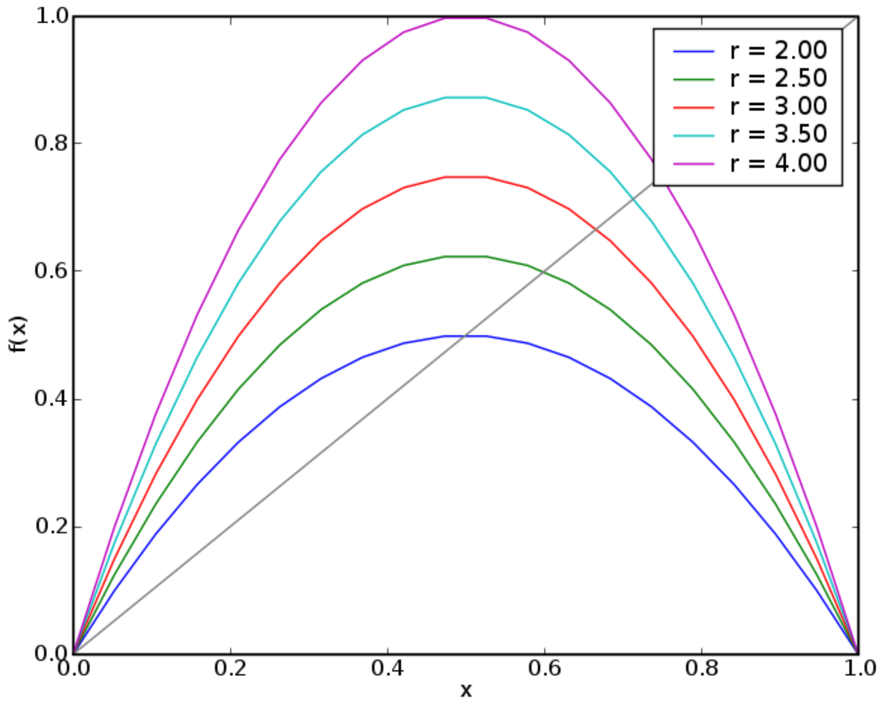
An intellect which at a certain moment would know all forces that set nature in motion, and all positions of all items of which nature is composed, if this intellect were also vast enough to submit these data to analysis, it would embrace in a single formula the movements of the greatest bodies of the universe and those of the tiniest atom; for such an intellect nothing would be uncertain and the future just like the past would be present before its eyes.

Logistic Difference Equation

x_n is the size of the population in the nth generation divided by the maximum sustainable population

$$x_{n+1} = r\,x_n\,(1 - x_n)$$

We take $0 \leq r \leq 4$



The graph of $f(x) = r\,x\,(1 - x)$
For different values of r

Logistic equation with r = 2 and starting at 0.1

The equation is:

$$x_{n+1} = 2x_n (1 - x_n)$$

When $x_1 = 0.1$ then $x_2 = 2 \times 0.1 \times (1 - 0.1)$

$= 0.18$

When $x_2 = 0.18$ then $x_3 = 2 \times 0.18 \times (1 - 0.18)$

$= 0.2952$

When $x_3 = 0.2952$ then $x_4 = 2 \times 0.2952 \times (1 - 0.2952)$

$= 0.4161$

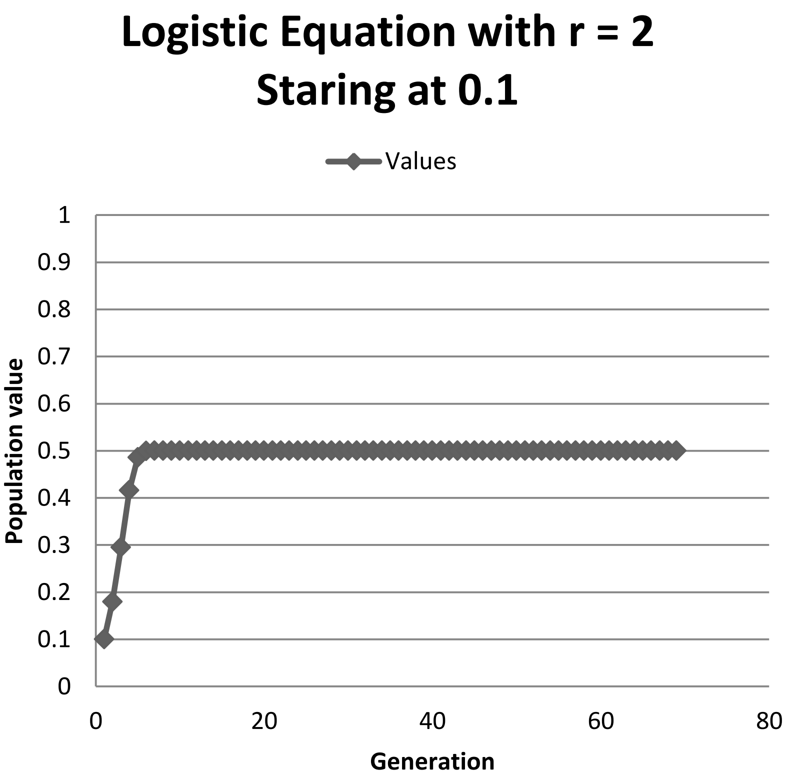
$x_4 = 0.4161$ *then* $x_5 = 0.4859$

$x_5 = 0.4859$ *then* $x_6 = 0.4996$

$x_6 = 0.4996$ *then* $x_7 = 0.4999$

$x_7 = 0.4999$ *then* $x_8 = 0.5$

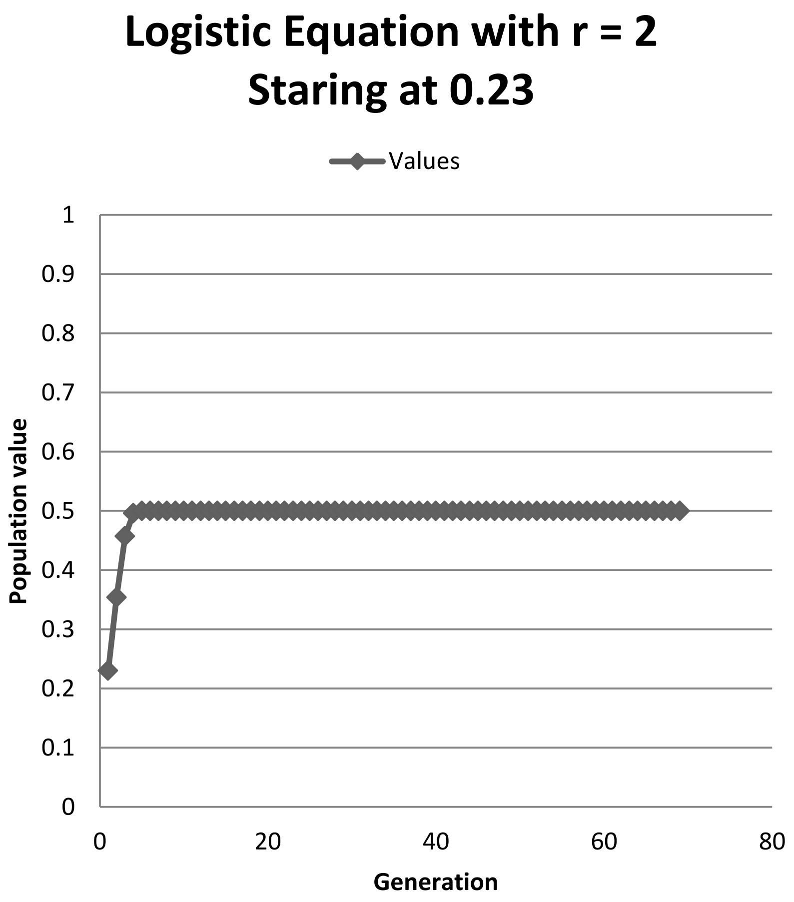
$x_8 = 0.5$ *then* $x_9 = 0.5$

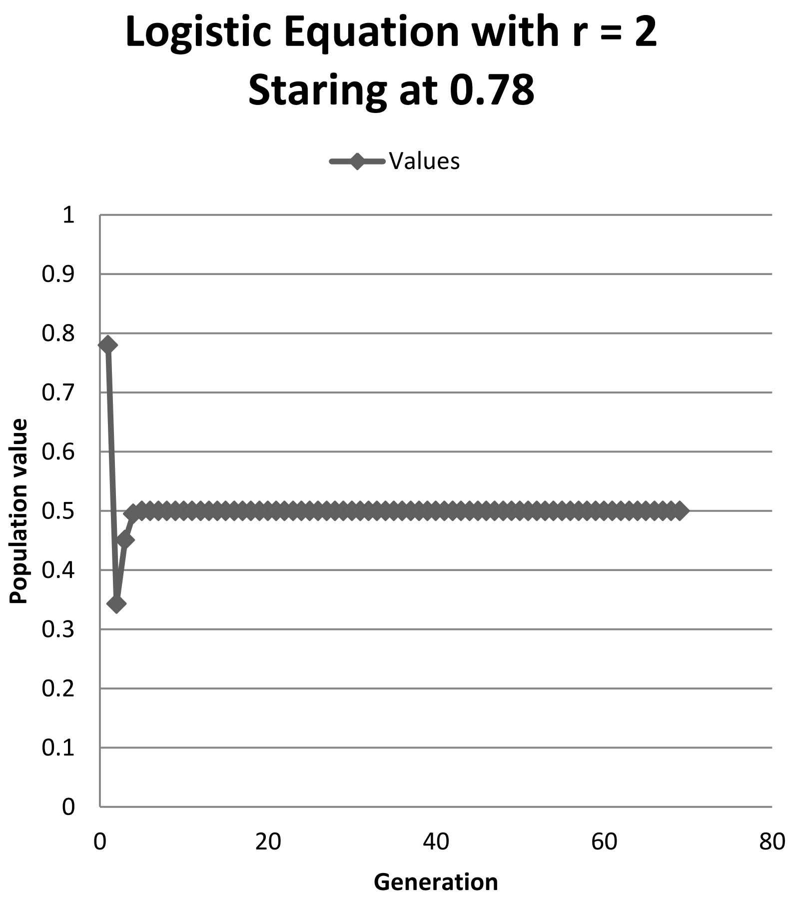


When r = 2 the value 0.5 is a fixed point since if $x_n = 0.5$ then

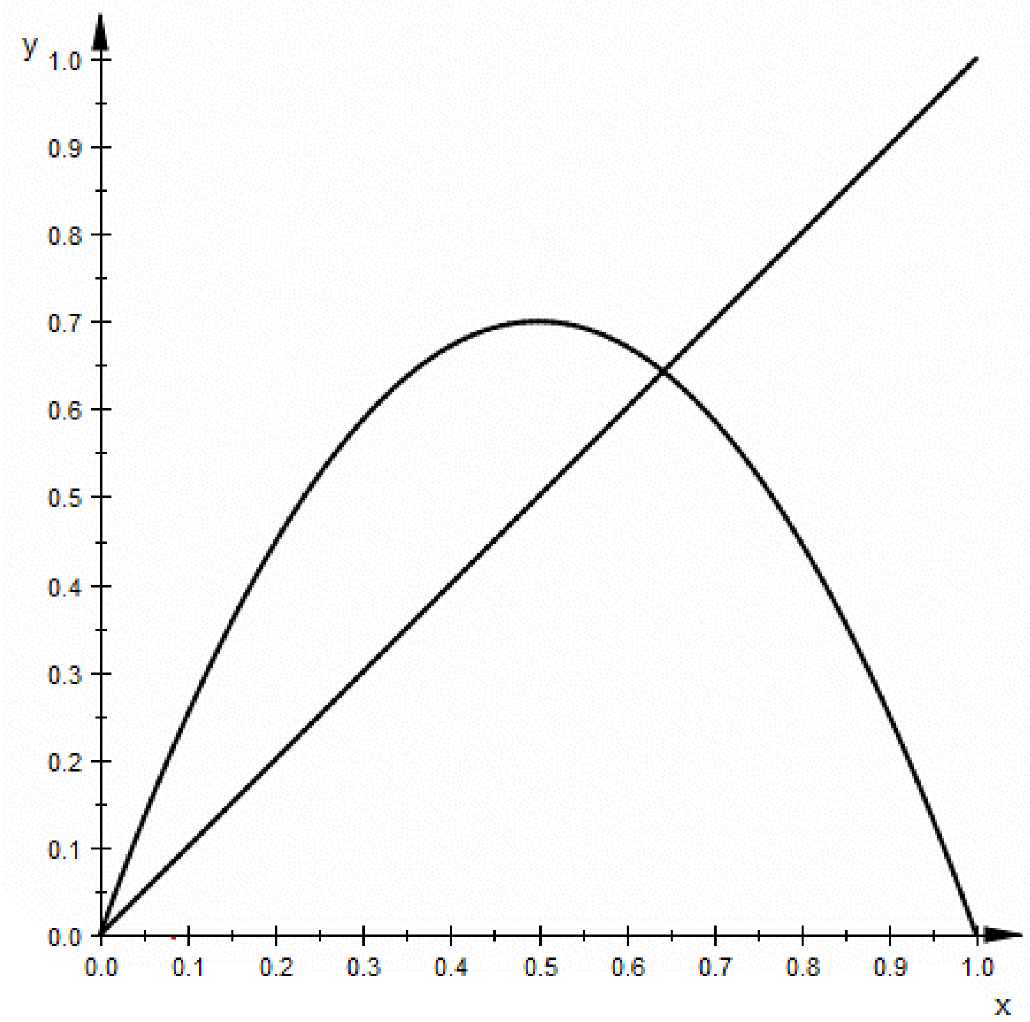
$$x_{n+1} = 2 \times 0.5 \times (1 - 0.5) = 0.5$$

