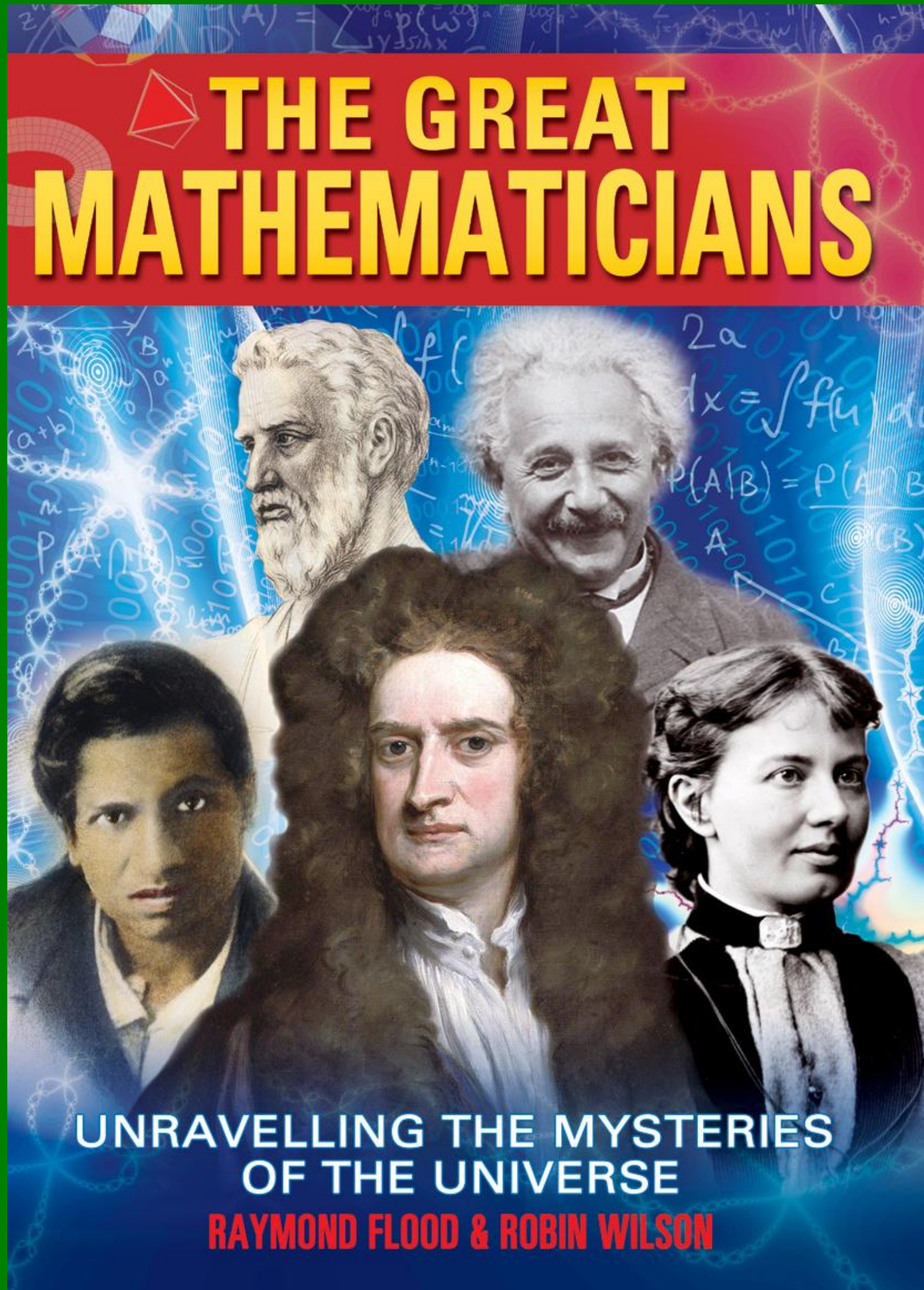

THE GREAT
MATHEMATICIANS
UNRAVELLING THE MYSTERIES
OF THE UNIVERSE
RAYMOND FLOOD & ROBIN WILSON

THE GREAT MATHEMATICIANS

- Which mathematician was killed in a duel?
- Which one published books, yet did not exist?
- Which one was crowned Pope?
- Who were Dr Mirabilis and Dr Profundus?
- Who learned calculus from her nursery wallpaper?
- Who was excited by a taxicab number?
- Who measured the chests of 5732 Scottish soldiers?

INTRODUCTION

The stories of Isaac Newton and the apple, and of Archimedes running naked along the street shouting 'Eureka', are familiar to many. But which mathematicians are the answers to the following questions?

- **Who was killed in a duel?**
- **Who published books, yet did not exist?**
- **Who was crowned Pope?**
- **Who were Dr Mirabilis and Dr Profundus?**
- **Who learned calculus from her nursery wallpaper?**
- **Who was excited by a taxicab number?**
- **Who measured the chests of 5732 Scottish soldiers?**

And what have Geoffrey Chaucer, Christopher Wren, Napoleon, Florence Nightingale and Lewis Carroll to do with mathematics?

As these questions may indicate, and as the pages of this book will show, mathematics has always been a human endeavour as people have found themselves grappling with a wide range of problems, both practical and theoretical. The subject has as long and interesting a history as literature, music or painting, and its origins were both international and multicultural.

For many who remember mathematics from their schooldays as a dull and dusty subject, largely incomprehensible and irrelevant to their everyday lives, this view of mathematics may come as a surprise. The subject has all too often been presented as a collection of rules to be learned and techniques to be applied, providing little understanding of the underlying principles or any appreciation of the nature of the subject as a whole – it is rather like teaching musical scales and intervals without ever playing a piece of music.

For wherever we look, mathematics pervades our daily lives. Our credit cards and the nation's defence secrets are kept secure by encryption methods based on the properties of prime numbers, and mathematics is intimately involved when one flies in a plane, starts a car, switches on the television, forecasts the weather, books a holiday on the internet, programs a computer, navigates heavy traffic, analyses a pile of statistical data, or seeks a cure for a disease. Without mathematics as its foundation there would be no science.

Mathematicians are often described as 'pattern-searchers' – whether they study abstract patterns in numbers and shapes or look for symmetry in the natural world around us. Mathematical laws shape the patterns of seeds in sunflower heads and guide the solar system that we live in. Mathematics analyses the minuscule structure of the atom and the massive extent of the universe.

But it can also be a great deal of fun. The logical thinking and problem-solving techniques that one learns in school can equally be put to recreational use. Chess is essentially a mathematical game, many people enjoy solving logical puzzles based on mathematical ideas, and thousands travel into work each day struggling with their sudoku puzzles, a pastime arising from combinatorial mathematics.

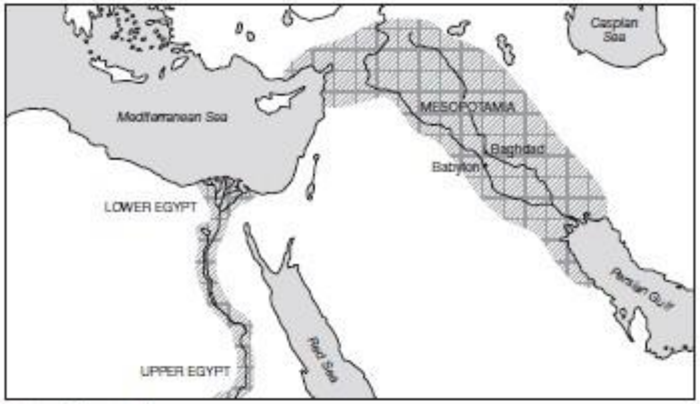
Mathematics is developing at an ever-increasing rate – indeed, more new mathematics has been discovered since the Second World War than was known up to that time. An outcome of all this activity has been the International Congresses of Mathematicians that are held every four years for the presentation and discussion of the latest advances.