But it also an ***attractor*** for the trajectories. No matter what our starting value we end at 0.5



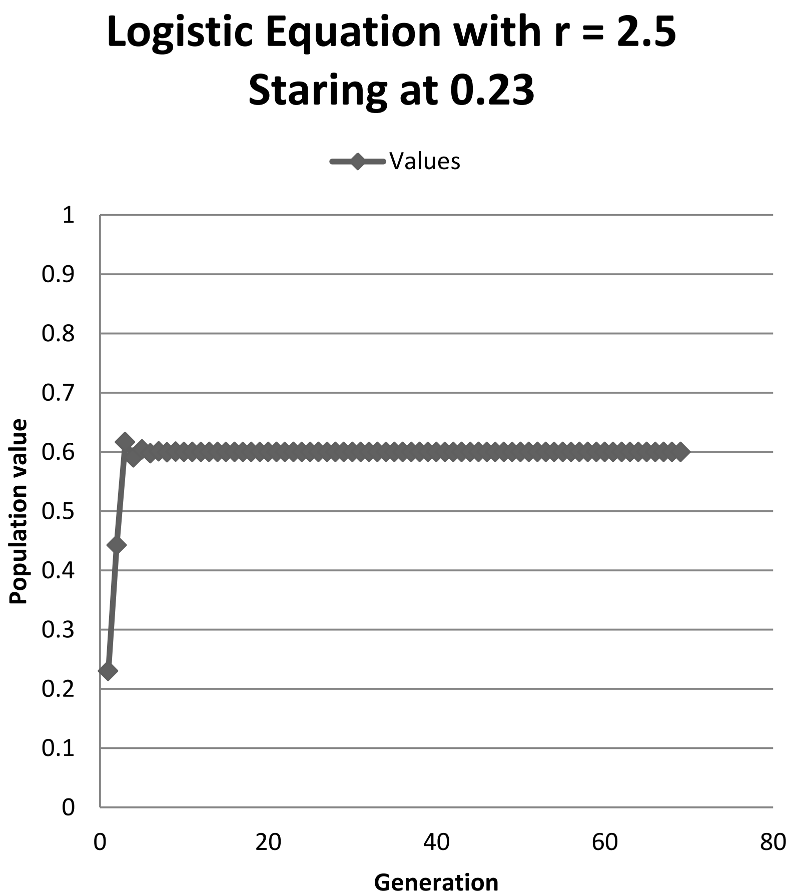


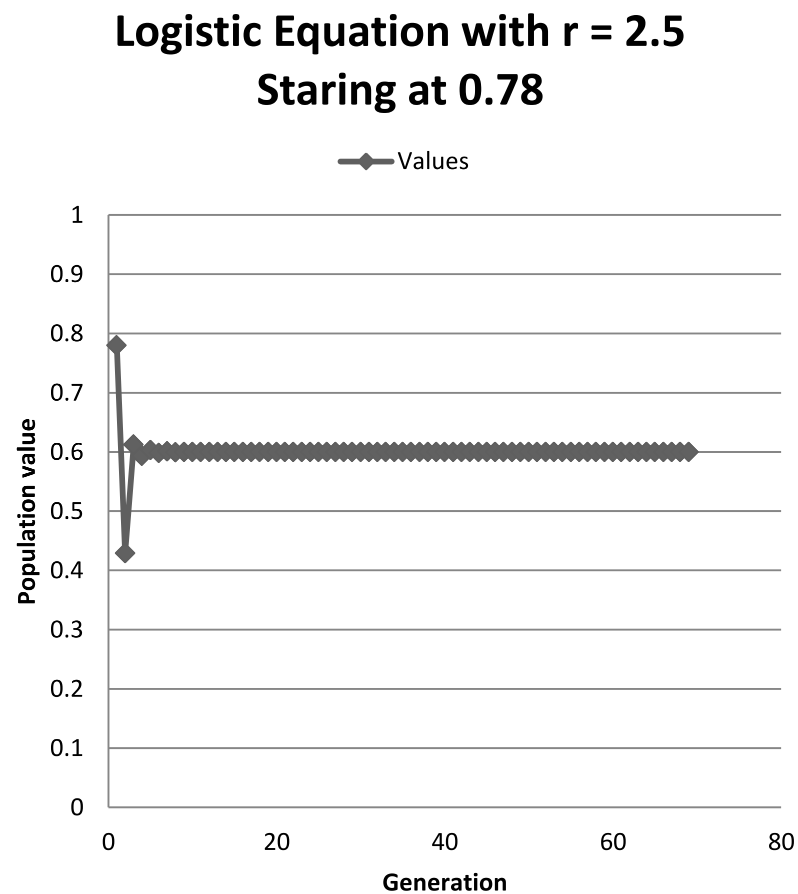
Cobweb construction
r = 2.8 start is 0.07



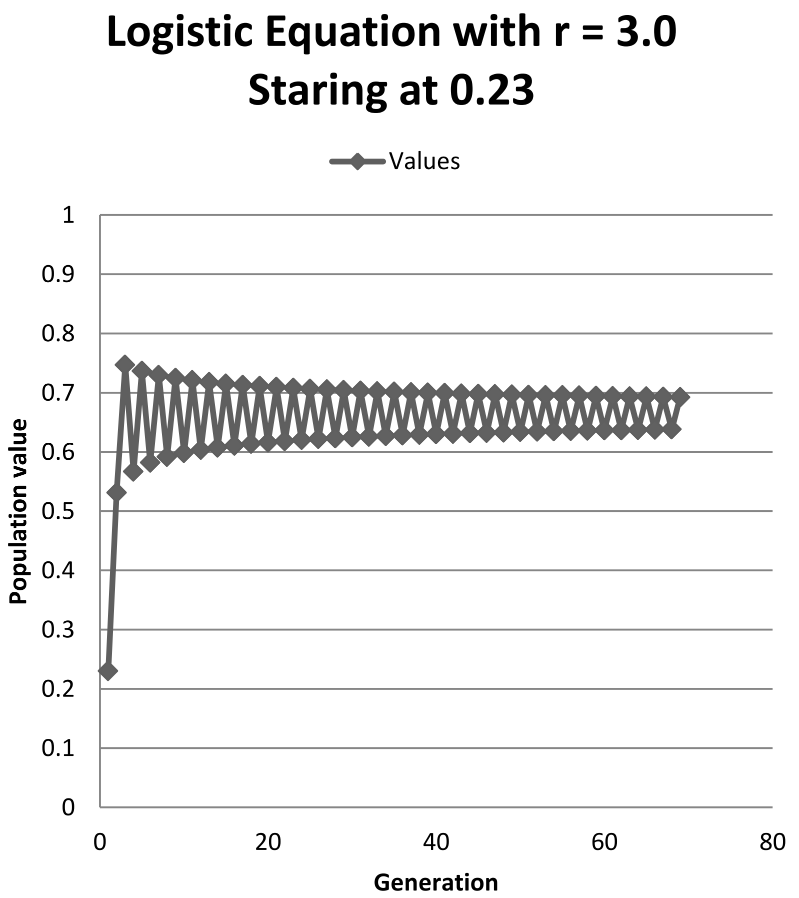
When $r = 2.5$ then 0.6 is the attractor.

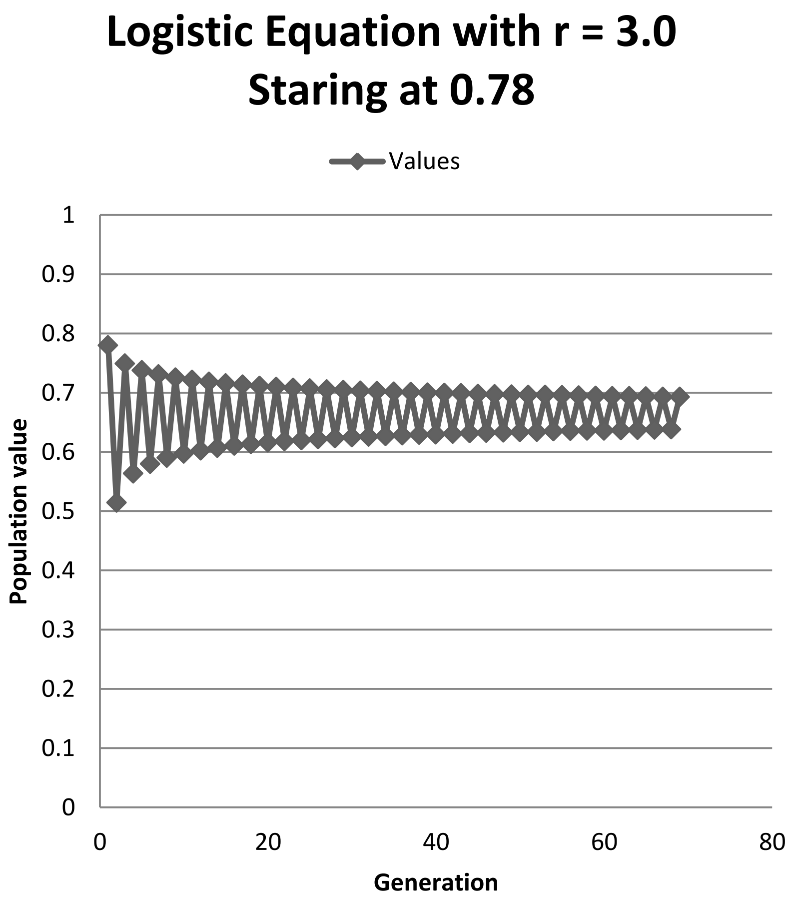
If $x_n = 0.6$ then $x_{n+1} = 2.5 \times 0.6 \times (1 - 0.6) = 0.6$





When r = 3.0
the attractor is now a pair of values and the system oscillates between them.

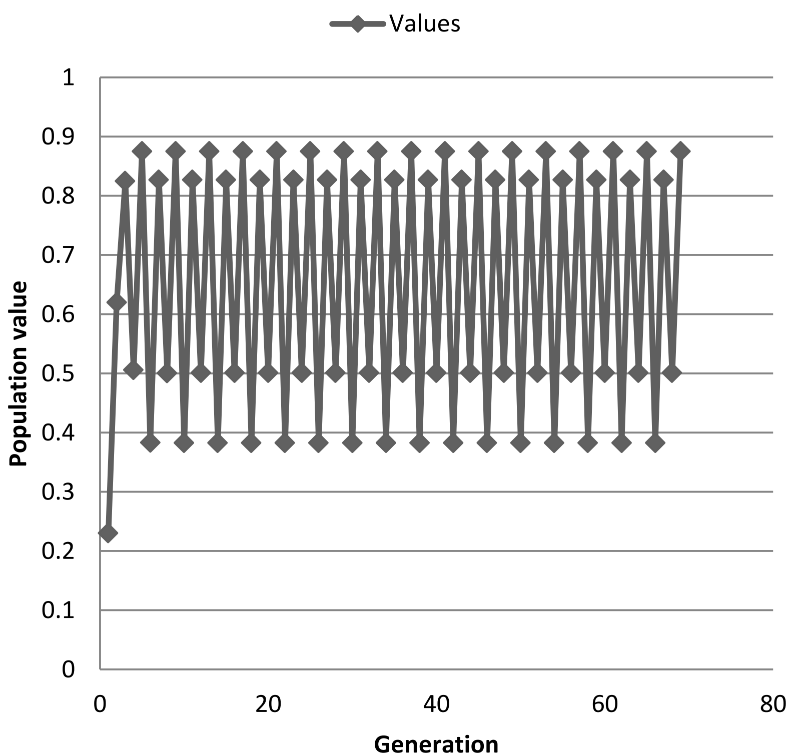




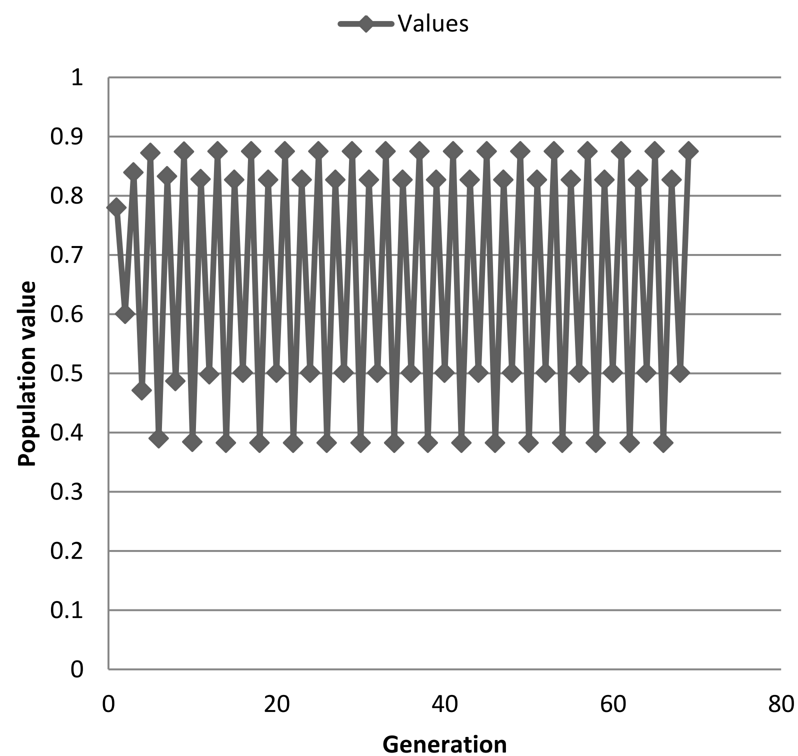
When r = 3.5

the attractor is now a set of four values and the system oscillates between them.

Logistic Equation with r = 3.5
Staring at 0.23



Logistic Equation with r = 3.0
Staring at 0.78



Going from order to chaos

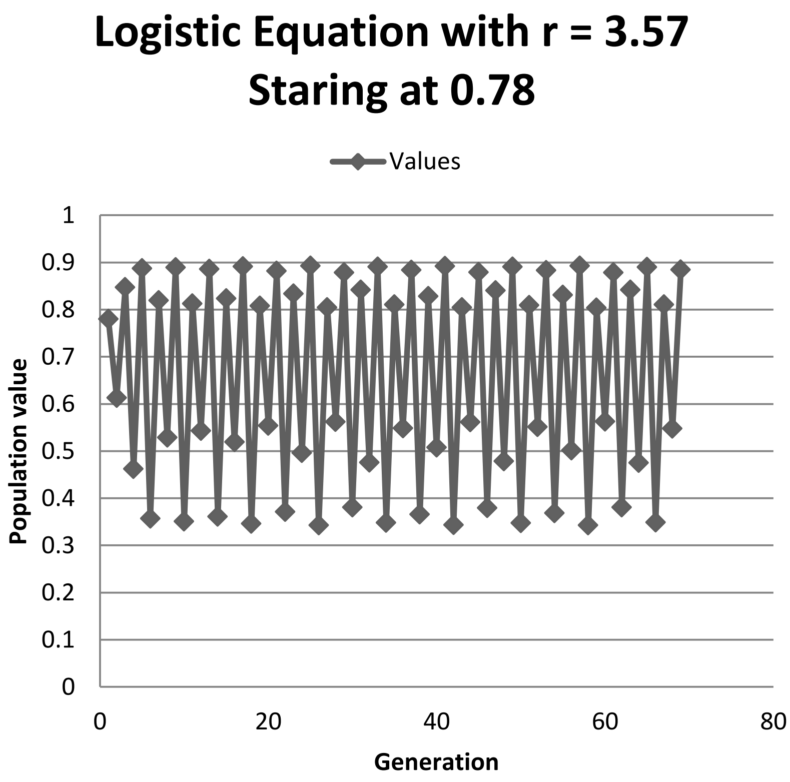
For r from 0 to 3 there is a *point* attractor.

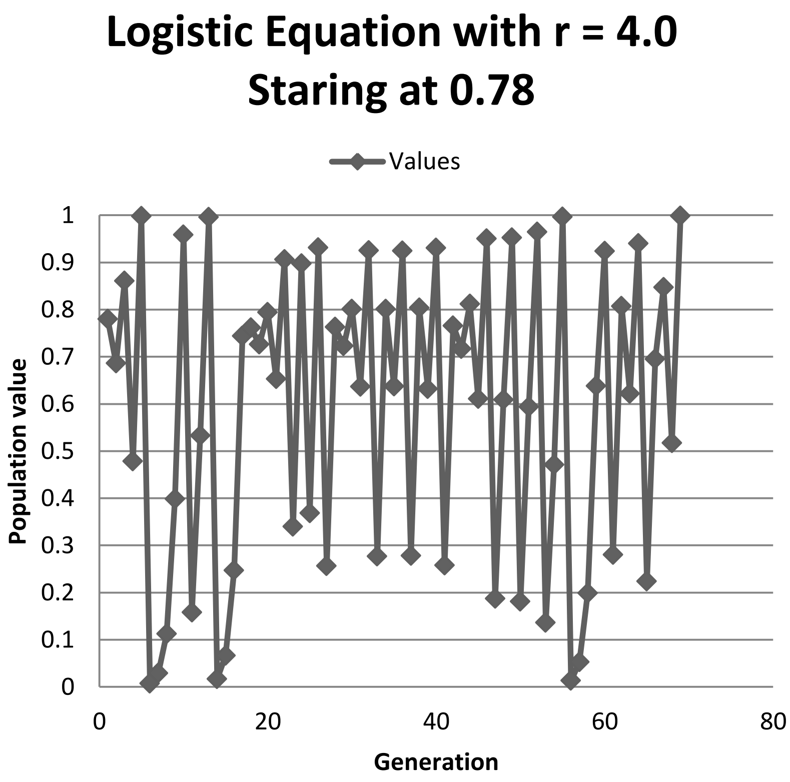
For r from 3 to $1 + \sqrt{6} = 3.449$ the attractor is of *period 2*

For r slightly above that the period doubles and the attractor is of *period 4*.

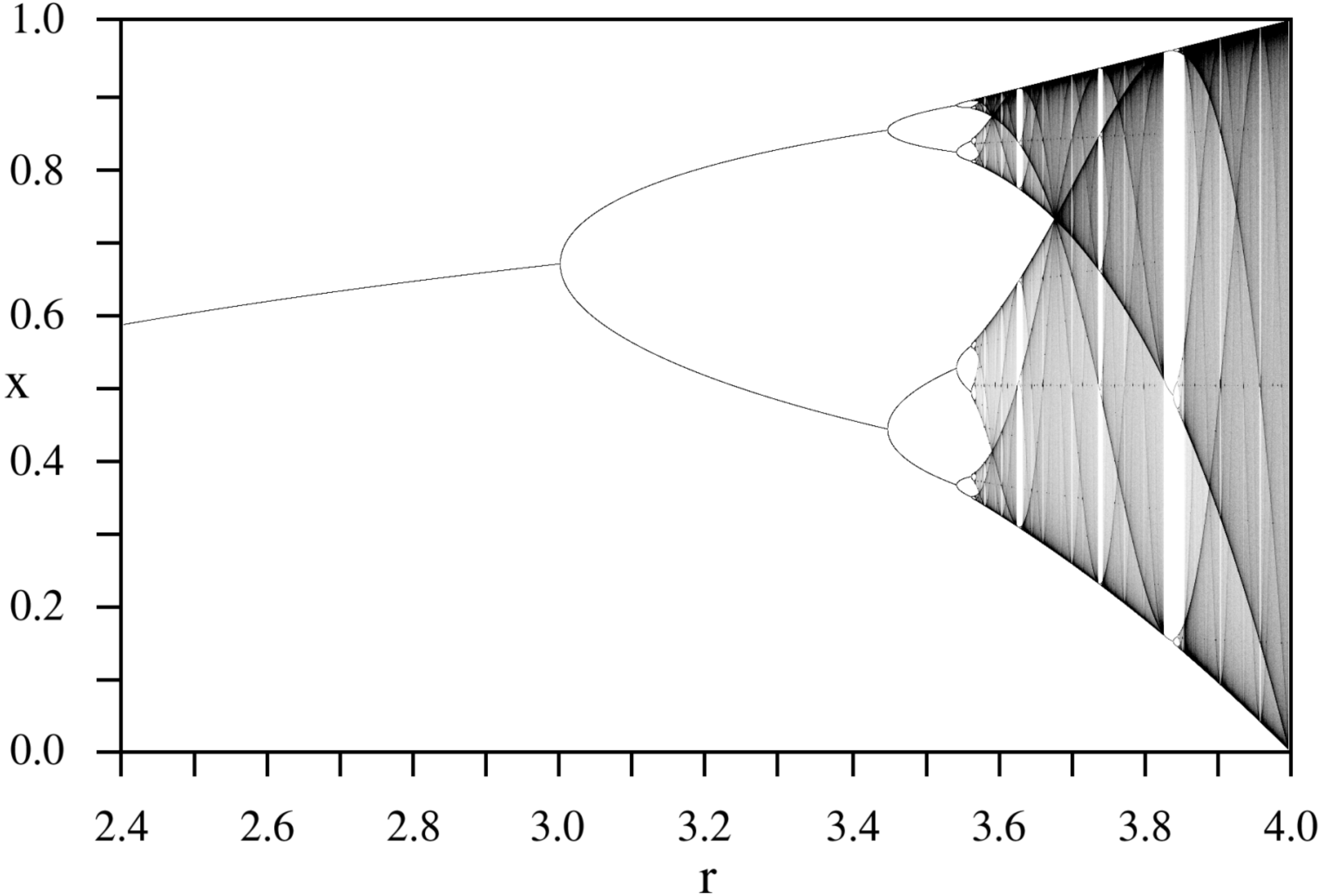
As r increases *period doubling 8, 16, 32* … occurs at ever more closely spaced values of r until at r = 3.57 the system is **no longer periodic – it is called chaotic.**