William Blake's 'Isaac Newton'

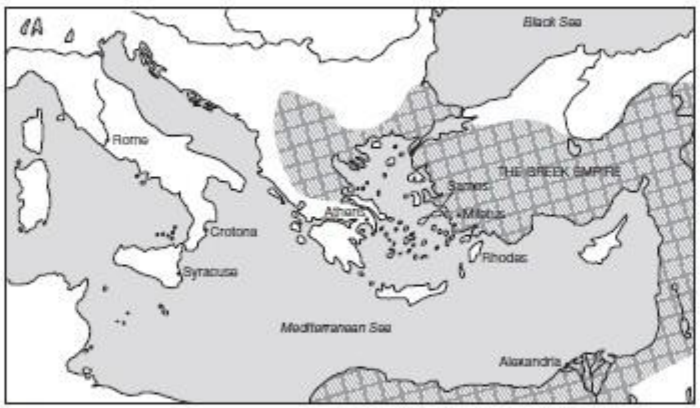
But none of this would have happened if it had not been for the mathematicians who created their subject.

In this book you will meet time-measurers like the Mayans and Huygens, logicians like Aristotle and Russell, astronomers like Ptolemy and Halley, textbook writers like Euclid and Bourbaki, geometers like Apollonius and Lobachevsky, statisticians like Bernoulli and Nightingale, architects like Brunelleschi and Wren, teachers like Hypatia and Dodgson, arithmeticians like Pythagoras and al-Khwarizmi, number-theorists like Fermat and Ramanujan, applied mathematicians like Poisson and Maxwell, algebraists like Viète and Galois, and calculators like Napier and Babbage. We hope that you find all their lives and achievements as fascinating as we do.

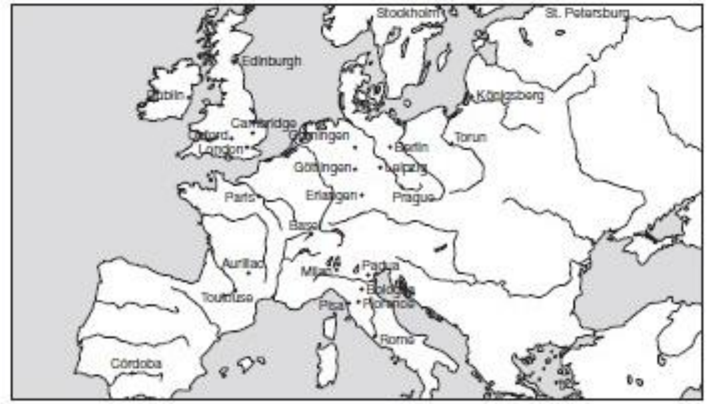
MAPS



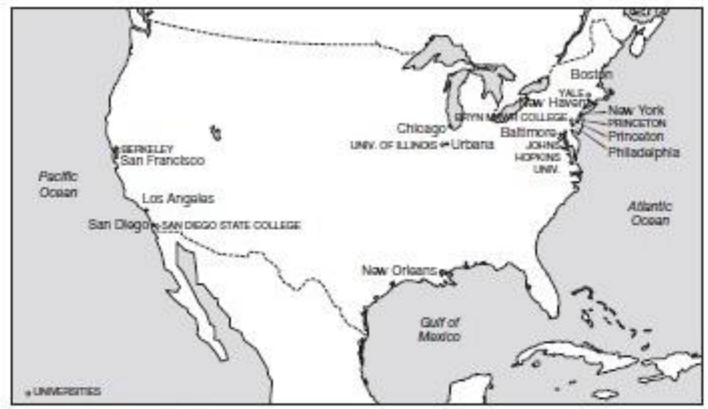
Egypt and Mesopotamia



Greece



European cities



American cities and universities

CHAPTER 1

ANCIENT MATHEMATICS

Mathematics is ancient and multicultural. Several examples of early counting devices on bone (such as tally sticks) have survived, and some of the earliest examples of writing (from around 5000BC) were financial accounts involving numbers. Much mathematical thought and ingenuity also went into the construction of such edifices as the Great Pyramids, the stone circles of Stonehenge, and the Parthenon in Athens.

In this chapter we describe the mathematical contributions of several ancient cultures: Egypt, Mesopotamia, Greece, China, India and Central America. The mathematics developed in each culture depended on need, which may have been practically inspired (for example, agricultural, administrative, financial or military), academically motivated (educational or philosophical), or a mixture of both.

SOURCE MATERIAL

Much of what we know about a culture depends on the availability of appropriate primary source material.

For the Mesopotamians we have many thousands of mathematical clay tablets that provide much useful information. On the other hand, the Egyptians and the Greeks wrote on papyrus, made from reeds that rarely survive the ravages of the centuries, although we do have two substantial Egyptian mathematical papyri and a handful of Greek extracts. The Chinese wrote their mathematics on bamboo and paper, little of which has survived. The Mayans wrote on stone pillars called *stelae* that contain useful material. They also produced codices, made of bark paper; a handful of these survive, but most were destroyed during the Spanish Conquest many centuries later.

A Mesopotamian clay tablet

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COUNTING SYSTEMS

All civilizations needed to be able to count, whether for simple household purposes or for more substantial activities such as the construction of buildings or the planting of fields.

As we shall see, the number systems developed by different cultures varied considerably. The Egyptians used a decimal system with different symbols for 1, 10, 100, 1000, etc. The Greeks used different Greek letters for the units from 1 to 9, the tens from 10 to 90, and the hundreds from 100 to 900. Other cultures developed place-value counting systems with a limited number of symbols: here the same symbol may play different roles, such as the two 3s in 3835 (referring to 3000 and 30). The Chinese used a decimal place-value system, while the Mesopotamians had a system based on 60 and the Mayans developed a system mainly based on 20.

Any place-value system needs the concept of zero; for example, we write 207, with a zero in the tens place, to distinguish it from 27. Sometimes the positioning of a zero was clear from the context. At other times a gap was left, as in the Chinese counting boards, or a zero symbol was specifically designed, as in the Mayan system.

The use of zero in a decimal place-value system eventually emerged in India and elsewhere, and rules were given for calculating with it. The Indian counting system was later developed by Islamic mathematicians and gave rise to what we now call the *Hindu-Arabic numerals*, the system that we use today.

A Central American stela featuring Mayan head-form numbers

So, starting from the natural numbers, 1, 2, 3, ... , generations of mathematicians obtained all the *integers* – the positive and negative whole numbers and zero. This was a lengthy process that took thousands of years to accomplish.

CHAPTER 2

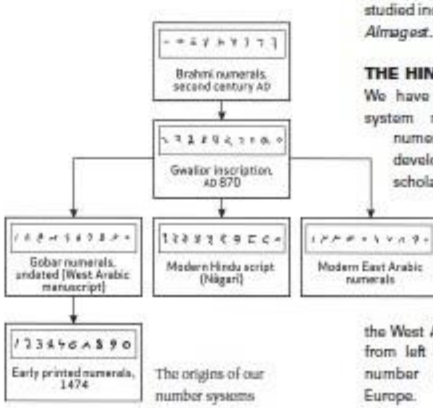
EARLY EUROPEAN MATHEMATICS

The revival of mathematical learning during the Middle Ages was largely due to three factors:

- **the translation of Arabic classical texts into Latin during the 12th and 13th centuries**
- **the establishment of the earliest European universities**
- **the invention of printing**

The first of these made the works of Euclid, Archimedes and other Greek writers available to European scholars, the second enabled groups of like-minded scholars to meet and discourse on matters of common interest, while the last enabled scholarly works to be available at modest cost to the general populace in their own language.

The first European university was founded in Bologna in 1088, and Paris and Oxford followed shortly after. The curriculum was in two parts. The first part, studied for four years by those aspiring to a Bachelor's degree, was based on the ancient 'trivium' of grammar, rhetoric and logic (usually Aristotelian). The second part, leading to a Master's degree, was based on the 'quadrivium', the Greek mathematical arts of arithmetic, geometry, astronomy and music; the works studied included Euclid's *Elements* and Ptolemy's *Almagest*.



The origins of our number systems

THE HINDU–ARABIC NUMERALS

We have seen how the decimal place-value system represented by the Hindu–Arabic numerals first arose in India and was later developed by al-Khwarizmi and other Islamic scholars working in Baghdad and elsewhere.

Gradually the numerals diverged into three separate types – the modern Hindu script, the East Arabic numerals (written from right to left), still found today in the countries of the Middle East, and the West Arabic numerals 1 to 9 and 0 (written from left to right) that eventually became the number system used throughout Western Europe.

Viewing the heavens with a joynt rule

But it took many centuries for the Western form of the Hindu–Arabic numerals to become fully established. They were certainly more convenient to calculate with than Roman numerals, but for practical use most people continued to use an abacus.

As time progressed the situation improved with the publication of influential books that promoted them, such as those by Fibonacci (in Latin), Pacioli (in Italian) and Recorde (in English). By the time that printed books had become widely available, the Hindu–Arabic numerals were in general use.

THE AGE OF DISCOVERY

The spirit of enquiry and inventiveness of the Middle Ages and the Renaissance led people to adopt a more critical view to ideas that had been accepted for centuries. It showed itself in many ways:

- the voyages of discovery to unknown lands
- the development and invention of scientific and mathematical instruments for a variety of purposes
- the use of geometrical perspective in painting and other visual arts
- the solution of cubic and quartic equations
- the development and standardization of mathematical terminology and notation
- the revolutionary approach to planetary motion
- the rediscovery and reinterpretation of classical texts
- the development of mechanics
- the removal of algebra from its dependence on geometry.

These all contributed to the development of a view that the universe is a book written in the language of mathematics. As instruments became ever more sophisticated, mathematics for practical purposes increased – particularly in navigation, map-making, astronomy and warfare.

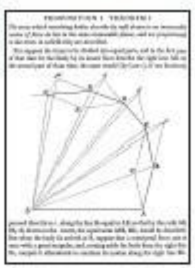
CHAPTER 3

AWAKENING AND ENLIGHTENMENT

The 17th and 18th centuries witnessed the beginnings of modern mathematics. New areas of the subject came into being – notably, analytic geometry and the calculus – while others, such as number theory, were reborn or took on a new lease of life. Fundamental problems, such as that of determining the orbits of the heavenly bodies, were solved or investigated with novel techniques.

It was the age of Newton in England, Descartes and Pascal in France, and Leibniz in Germany, followed by a succession of Continental 'greats': the Bernoulli brothers, Euler, Lagrange and Laplace.

It was also the age of gatherings – the formation of national scientific societies, such as London's Royal Society and the Academy of Sciences in Paris, and the founding of scholarly institutions such as the St Petersburg Academy and the Academy of Sciences in Berlin.



The area swept out by a moving body: Newton's use of geometry (right) contrasts with Laplace's analytical approach (far right)

CALCULUS AND DISCOVERY

Initially, the problems that mathematicians solved were geometrical, as were their answers, although the techniques they used (including the calculus) were not necessarily of this kind, being seen as methods of proceeding from a geometrical problem to a geometrical answer. The 18th century then led to a new conception of mathematics, with its most striking characteristic being its algebraic appearance.

The objects of mathematics were now described by formulas with symbols for

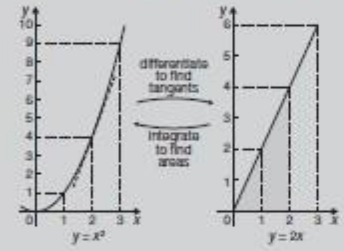
> If we project the body *m*, on the plane of *x* and *y*, the differential $(xdy - ydx)/2$, will represent the area which the radius vector, drawn from the origin of the coordinates to the projection of *m*, describes in the time *dt*, consequently the sum of the areas, multiplied respectively by the masses of the bodies, is proportional to the element of time, from which it follows that in a finite time, it is proportional to the time. It is this which constitutes the principle of the conservation of areas.

WHAT IS THE CALCULUS?

The calculus is made up from two seemingly unrelated strands, now called *differentiation* and *integration*. Differentiation is concerned with how fast things move or change, and is used in the finding of velocities and tangents to curves. Integration is used to find areas of shapes in two-dimensional space or volumes in three dimensions.

As the 17th century progressed, it was gradually realized that these two strands are intimately related. As both Newton and Leibniz explained, they are inverse processes — if we follow either by the other, we return to our starting point.

However, Newton and Leibniz had different motivations, with Newton focusing on motion and Leibniz concerned with tangents and areas.



Differentiation and integration

variables and constants. A main reason for doing so was that the machinery of the calculus could then be applied both to them and to practical situations. This hastened the development of new mathematical descriptions and techniques, such as in the emerging area of *differential equations*.

This shift towards the algebraic type of description also led to a good way of discovering new objects. Books were written in the algebraic style, and mathematicians formulated, thought about and solved problems in this way. Increasingly, algebra came to be seen as a logical language suitable for the investigation of all the sciences.

Mechanics and astronomy were the main areas of practical investigation. They both applied the calculus to functions of more than one variable, such as

$u(x, y) = x^6 + x^2y^2 + y^4$;

here, $u(x, y)$ can be thought of as the height of a surface above the point with coordinates (x, y) in the plane.

The equations that arose were called *partial differential equations*, because they involved 'partial differentiation'. The partial derivative $\partial u/\partial x$ is the rate of change of *u* in the *x*-direction, while the partial derivative $\partial u/\partial y$ is the rate of change of *u* in the *y*-direction.

Louis XIV visits the Paris Academy of Sciences, 1671

CHAPTER 4

THE AGE OF REVOLUTIONS

The 19th century saw the development of a mathematics profession in which people earned their living from teaching, examining and researching. The mathematical centre of gravity moved from France to Germany, while Latin gave way to national languages for publishing mathematical work. There was also a dramatic increase in the number of textbooks and journals.

Because of this increase in mathematical activity, mathematicians began to (indeed, needed to) specialize. While one would use the term *mathematician* in the 18th century, one now had *analysts, algebraists, geometers, number theorists, logicians* and *applied mathematicians*. This need for specialization was avoided only by the very greatest: Gauss, Hamilton, Riemann and Klein.

The University of Göttingen, where Gauss, Riemann and Klein worked.

In each discipline there was a revolution (as well as an evolution) in the depth, extent, and even the very existence of the discipline. But each discipline experienced a movement towards an increasingly abstract style with an increased emphasis on putting mathematics on a sound and rigorous basis and examining its foundations. We illustrate this by considering the revolutions in three areas – analysis, algebra and geometry.

FROM CALCULUS TO ANALYSIS

In the 1820s Augustin-Louis Cauchy, the most prolific mathematician of the century, rigorized the calculus by basing it on the concept of a *limit*. He then used this idea to develop the areas of real and complex analysis. This increase in rigour necessitated the formulation of a foolproof definition of the real numbers, which in turn led to a study of infinite sets by Georg Cantor and others.

Joseph Fourier's work on heat conduction also gave rise to infinite processes – in this case, infinite series – thereby stimulating Bernhard Riemann in his work on integration. Analytical techniques came to be applied to a wide range of problems – in electricity and magnetism by William Thomson (Lord Kelvin) and James Clerk Maxwell, in hydrodynamics by George Gabriel Stokes, and in probability and number theory by Pafnuty Chebyshev.

Revolutions did not happen only in mathematics: this is a miners' riot that took place in Belgium, 1868

FROM EQUATIONS TO STRUCTURES

Algebra also changed dramatically throughout the 19th century. In 1800 the subject was about solving equations, but by 1900 it had become the study of mathematical structures – sets of elements that are combined according to specified rules, called axioms.

At the beginning of the century, Gauss laid down the basics of number theory and introduced modular arithmetic, an early example of a new algebraic structure called a *group*.

A long-standing problem had to do with finding a general method for solving polynomial equations of degree 5 or more, using only arithmetical operations and the taking of roots. Niels Abel showed that there can be no such general solution, and Évariste Galois developed his ideas by examining groups of permutations of the roots of an equation.