The system is no longer periodic it is *chaotic*

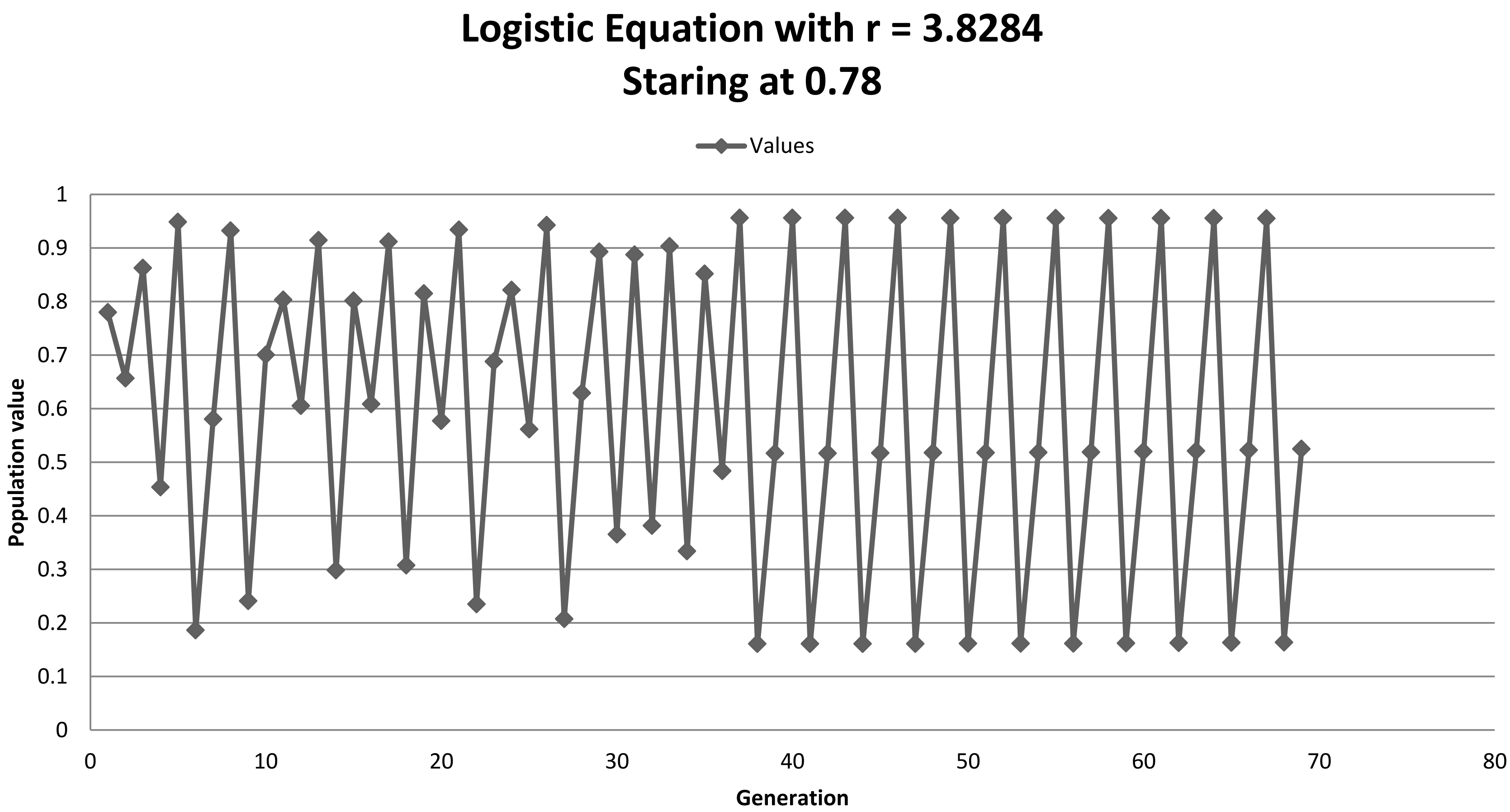




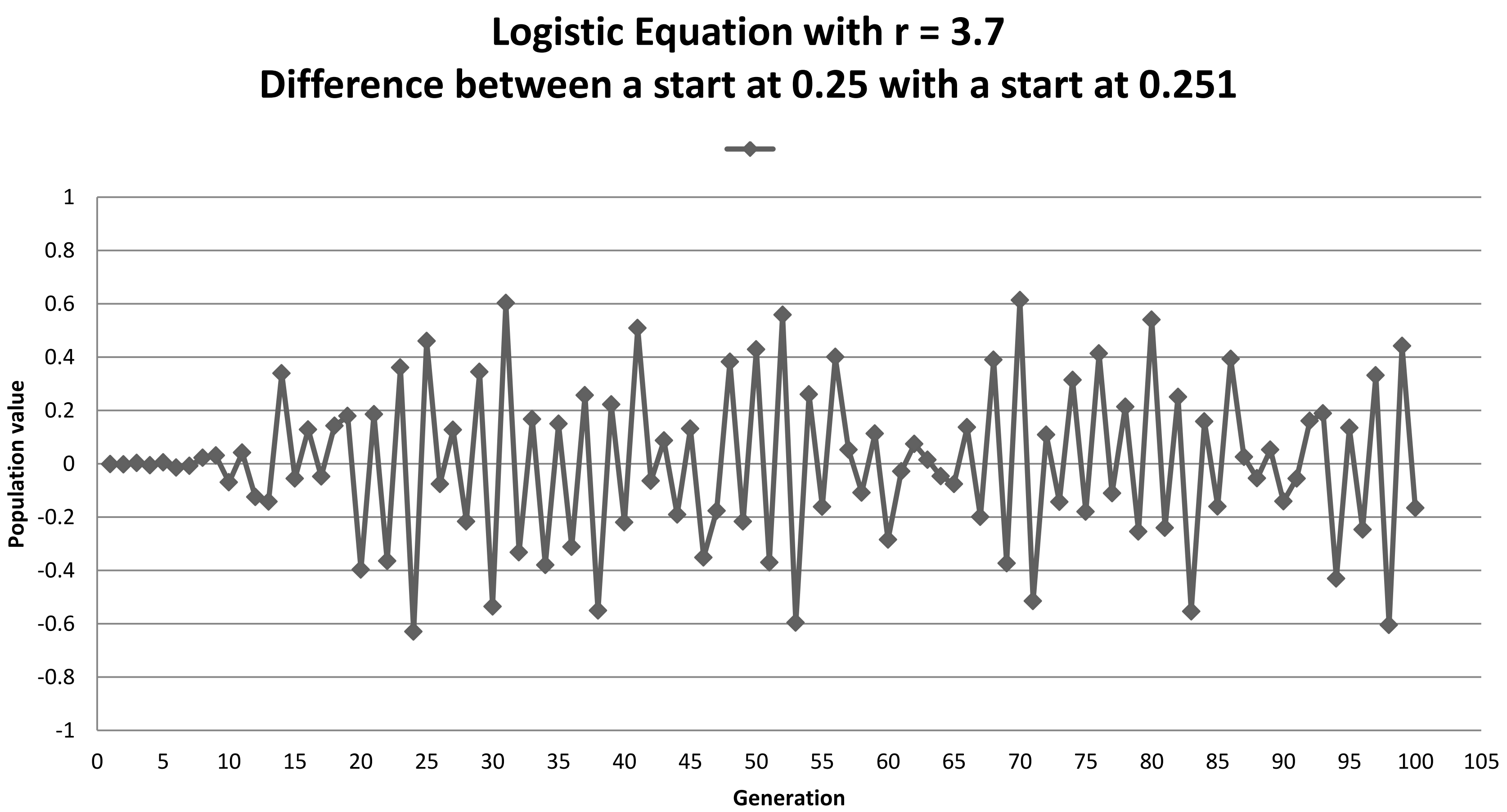
Period doubling road to chaos



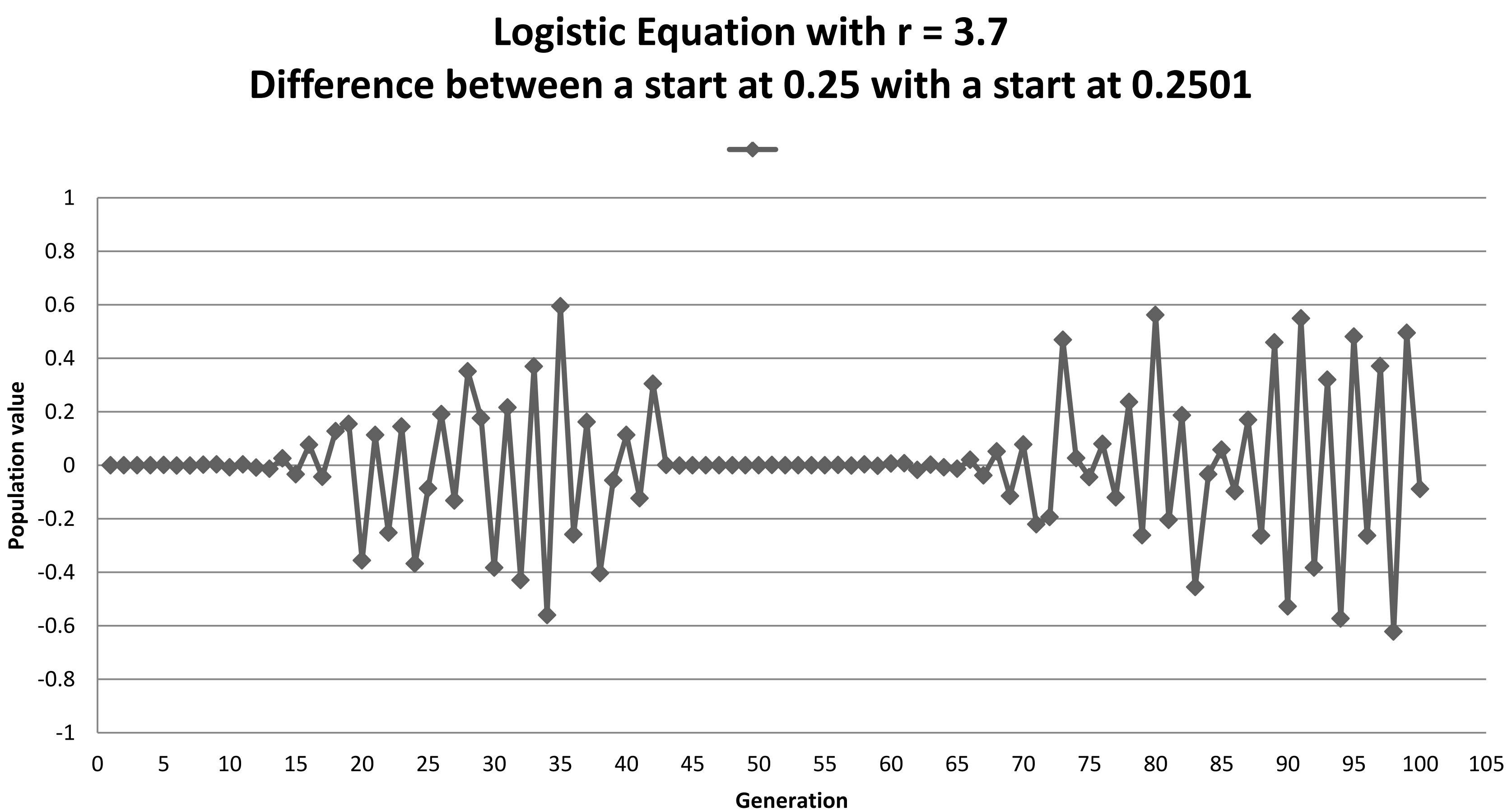
Period three implies chaos



Sensitivity to initial conditions



Sensitivity to initial conditions



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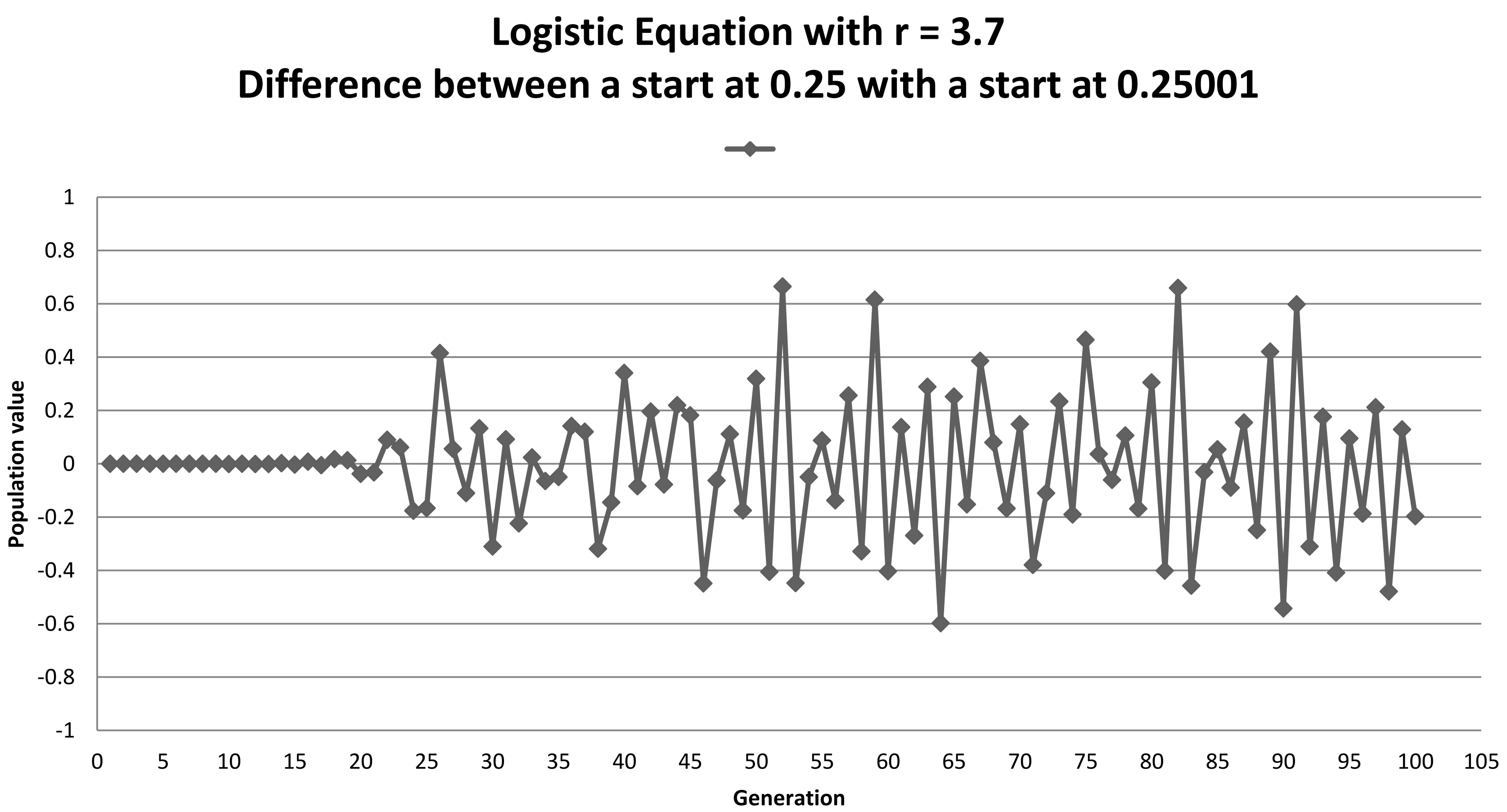