The mystique concerning complex numbers was at last removed by William Rowan Hamilton, who defined them as pairs of real numbers with certain operations. Other algebraic structures were discovered: Hamilton introduced the algebra of quaternions, George Boole created an algebra for use in logic and probability, and Cayley studied the algebra of rectangular arrays of symbols, called *matrices*.

FROM ONE TO MANY GEOMETRIES

Over the space of one hundred years the study of geometry was completely transformed. In 1800 the only 'true' geometry had been Euclidean geometry, although there were some scattered results on spherical and projective geometry. By the end of the century, infinitely many geometries were known, while geometry had become closely linked to group theory and placed on a more rigorous foundation.

Gauss studied surfaces and their curvature, finding a relationship between curvature and the sum of the angles of a triangle on the surface, and this turned out to be related to the investigations into the parallel postulate in Euclidean geometry. Nikolai Lobachevsky and János Bolyai independently developed non-Euclidean geometry, in which the parallel postulate does not hold.

It took time, however, for the ideas on non-Euclidean geometry to become absorbed, and it was the mid-century work of Riemann that showed the importance of the new ideas and extended the work of Gauss. Through such abstract techniques, geometry was also moving out of two and three dimensions and into higher ones. Later, Felix Klein used groups to examine and classify different types of geometry.

CHAPTER 5

THE MODERN AGE

In our final chapter we meet the mathematicians who have:

- **examined the limits of what we can prove and shown why some tasks are impossible to carry out**
- **laid the foundations of our current scientific knowledge**
- **carried out mathematical work of historical, social and political impact, and thereby changed the world we live in**
- **developed computers, both theoretical and practical, that enable us to simulate, model and prove things that we could not do otherwise, while raising questions about our identity.**

PARADOXES AND PROBLEMS

In previous chapters we have seen a developing desire to place mathematics on a sounder foundation, with the story going from the underpinning notions of the calculus to arithmetic and the theory of sets. As 20th-century mathematicians examined more carefully the nature of infinity and the problems connected with sets they met a number of problems and paradoxes. One of the most famous of these was formulated by Bertrand Russell in 1902, and necessitated a much more thorough treatment of the very foundations of set theory and of the exact nature of deductive proof.

The Princeton Institute for Advanced Study, established in 1930: 25 Nobel Laureates and 38 (out of 52) Fields Medallists have been affiliated with it

G. H. Hardy with his research student Mary Cartwright, later to be the first woman appointed President of the London Mathematical Society

Another approach was taken by David Hilbert, whose attempt to make arithmetic secure was to make it *axiomatic*, an approach that he had already used with success when dealing with the foundations of geometry. Instead of defining all the basic terms, such as point or line, he gave a set of rules (or axioms) that they had to satisfy.

Although Hilbert's approach was influential, his objectives were eventually proved to be unattainable, as demonstrated in the 1930s by Kurt Gödel and Alan Turing, who obtained a number of amazing and unexpected results about the limits of what can be proved or decided.

ABSTRACTION AND GENERALIZATION

The 19th-century trend towards increasing generalization and abstraction continued to accelerate dramatically throughout the 20th century. For example, Albert Einstein used the abstract formulations of geometry and calculus for his general theory of relativity, while algebra became an abstract and axiomatic subject, being particularly influenced by the work of Emmy Noether. Advances also continued to be made in number theory, with Hardy (and his co-workers Littlewood and Ramanujan) and Andrew Wiles making major contributions.

Meanwhile, new areas of the subject came into being, such as algebraic topology and the theory of 'Hilbert spaces', while machine computation entered the mainstream of the subject, as spectacularly illustrated by Appel and Haken in their proof of the four-colour theorem.

SPREAD AND DEVELOPMENT

The 20th century saw mathematics becoming a major profession throughout the world, with jobs in education and industry and numerous areas of specialization and application.

With mathematics developing at such a fast rate, many new journals have been created, and national and international conferences have become widespread. Most important among these meetings are the International Congresses of Mathematicians, held every four years, when the prestigious Fields medals are awarded and many thousands of mathematicians gather to learn about the latest developments in their subject.

CHAPTER 1

ANCIENT MATHEMATICS

Mathematics is ancient and multi-cultural. Several examples of early counting devices on bone (such as tally sticks) have survived, and some of the earliest examples of writing (from around 5000BC) were financial accounts involving numbers. Much mathematical thought and ingenuity also went into the construction of such edifices as the Great Pyramids, the stone circles of Stonehenge, and the Parthenon in Athens.

In this chapter we describe the mathematical contributions of several ancient cultures: Egypt, Mesopotamia, Greece, China, India and Central America. The mathematics developed in each culture depended on need, which may have been practically inspired (for example, agricultural, administrative, financial or military), academically motivated (educational or philosophical), or a mixture of both.

A Mesopotamian clay tablet

SOURCE MATERIAL

Much of what we know about a culture depends on the availability of appropriate primary source material.

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All civilizations needed to be able to count, whether for simple household purposes or for more substantial activities such as the construction of buildings or the planting of fields.

As we shall see, the number systems developed by different cultures varied considerably. The Egyptians used a decimal system with different symbols for 1, 10, 100, 1000, etc. The Greeks used different Greek letters for the units from 1 to 9, the tens from 10 to 90, and the hundreds from 100 to 900. Other cultures developed place-value counting systems with a limited number of symbols: here the same symbol may play different roles, such as the two 3s in 3835 (referring to 3000 and 30). The Chinese used a decimal place-value system, while the Mesopotamians had a system based on 60 and the Mayans developed a system mainly based on 20.

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A Central American stela featuring Mayan head-form numbers

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PYTHAGORAS

The semi-legendary figure of Pythagoras (c.570–490BC) was born on the island of Samos, in the Aegean Sea. In his youth he studied mathematics, astronomy, philosophy and music. Possibly around 520BC, he left Samos to go to the Greek seaport of Crotona (now in Southern Italy) and formed a philosophical school, now known as the *Pythagoreans*.

The inner members of the Pythagoreans (the *mathematikoi*) apparently obeyed a strict regime, having no personal possessions and eating only vegetables (except beans); the sect was open to both men and women.

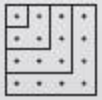
The Pythagoreans studied mathematics, astronomy and philosophy. They believed that *everything is created from whole numbers*, and that anything worthy of study can be quantified. They are said to have subdivided the mathematical sciences into four parts: *arithmetic*, *geometry*, *astronomy* and *music* (later called the *quadrivium*). These subjects, in combination with the *trivium* (the liberal arts of grammar, rhetoric and logic), comprised the 'liberal arts' – the curriculum of academies and universities over the next 2000 years.

Pythagoras, from Raphael's fresco *The School of Athens*

NUMBER PATTERNS

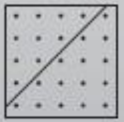
For the Pythagoreans, 'arithmetic' meant studying whole numbers, which they sometimes represented geometrically; for example, they considered *square numbers* as being formed by square patterns of dots or pebbles.

Using such pictures they could show that square numbers can be obtained by adding consecutive odd numbers, starting from 1 – for example, $16 = 1 + 3 + 5 + 7$.



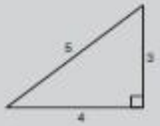
They also studied *triangular numbers*, formed by triangular patterns of dots. The first few triangular numbers are 1, 3, 6, 10, 15 and 21. Notice that $3 = 1 + 2$, $6 = 1 + 2 + 3$, $10 = 1 + 2 + 3 + 4$, etc.

Using such pictures they could show that the sum of any two consecutive triangular numbers is a square number – for example, $10 + 15 = 25$.

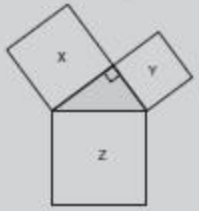


THE PYTHAGOREAN THEOREM

Important in geometry are *right-angled triangles*, where one of the angles is 90°; an example is the triangle with sides 3, 4, 5.

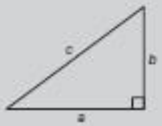


The most important result concerning them is known as the *Pythagorean theorem*, although no contemporary historical evidence links it to Pythagoras himself. Although it was known by the Mesopotamians 1000 years earlier, the Greeks were probably the first to prove it.



Geometrically, the Pythagorean theorem says that if we take a right-angled triangle and draw squares on each side of it, then *The area of the square on the longest side is equal to the sum of the areas of the squares on the other two sides*
— that is, (area of Z) = (area of X) + (area of Y)

So, for a right-angled triangle with sides of lengths a, b and c (where c is the length of the longest side), we have $a^2 + b^2 = c^2$
— for example, for the triangle with sides 3, 4, 5,
$3^2 + 4^2 = 9 + 16 = 25 = 5^2$.
Other examples are the right-angled triangles with sides 5, 12, 13 and 8, 15, 17.

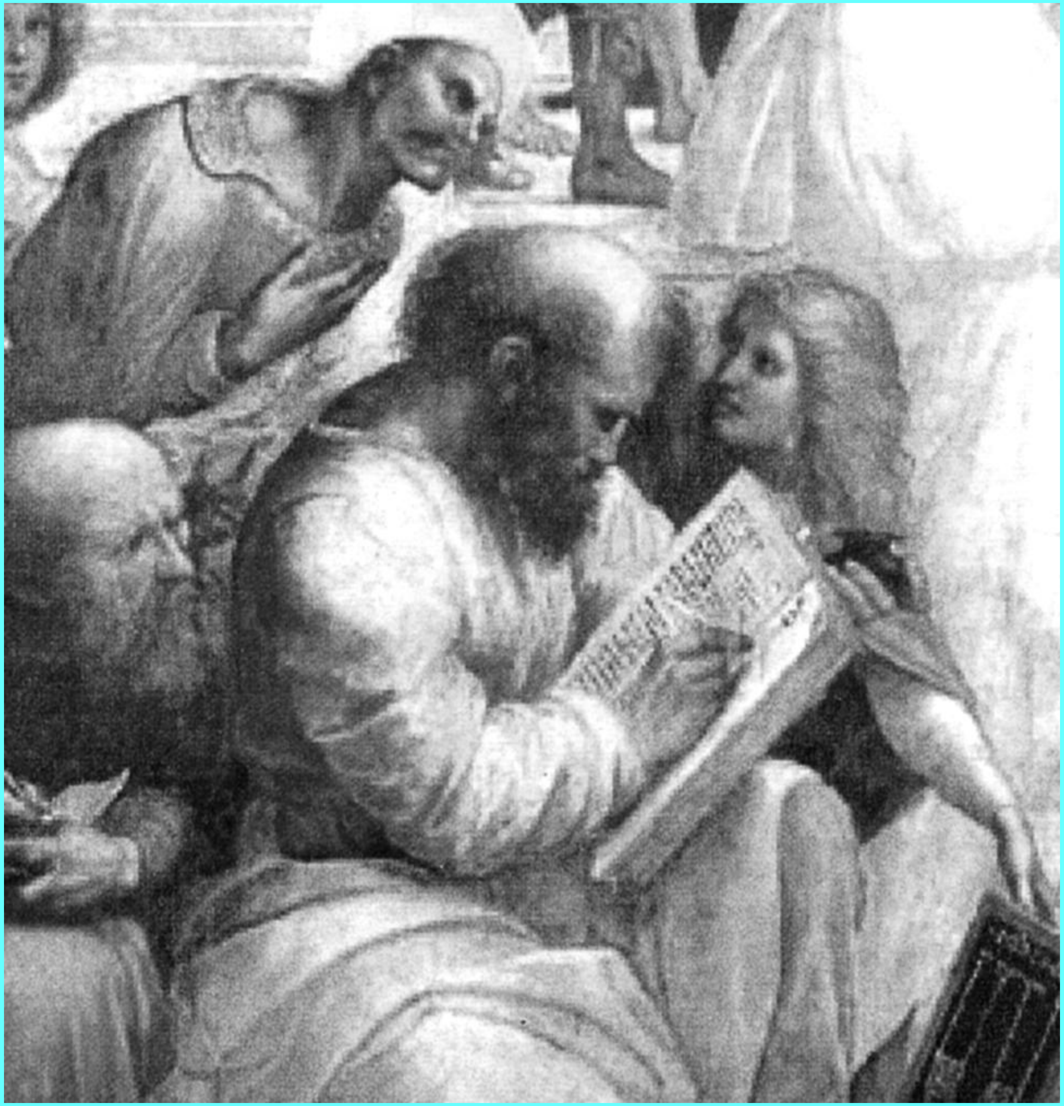


MATHEMATICS AND MUSIC

The Pythagoreans also experimented with music – in particular, linking certain musical intervals to simple ratios between small numbers.

It is likely that they discovered these ratios by plucking strings of different lengths and comparing the notes produced; for example, the harmonious interval of an *octave* results from halving the length of a string, giving a frequency ratio of 2 to 1, while another harmonious interval, a *perfect fifth*, results from stopping a string at two-thirds of its length, giving a ratio of 3 to 2.

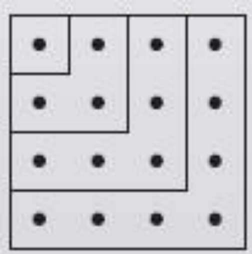
A 1492 woodcut featuring some of Pythagoras's musical experiments.



NUMBER PATTERNS

For the Pythagoreans, 'arithmetic' meant studying whole numbers, which they sometimes represented geometrically; for example, they considered *square numbers* as being formed by square patterns of dots or pebbles.

Using such pictures they could show that square numbers can be obtained by adding consecutive odd numbers, starting from 1 – for example, $16 = 1 + 3 + 5 + 7$.

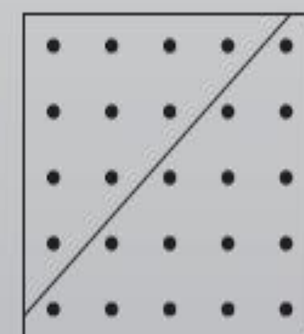


They also studied *triangular numbers*, formed by triangular patterns of dots. The first few triangular numbers are

1, 3, 6, 10, 15 and 21.

Notice that $3 = 1 + 2$, $6 = 1 + 2 + 3$, $10 = 1 + 2 + 3 + 4$, etc.

Using such pictures they could show that the sum of any two consecutive triangular numbers is a square number – for example, $10 + 15 = 25$.




IV BAL
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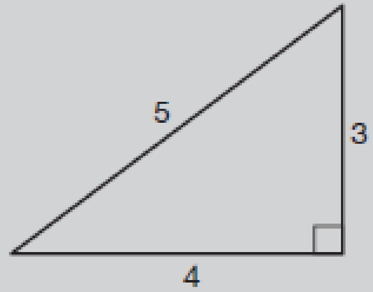
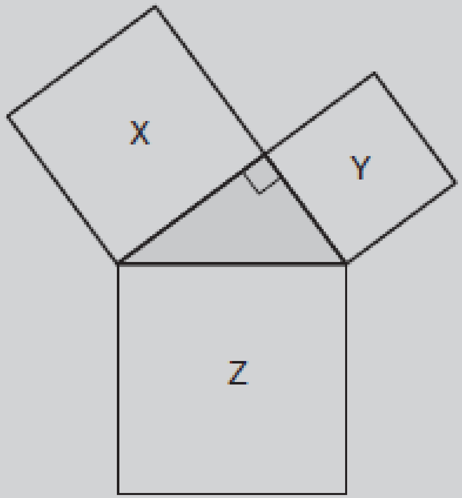
THE PYTHAGOREAN THEOREM

Important in geometry are *right-angled triangles,* where one of the angles is 90°; an example is the triangle with sides 3, 4, 5.



The most important result concerning them is known as the *Pythagorean theorem,* although no contemporary historical evidence links it to Pythagoras himself. Although it was known by the Mesopotamians 1000 years earlier, the Greeks were probably the first to prove it.