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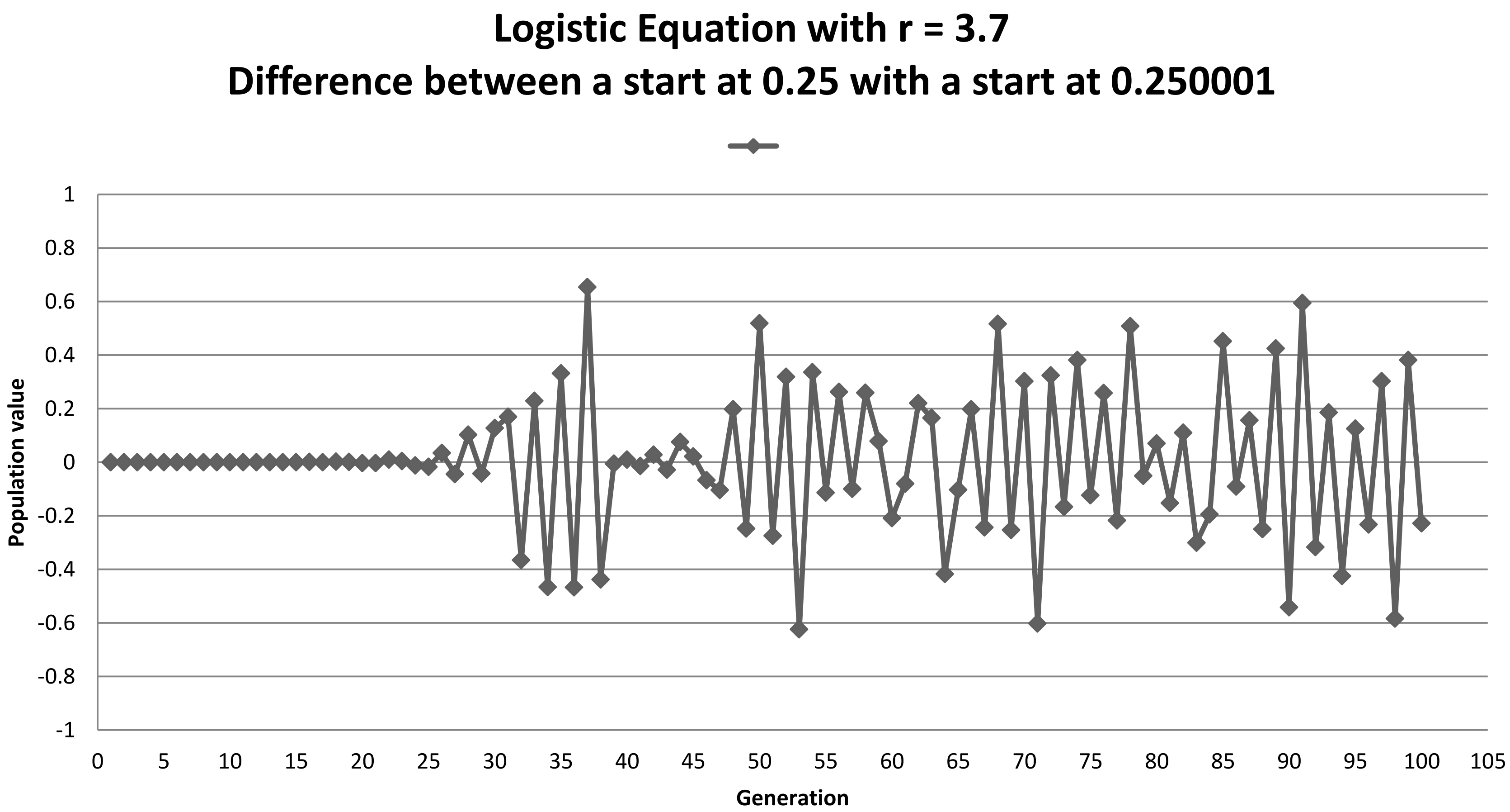

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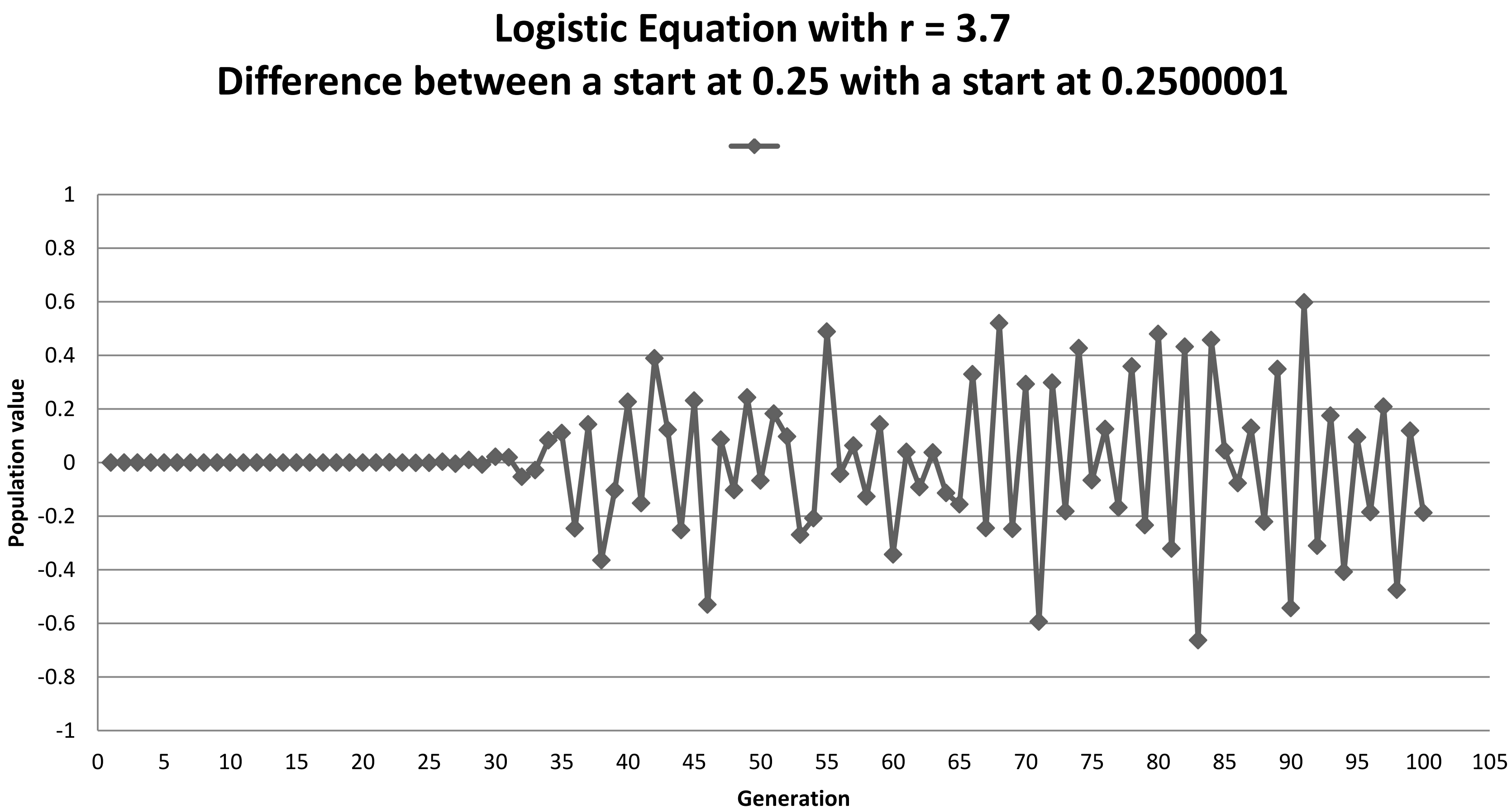
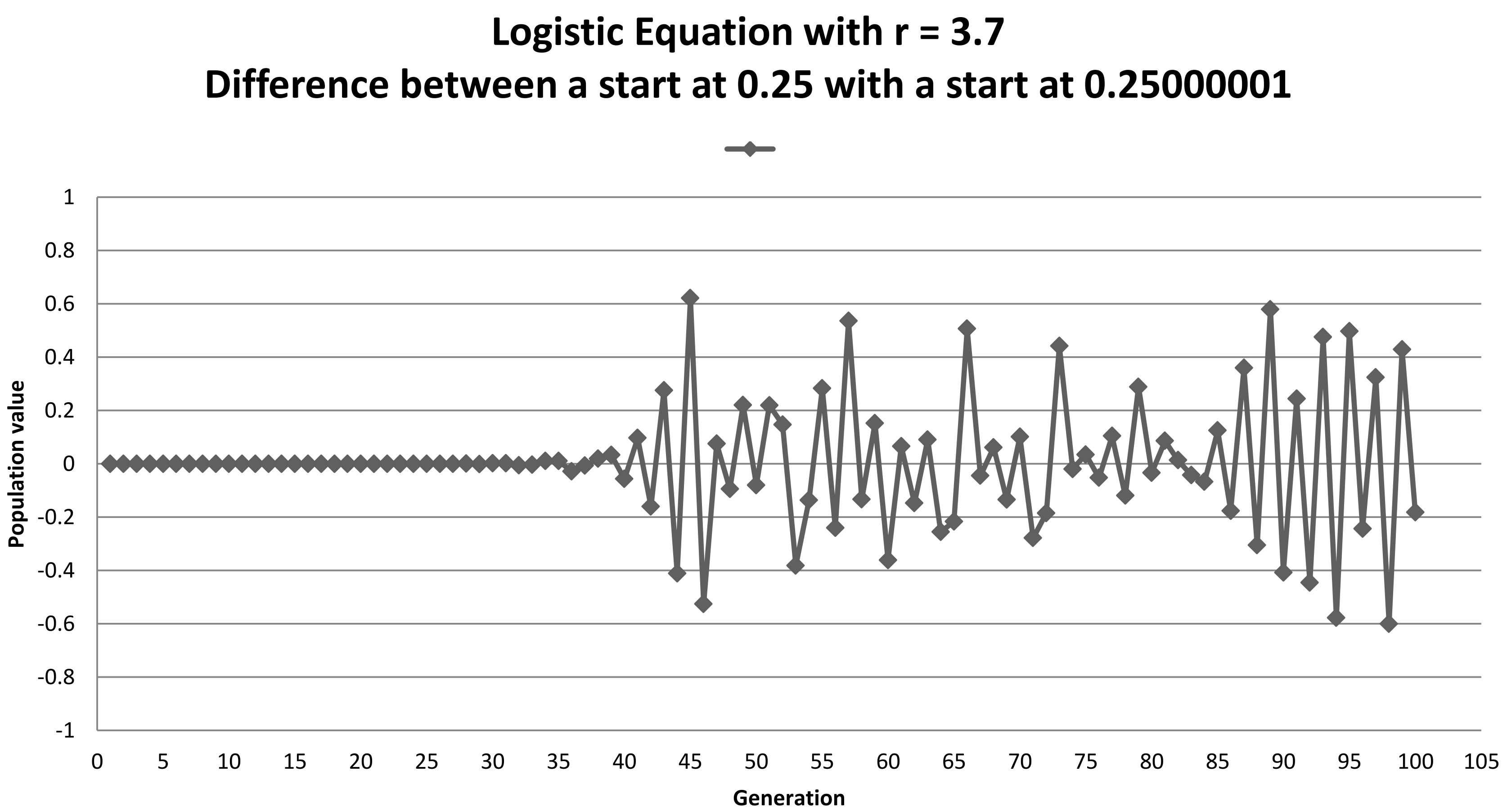


Sensitivity to initial conditions



Sensitivity to initial conditions Predictability Horizon

Starting Value	Number of generations *in step* with the starting value 0.25
0.251	5
0.2501	13
0.25001	17
0.250001	22
0.2500001	26
0.25000001	32

Sensitivity to initial conditions
Predictability Horizon

Starting Value	Number of generations *in step* with the starting value 0.25
0.251	5
0.2501	13
0.25001	17
0.250001	22
0.2500001	26
0.25000001	32

TEN fold increase in accuracy of starting values

only gives

LINEAR increase in agreement of population sizes

From this we can estimate the ***Lyapunov exponent*** which is a measure of the average speed with which infinitesimally close states separate

Simple mathematical models with very complicated dynamics
Robert M. May, *Nature*, 1976

Not only in research, but in the world of politics and economics, we would all be better off if more people realised that simple non-linear systems do not necessarily possess simple dynamical properties.