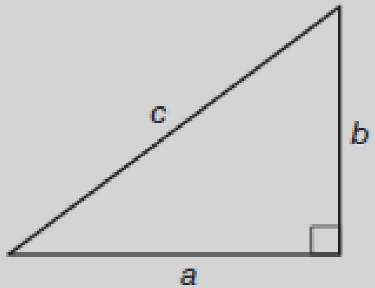


Geometrically, the Pythagorean theorem says that if we take a right-angled triangle and draw squares on each side of it, then *The area of the square on the longest side is equal to the sum of the areas of the squares on the other two sides*
— that is, (area of Z) = (area of X) + (area of Y)

So, for a right-angled triangle with sides of lengths a, b and c (where c is the length of the longest side), we have $a^2 + b^2 = c^2$
— for example, for the triangle with sides 3, 4, 5,

$3^2 + 4^2 = 9 + 16 = 25 = 5^2$.

Other examples are the right-angled triangles with sides 5, 12, 13 and 8, 15, 17.

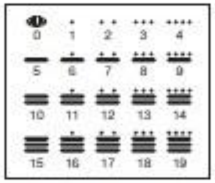


THE MAYANS

One of the most interesting counting systems is that of the Mayans of Central America, used between their most productive years from AD300 to 1000. The Mayans were situated over a large area centred on present-day Guatemala and Belize and extending from the Yucatan peninsula of Mexico in the north to Honduras in the south. Most of their calculations involved the construction of calendars, for which they developed a place-value system based mainly on the number 20.

Our knowledge of the Mayan counting system and of their calendars is derived mainly from writings on the walls of caves and ruins, hieroglyphic inscriptions on carved pillars (*stelae*), and a handful of painted manuscripts (*codices*). The codices were intended to guide Mayan priests in ritual ceremonies involving hunting, planting and rainmaking, but many codices were destroyed by the Spanish conquerors who arrived in this area after the year 1500.

The most notable of the surviving codices is the beautiful *Dresden codex*, dating from about 1200. It is painted in colour on a long strip of glazed fig-tree bark and contains many examples of Mayan numbers.



The Mayan symbols for the numbers from 0 to 19.

THE MAYAN NUMBER SYSTEM

The Mayan counting system was a place-value system with a dot to represent 1, a line to represent 5, and a special symbol (a shell) to represent 0. These were combined to give the numbers from 0 to 19.

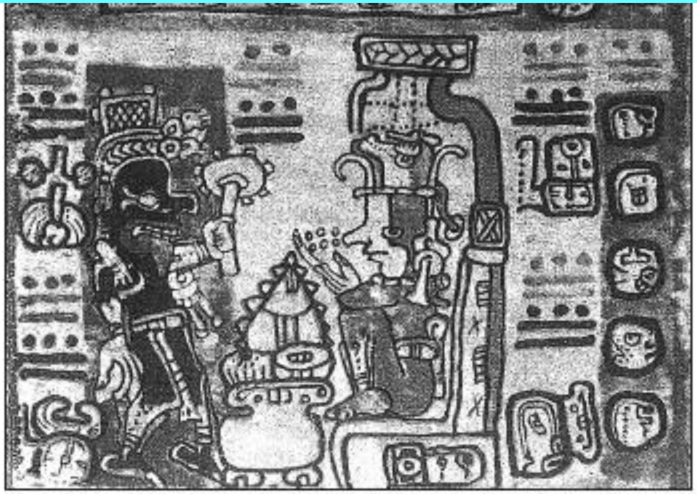
To obtain larger numbers they combined these numbers, writing them vertically; for example, the illustrated codex depicts the symbol for 12 above the symbol for 13 — this represents the number 253 (12 twenties + 13).

THE HEAD FORM

An interesting feature of the Mayan numerals is that there was an alternative form for each number, a pictorial form or glyph known as the *head-form*, with a pictorial representation of the head of a man, animal, bird or deity. These pictures appear on various pillars: below are the head-forms of various numbers.

THE MAYAN CALENDARS

In order to keep track of the passage of time, the Mayans employed two types of calendar, with 260 days and 365 days.



Part of a Mayan codex

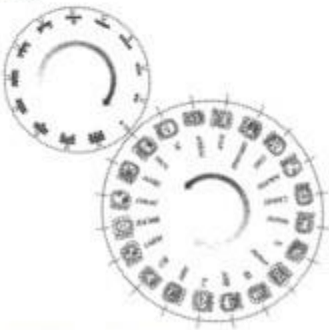
The 260-day calendar was a ritual one, used for forecasting and known as the *tzolkin*, or 'sacred calendar'. It consisted of thirteen months of twenty days. Each day combined a month-number (from 1 to 13) with one of twenty day-pictures named after deities (such as Imix, Ik and Akbal). These two systems then intermeshed, as illustrated — for example, the day 1 Imix was not followed by 2 Imix and 3 Imix, but by 2 Ik and 3 Akbal, etc. — eventually yielding a cycle of 13 × 20 = 260 days.

For their 365-day calendar, they modified their number system to take account of the number of days in the calendar year. To do so, they introduced an 18 into their 20-based system (since 18 × 20 = 360), and then added five extra 'inauspicious' days to make up the full 365 days. So their counting system was based on the following scheme:

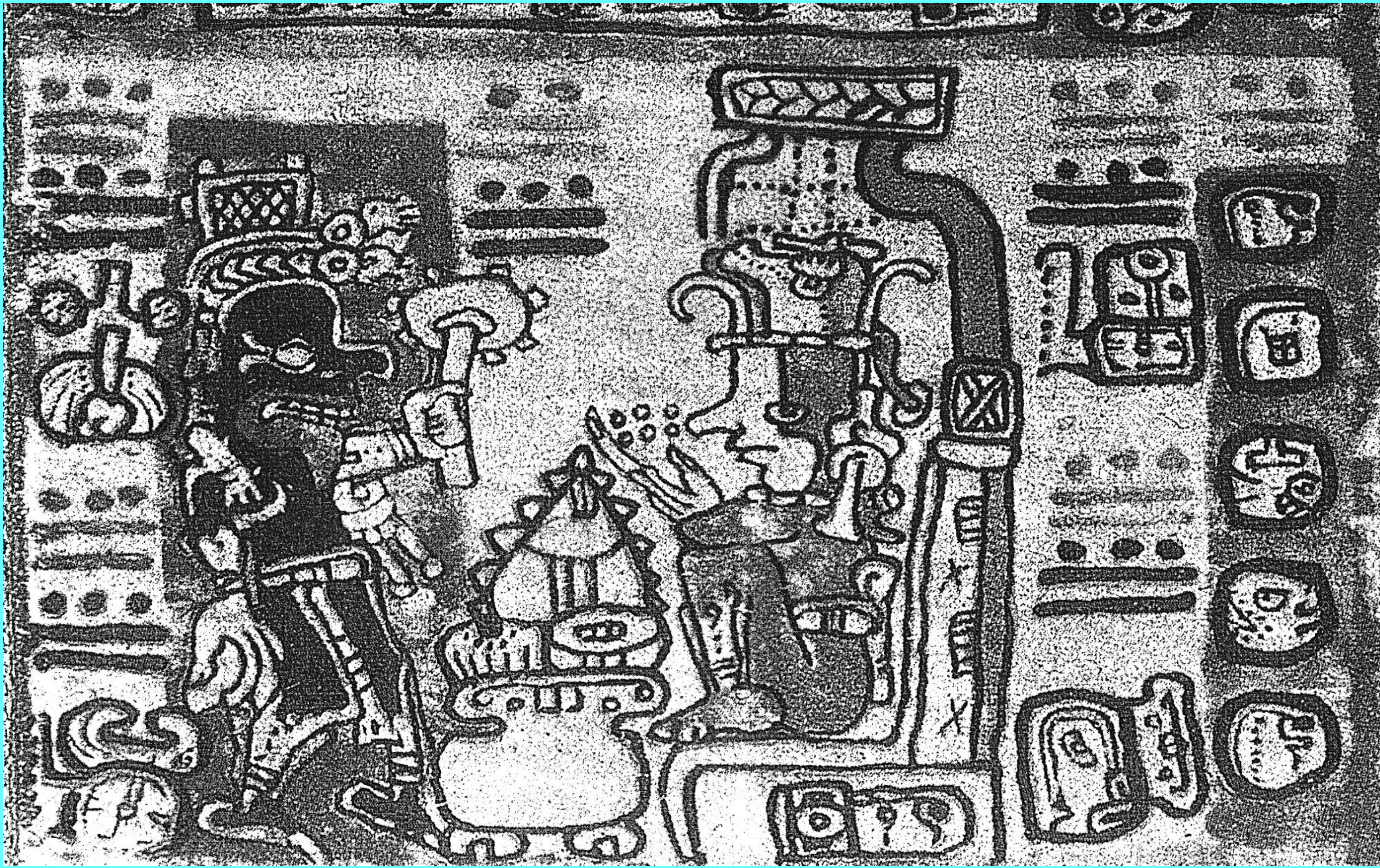
1 kin = 1 day
20 kins = 1 uinal = 20 days
18 uinals = 1 tun = 360 days
20 tuns = 1 katun = 7200 days
20 katuns = 1 baktun = 144,000 days,