Lorenz system

Three variables x, y, and z

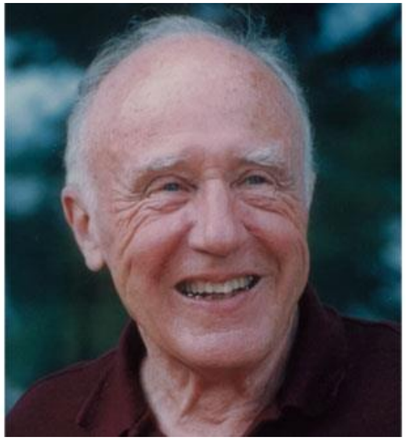
Three parameters σ, ρ and β

$$\frac{dx}{dt} = \sigma(y - x),$$

$$\frac{dy}{dt} = x(\rho - z) - y,$$

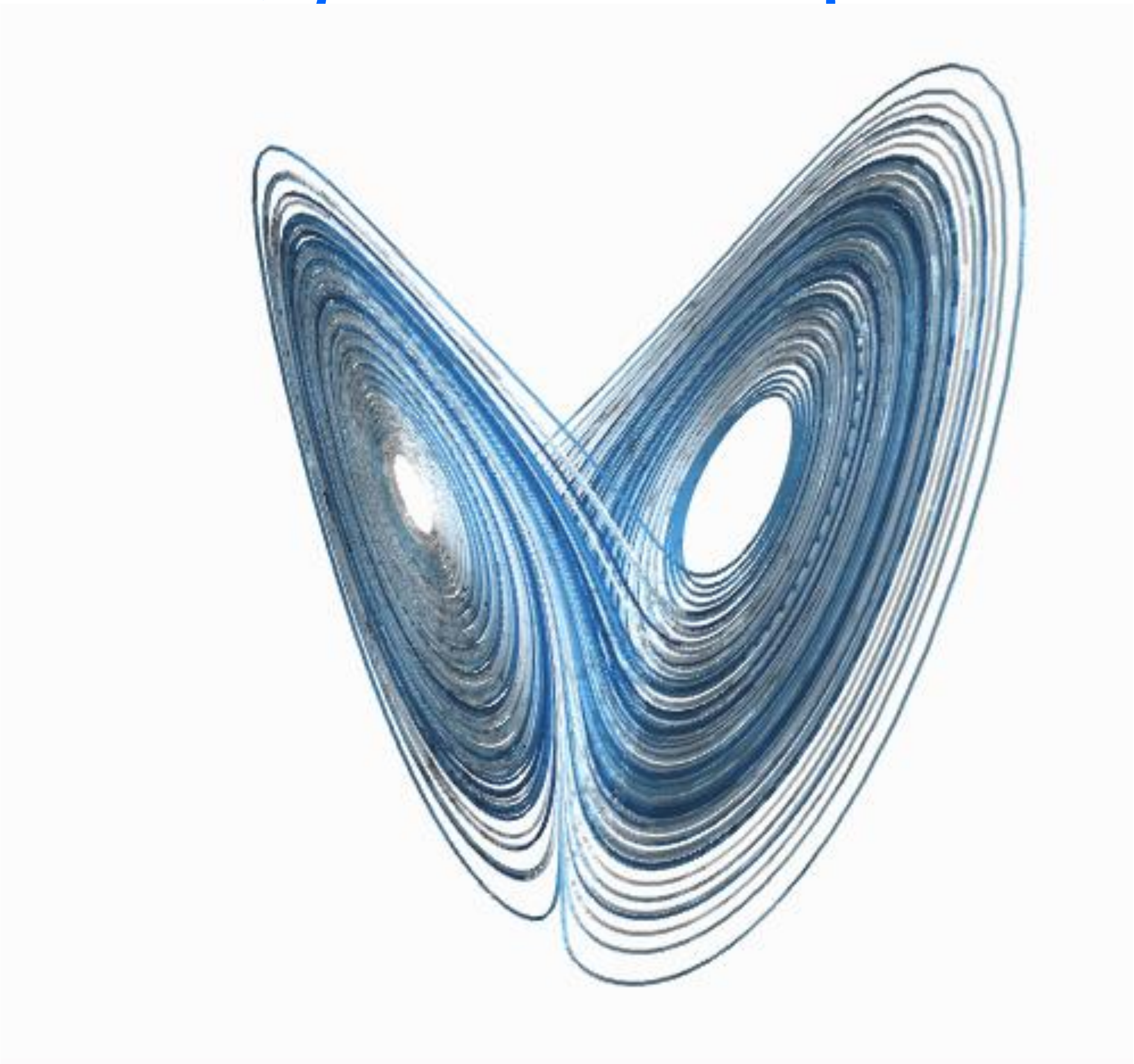
$$\frac{dz}{dt} = xy - \beta z.$$

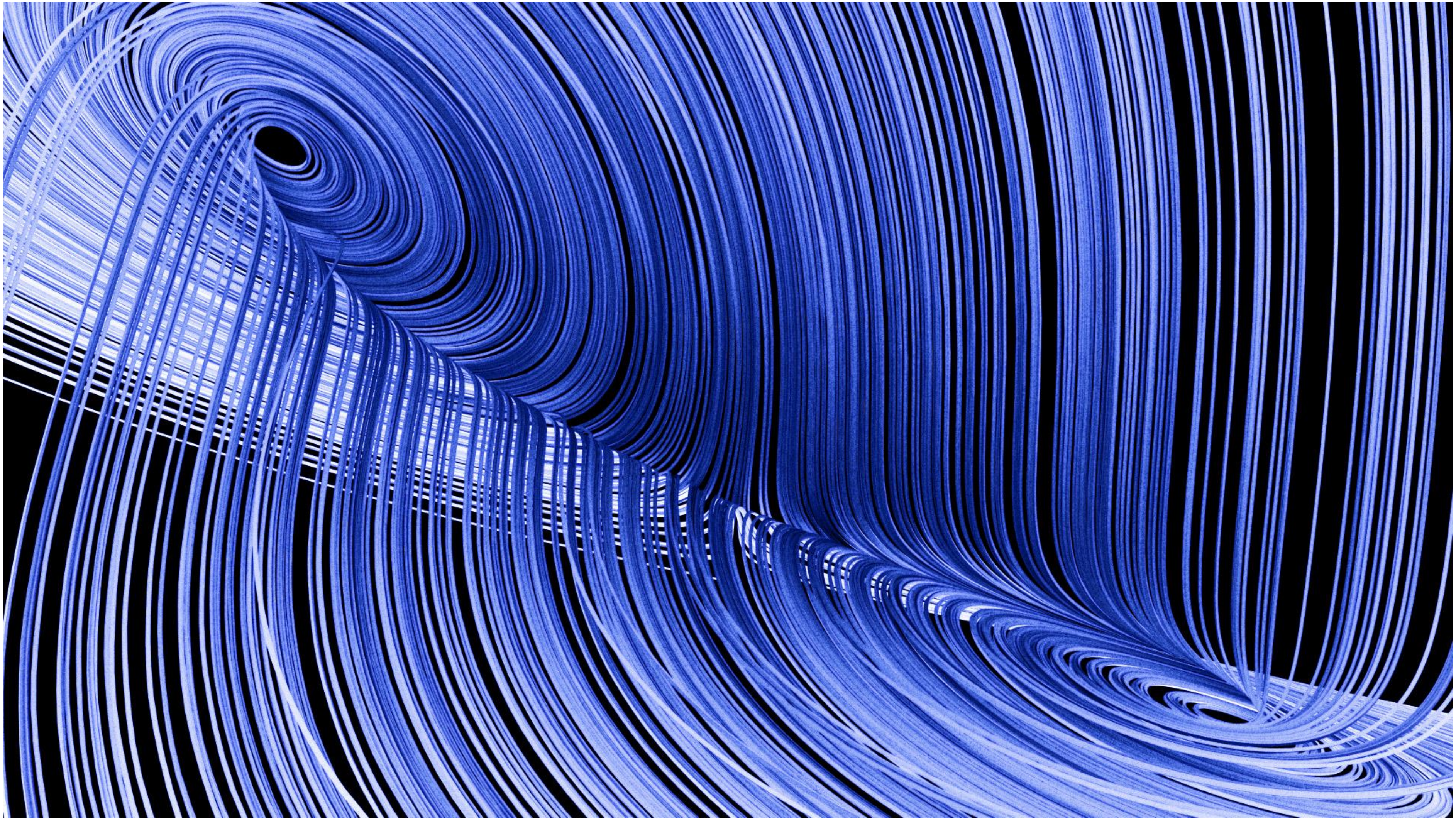
Edward Lorenz 1917 - 2008



Lorenz Attractor

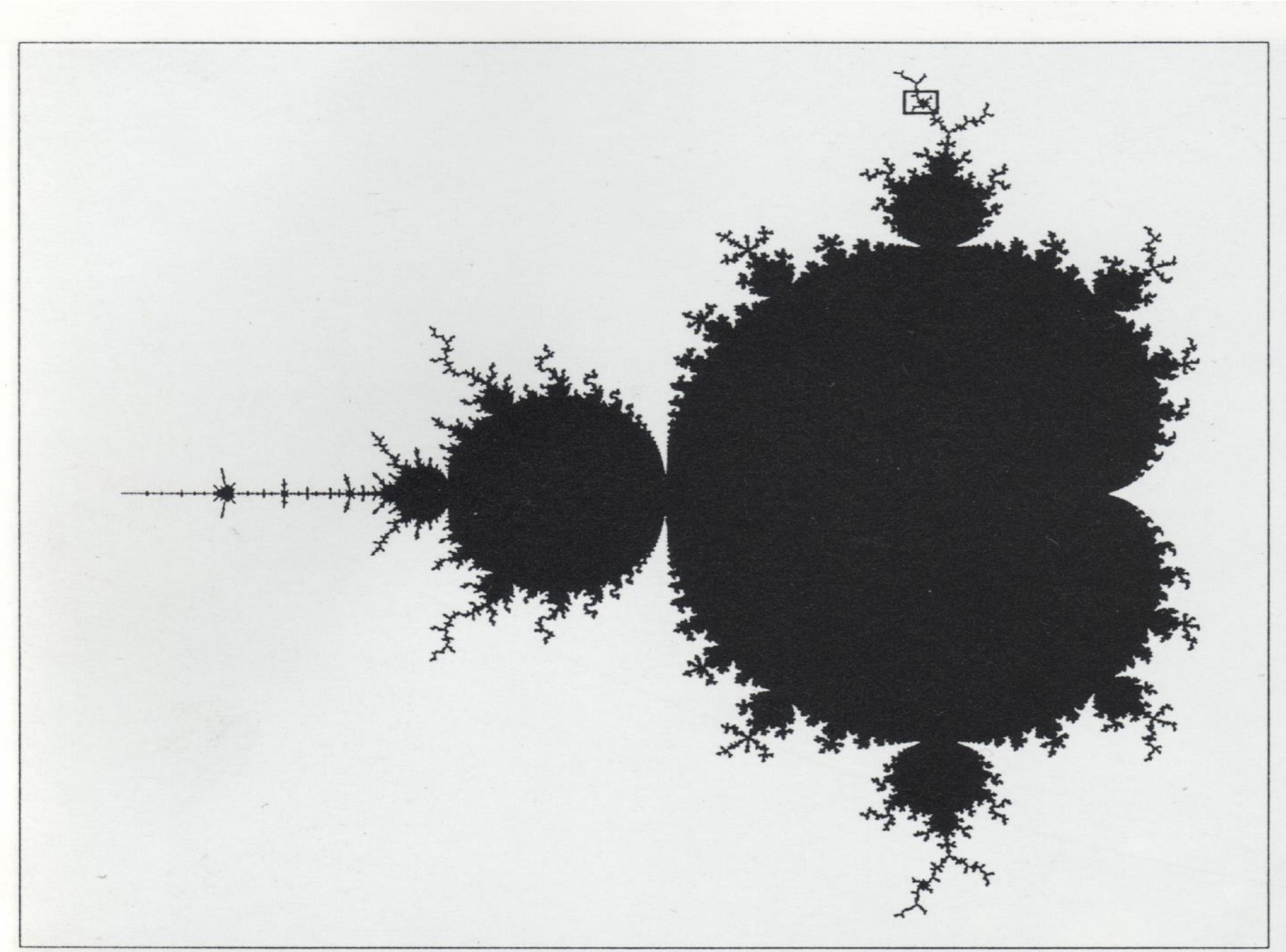
$\sigma = 10$, $\rho = 28$ and $\beta = 8/3$



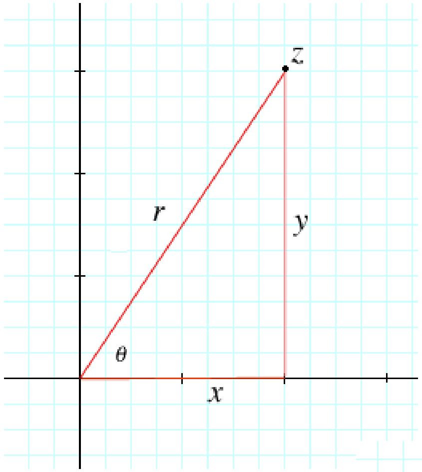


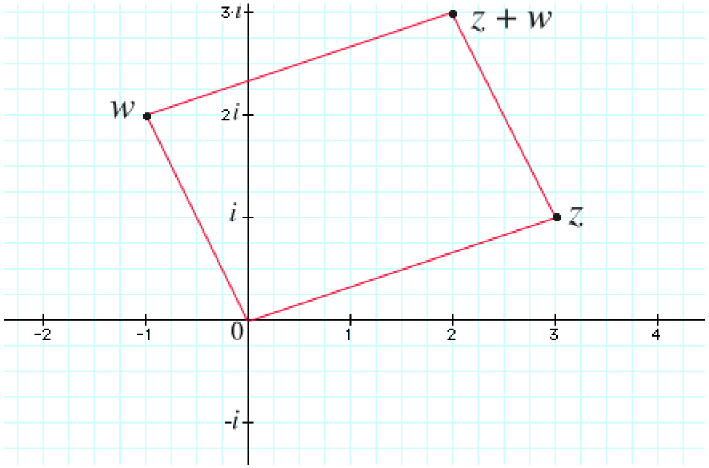
Source: http://www.ylilammi.com/lorenzattractor.shtml

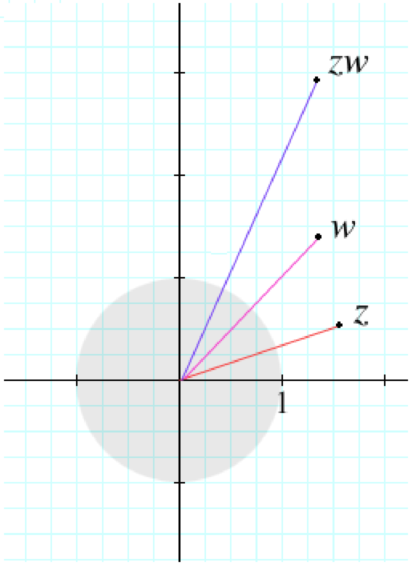
Mandelbrot set



Complex numbers
Representing, adding and multiplying

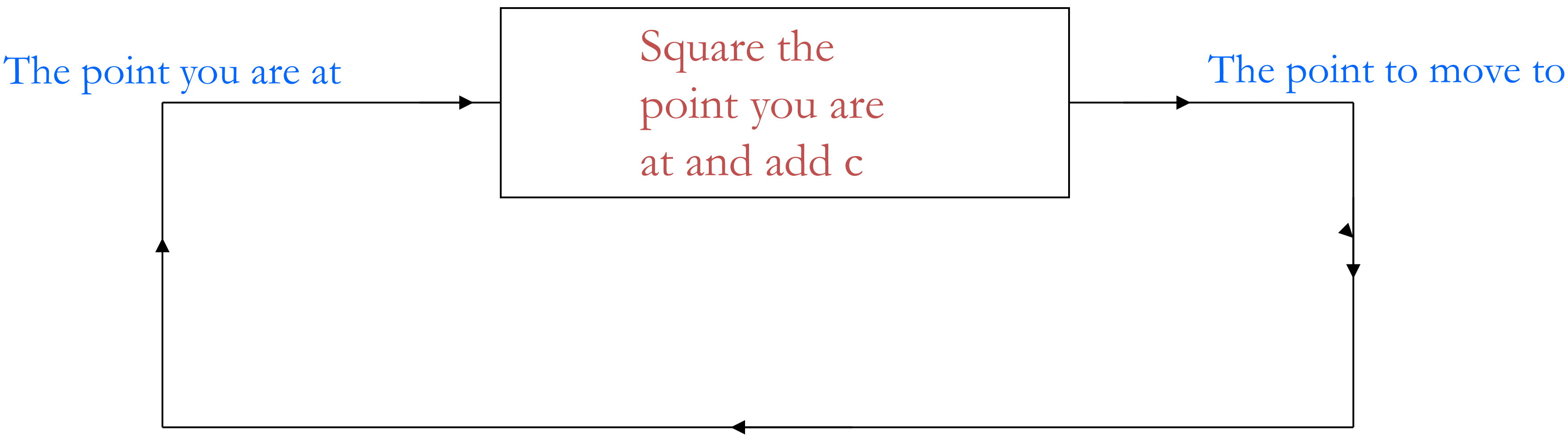






Hopping

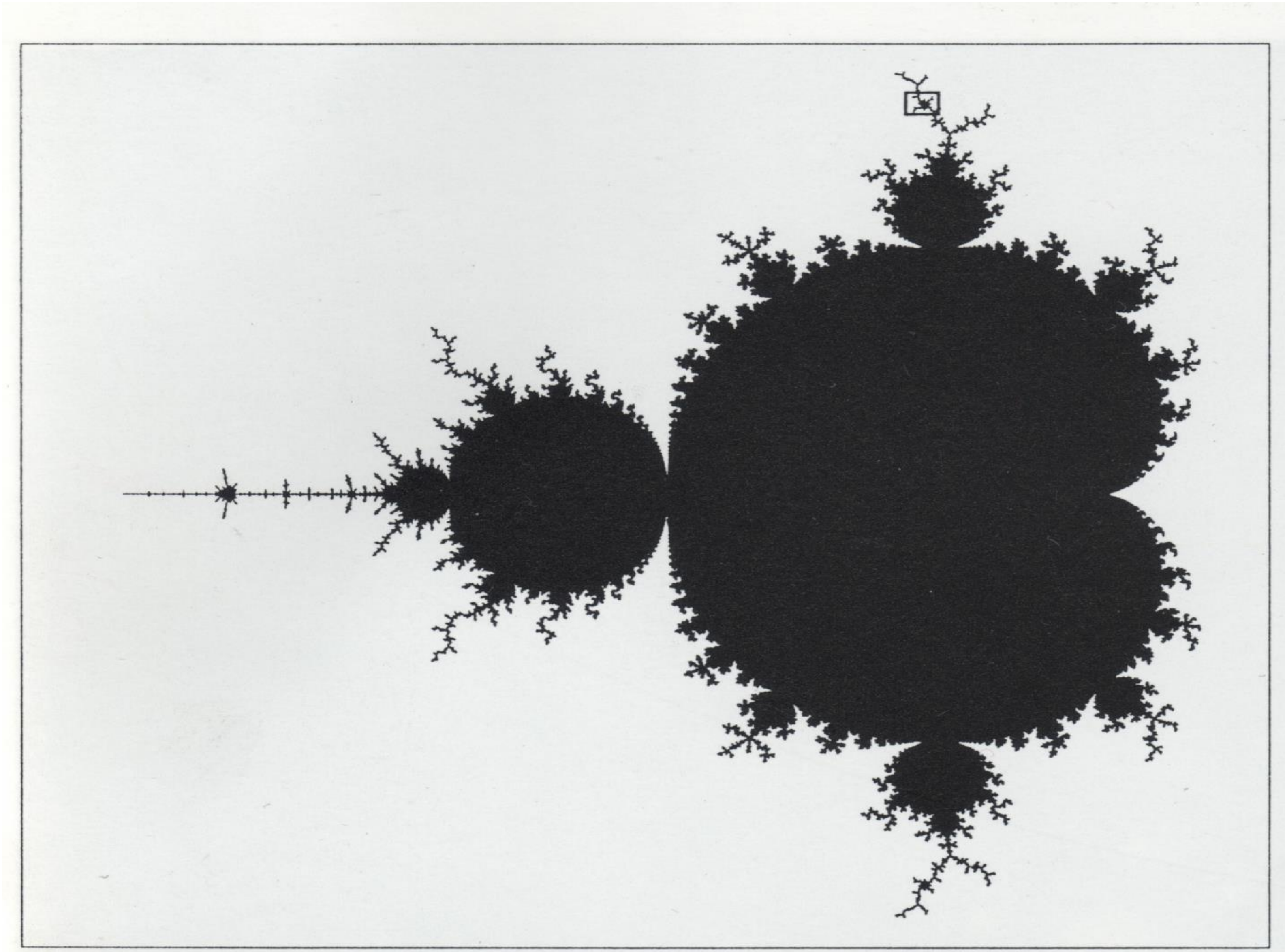
- Pick a point, c, on the plane.
- Start at the point c
- Hop according to the rule

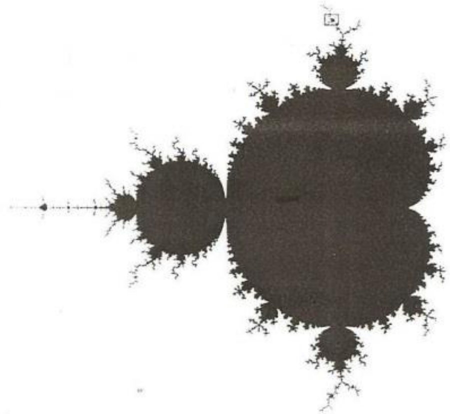


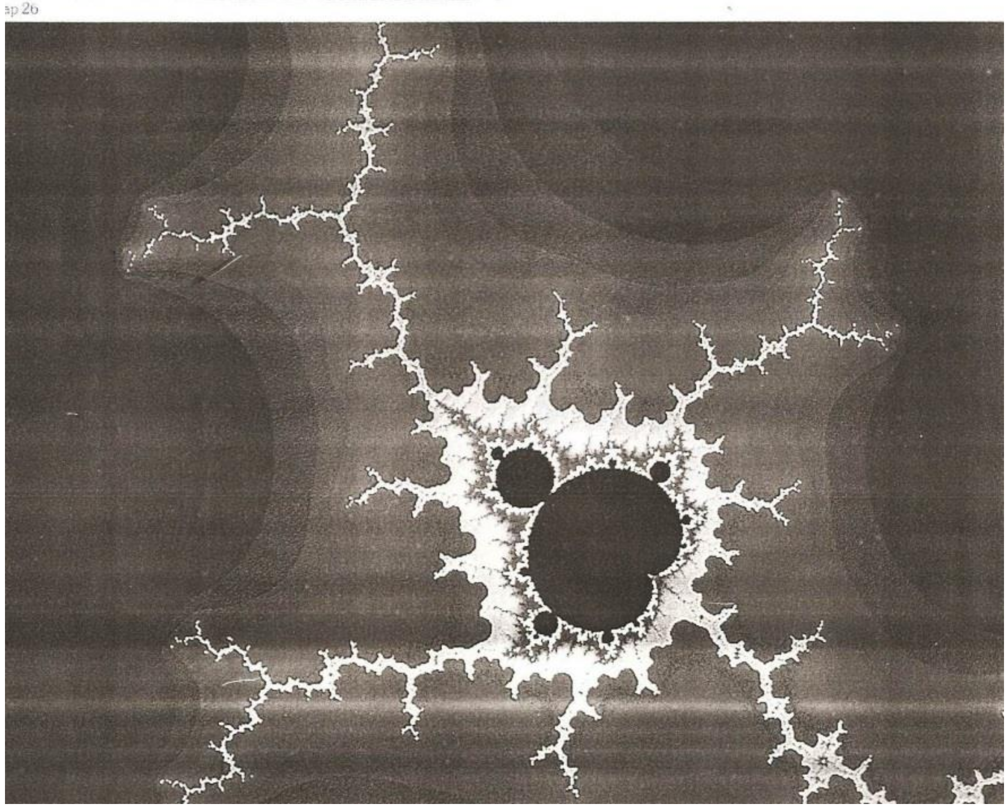
If you hop off to infinity
colour the staring point c white
otherwise
colour it black

Iterate $z_{n+1} = z_n^2 + c$

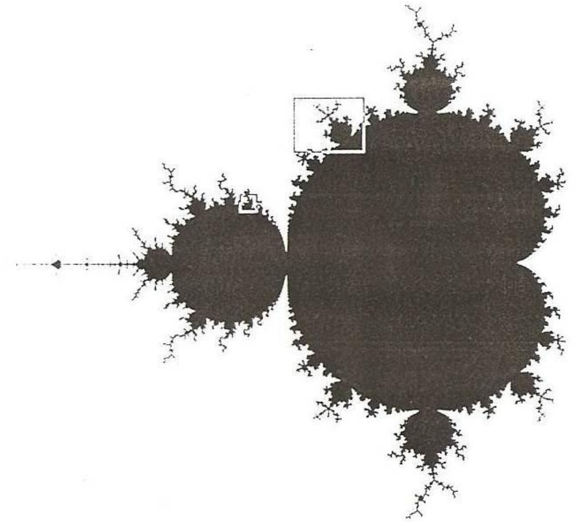
Colour black those starting points which do not go to infinity