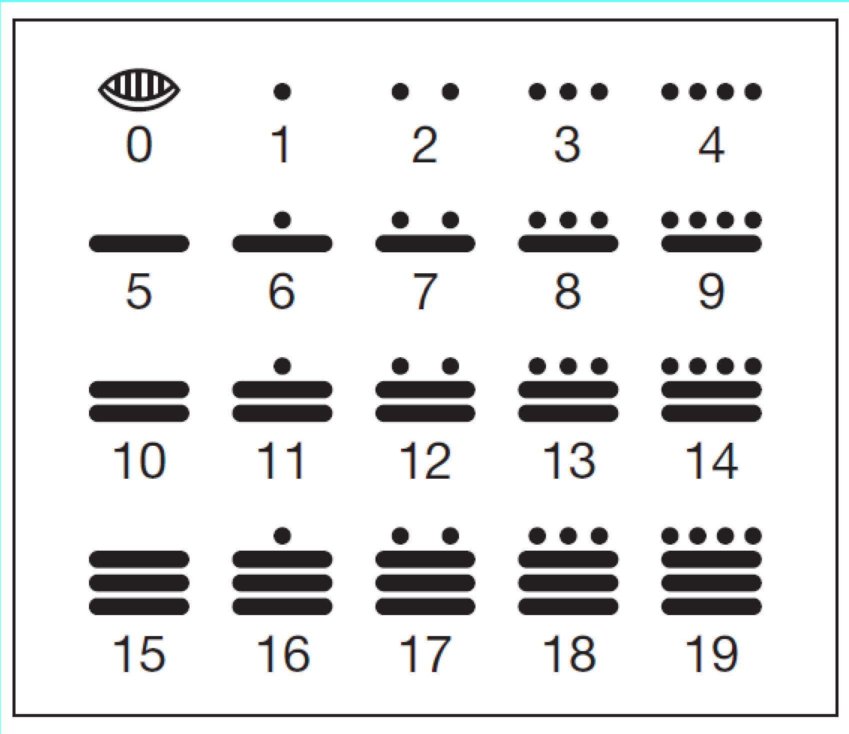
and so on. They had no problems in calculating with such large numbers.

These two calendars operated independently, and were also combined to give a *calendar round*, in which the number of days was the least common multiple of 260 and 365, which is 18,980 days, or 52 calendar years. These periods of 52 years were then packaged into even longer time periods. The longest time period used by the Mayans was the *long count* calendar of 5125 years.



The Mayan 260-day calendar

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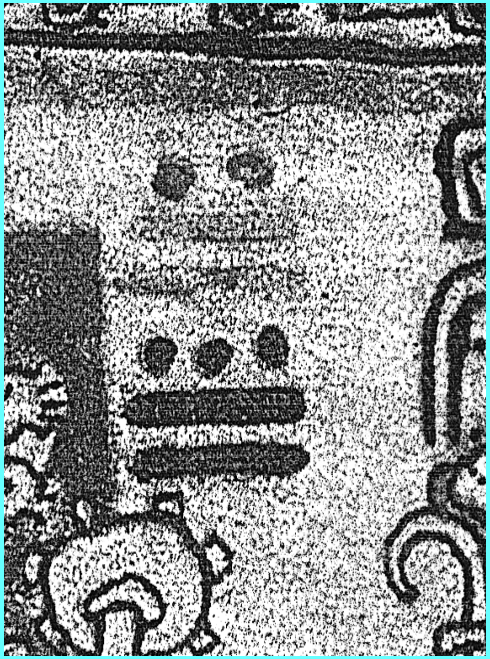