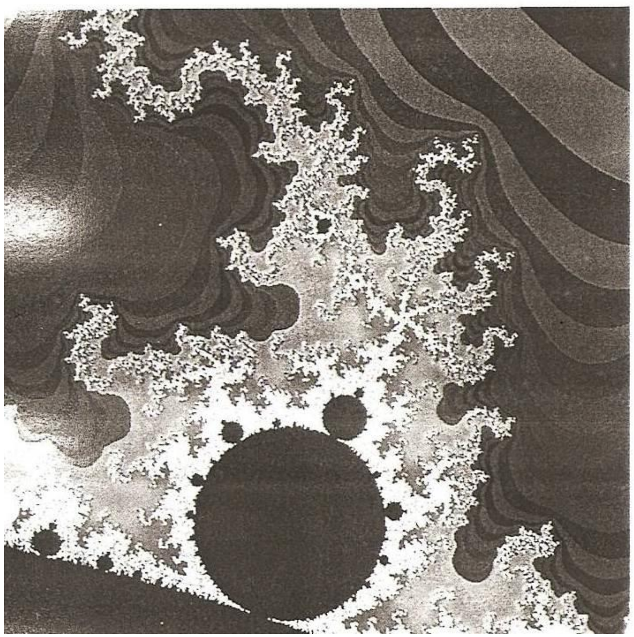


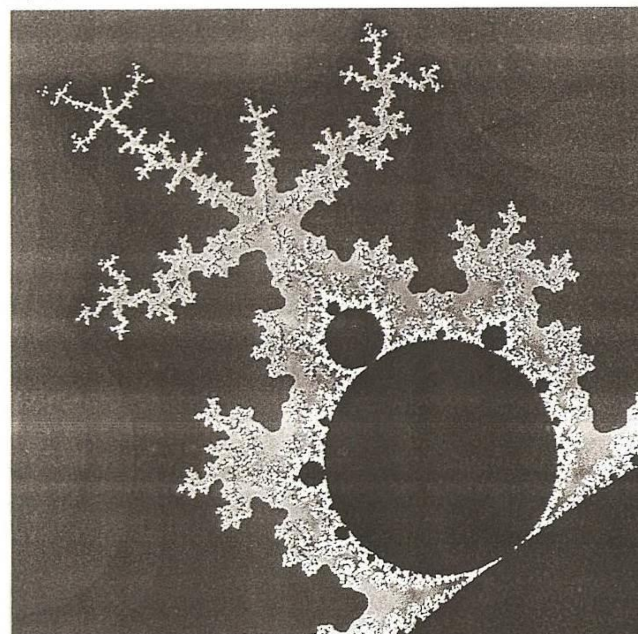


Peitgen and Richter *The Beauty of Fractals*



Map 28

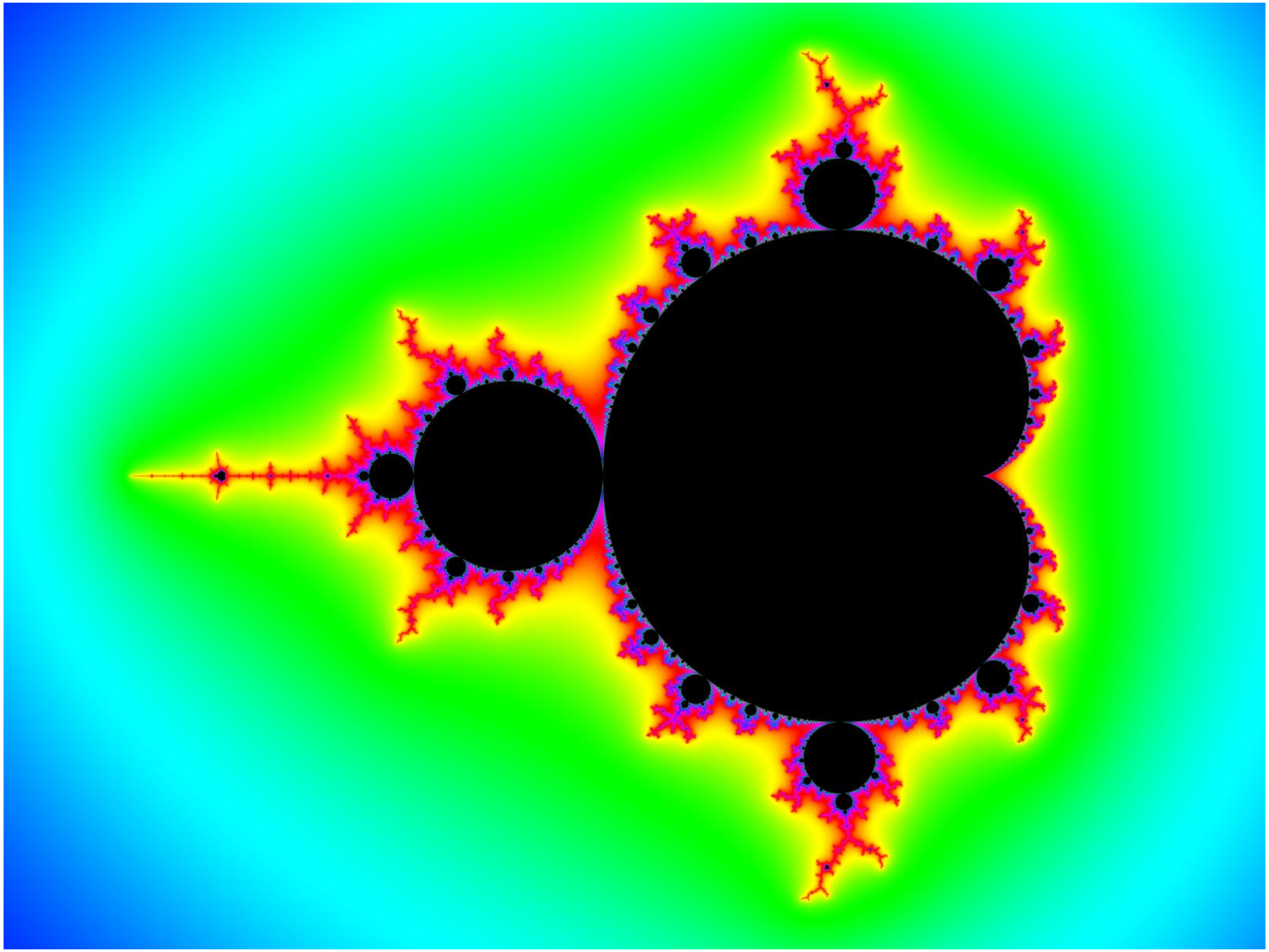




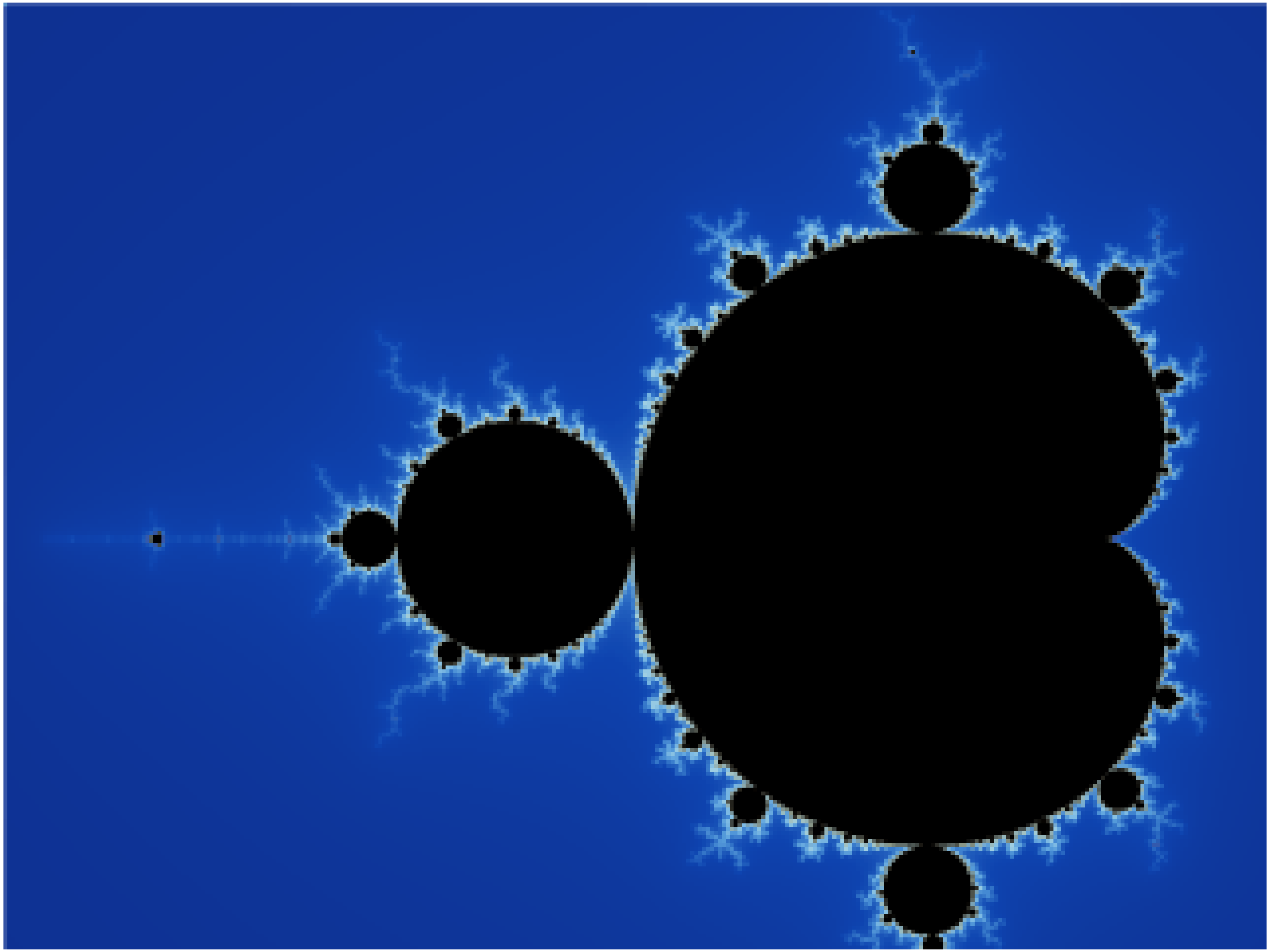
Peitgen and Richter *The Beauty of Fractals*

Peitgen and Richter *The Beauty of Fractals*

Colours indicate how quickly the point *goes to infinity*



Zooming in!



Benoît Mandelbrot: *How Long Is the Coast of Britain? Statistical Self-Similarity and Fractional Dimension*

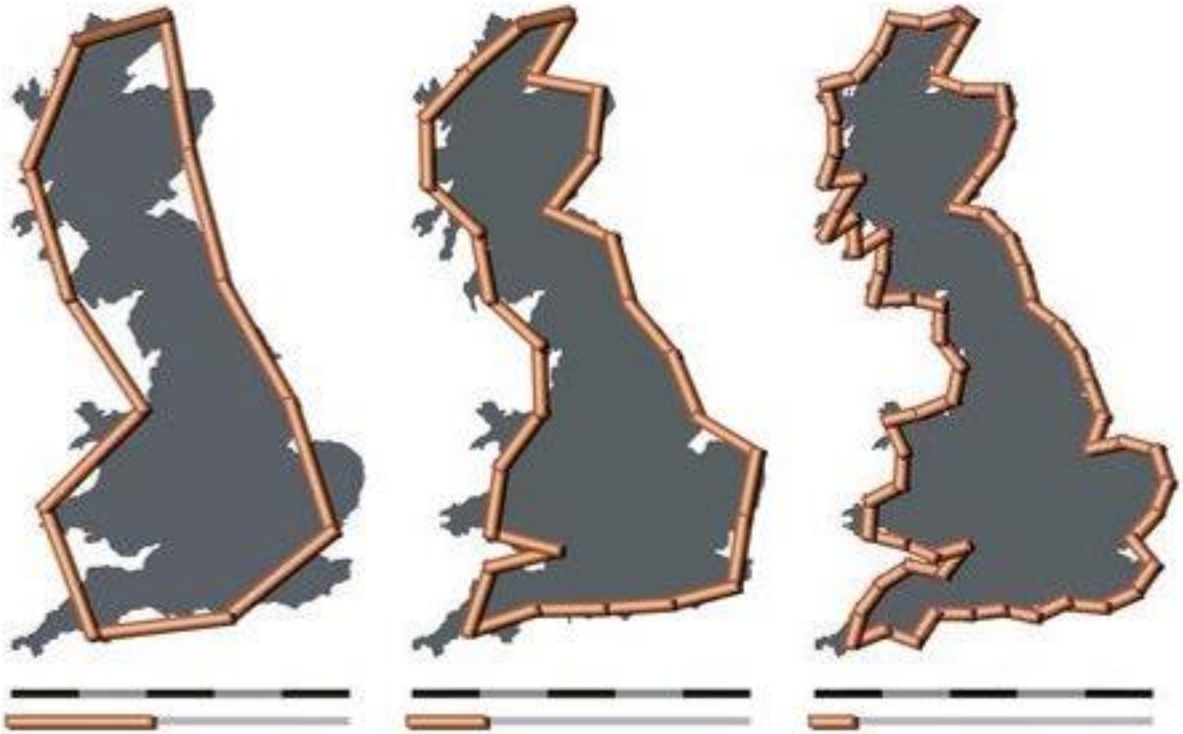
Source: Science, New Series, Vol. 156, No. 3775 (May 5, 1967), pp. 636-638

Abstract: Geographical curves are so involved in their detail that their lengths are often infinite or, rather, undefinable. However, many are statistically "self-similar," meaning that each portion can be considered a reduced-scale image of the whole. In that case, the degree of complication can be described by a quantity D that has many properties of a "dimension," though it is fractional; that is, it exceeds the value unity associated with the ordinary, rectifiable, curves.

Characterizing the dimension of a straight line

If you measure a straight line by laying rulers along it then if you halve the length of the rulers you use you will need twice as many of them.
We use this scaling to arrive at the dimension of the line as 1.

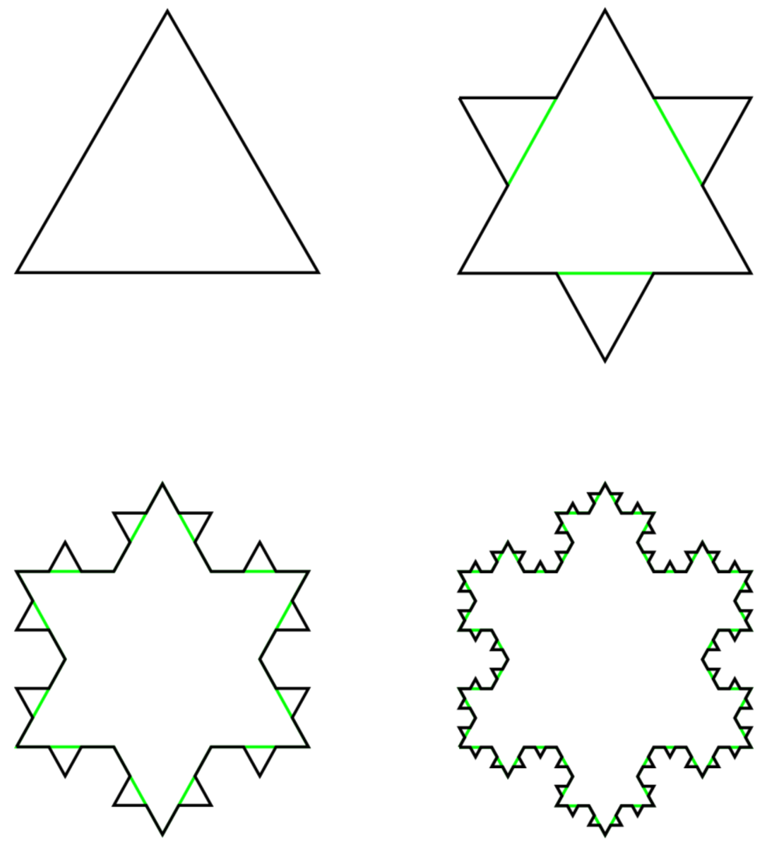
How many rulers needed for Britain?



If ruler is of length 200km need 11.5 of them = 2300 km
If ruler is of length 100km need 28 of them = 2800km
If ruler is of length 50km need 70 of them = 3500 km
As the length of the measuring stick is scaled smaller and smaller, the total length of the coastline measured increases and the number of rulers needed is increasing by more than a factor of 2

Mandelbrot took the *fract* in *fraction* as the root of the word *fractal*.
A *fractal* has *fractional dimension*

- Construction of Koch curve
- Continue this ad infinitum
- The dimension is ln 4/ln 3 ≈ 1.26



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