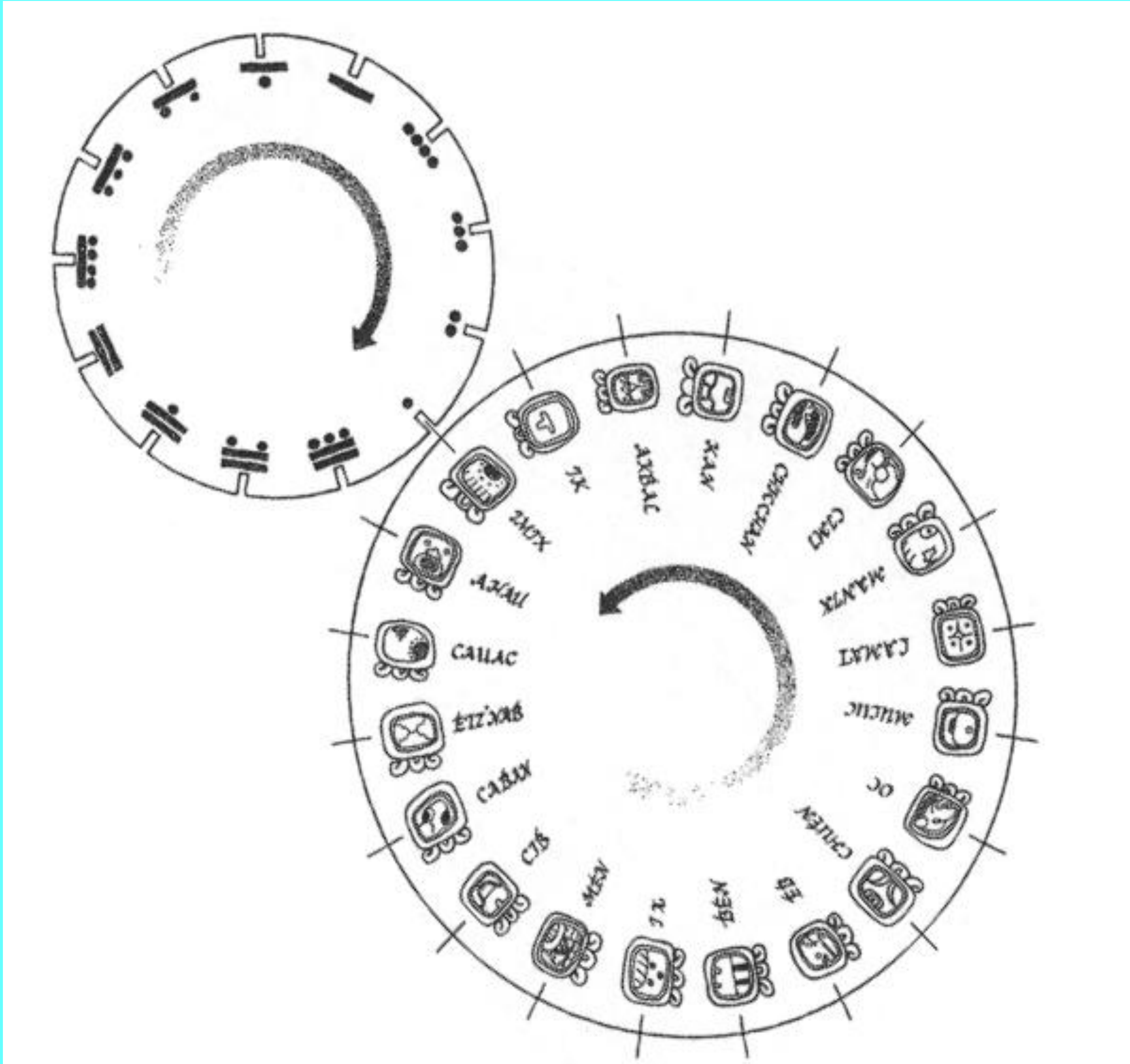

IK
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IMIX

CHAPTER 2

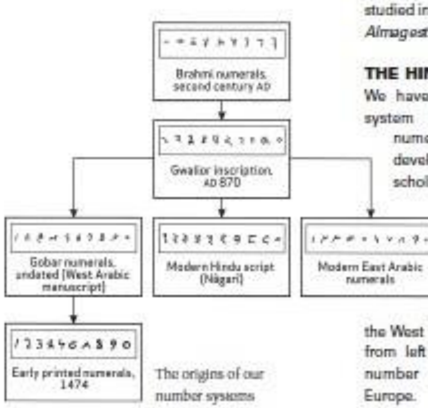
EARLY EUROPEAN MATHEMATICS

The revival of mathematical learning during the Middle Ages was largely due to three factors:

- **the translation of Arabic classical texts into Latin during the 12th and 13th centuries**
- **the establishment of the earliest European universities**
- **the invention of printing**

The first of these made the works of Euclid, Archimedes and other Greek writers available to European scholars, the second enabled groups of like-minded scholars to meet and discourse on matters of common interest, while the last enabled scholarly works to be available at modest cost to the general populace in their own language.

The first European university was founded in Bologna in 1088, and Paris and Oxford followed shortly after. The curriculum was in two parts. The first part, studied for four years by those aspiring to a Bachelor's degree, was based on the ancient 'trivium' of grammar, rhetoric and logic (usually Aristotelian). The second part, leading to a Master's degree, was based on the 'quadrivium', the Greek mathematical arts of arithmetic, geometry, astronomy and music; the works studied included Euclid's *Elements* and Ptolemy's *Almagest*.



The origins of our number systems

THE HINDU–ARABIC NUMERALS

We have seen how the decimal place-value system represented by the Hindu–Arabic numerals first arose in India and was later developed by al-Khwarizmi and other Islamic scholars working in Baghdad and elsewhere. Gradually the numerals diverged into three separate types – the modern Hindu script, the East Arabic numerals (written from right to left), still found today in the countries of the Middle East, and the West Arabic numerals 1 to 9 and 0 (written from left to right) that eventually became the number system used throughout Western Europe.

Viewing the heavens with a joynt rule

But it took many centuries for the Western form of the Hindu–Arabic numerals to become fully established. They were certainly more convenient to calculate with than Roman numerals, but for practical use most people continued to use an abacus.

As time progressed the situation improved with the publication of influential books that promoted them, such as those by Fibonacci (in Latin), Pacioli (in Italian) and Recorde (in English). By the time that printed books had become widely available, the Hindu–Arabic numerals were in general use.

THE AGE OF DISCOVERY

The spirit of enquiry and inventiveness of the Middle Ages and the Renaissance led people to adopt a more critical view to ideas that had been accepted for centuries. It showed itself in many ways:

- the voyages of discovery to unknown lands
- the development and invention of scientific and mathematical instruments for a variety of purposes
- the use of geometrical perspective in painting and other visual arts
- the solution of cubic and quartic equations
- the development and standardization of mathematical terminology and notation
- the revolutionary approach to planetary motion
- the rediscovery and reinterpretation of classical texts
- the development of mechanics
- the removal of algebra from its dependence on geometry.

These all contributed to the development of a view that the universe is a book written in the language of mathematics. As instruments became ever more sophisticated, mathematics for practical purposes increased – particularly in navigation, map-making, astronomy and warfare.

FIBONACCI

Leonardo of Pisa (c.1170–1240), known since the 19th century as Fibonacci (son of Bonaccio), is remembered mainly for his *Liber Abaci* (Book of Calculation), which he used to popularize the Hindu-Arabic numerals, and for a number sequence named after him. His work was crucial in bringing Arabic mathematics to wider recognition in Western Europe.

Fibonacci was born in Pisa. After travelling widely throughout the Mediterranean, he returned home and wrote works expanding on what he had learned, to help his countrymen deal with calculation and commerce.

THE *LIBER ABACI*

Most of our knowledge about Fibonacci comes from the prologue of his influential book *Liber Abaci*. The first edition of this book appeared in 1202. It covers four main areas starting with the use of Hindu-Arabic numerals in calculation and then using them for the mathematics needed in business. The largest part of the book deals with recreational mathematical problems, finishing with operations on roots and a little geometry.

Leonardo Fibonacci

PROBLEMS FROM THE *LIBER ABACI*

Fibonacci's *Liber Abaci* contains a wide range of mathematical problems, including the following three that may be similar to ones you remember from your school days!

There is a tree, $\frac{1}{4}$ and $\frac{1}{3}$ of which lie below ground. If the part below ground is 21 palmi, how tall is the tree?

If a lion can eat a sheep in 4 hours, a leopard can eat it in 5 hours, and a bear can eat it in 6 hours, how long would they take eating it together?

I can buy 3 sparrows for a penny, 2 turtle-doves for a penny or doves for 2 pence each. If I spent 30 pence buying 30 birds and bought at least one bird of each kind, how many of each kind did I buy?

Another problem involves adding powers of 7:

7 old women are going to Rome;
each has 7 mules; each mule carries 7 sacks;
each sack contains 7 loaves; each loaf has
7 knives; each knife has 7 sheaths;
what is the total number of things?

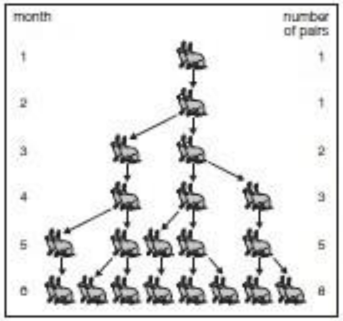
This is reminiscent of a problem from the Egyptian Rhind papyrus:

houses 7; cats 49; mice 343; spelt 2401; hekat 16,807. Total 19,607

and also of the more recent nursery rhyme:

As I was going to St Ives I met a man with 7 wives ... Kits, cats, sacks and wives,
How many were going to St Ives?

Such examples dramatically illustrate the fact that the same mathematical idea can resurface in different guises over thousands of years.



THE RABBITS PROBLEM

The most famous problem in the *Liber Abaci* is the problem of the rabbits:

A farmer has a pair of baby rabbits. Rabbits take two months to reach maturity and then give birth to another pair each month. How many pairs of rabbits are there after a year?

To solve this, we note that:

- In months 1 and 2 the farmer has only the original pair,
- In month 3, a new pair arrives, so he now has two pairs,
- In month 4, the original pair produces another pair, and the new pair has not yet produced, so he now has three pairs,
- In month 5, the original pair and the new pair both produce another pair; and so on.

The result of the problem is that the number of rabbits in each month follows the so-called *Fibonacci sequence*:

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ...,

in which each successive number (after the first two) is the sum of the previous two; for example, 89 = 34 + 55. The answer to the problem is the 12th number, which is 144.

SPIRALS AND THE GOLDEN NUMBER

The ratios of successive terms of the Fibonacci sequence are

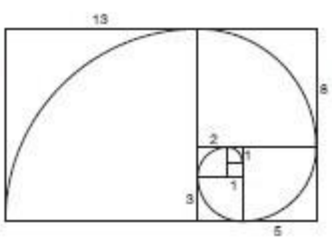
$\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \ldots$.

These tend to the 'golden number'

$\varphi = \frac{1}{2}(1 + \sqrt{5}) = 1.618\ldots,$

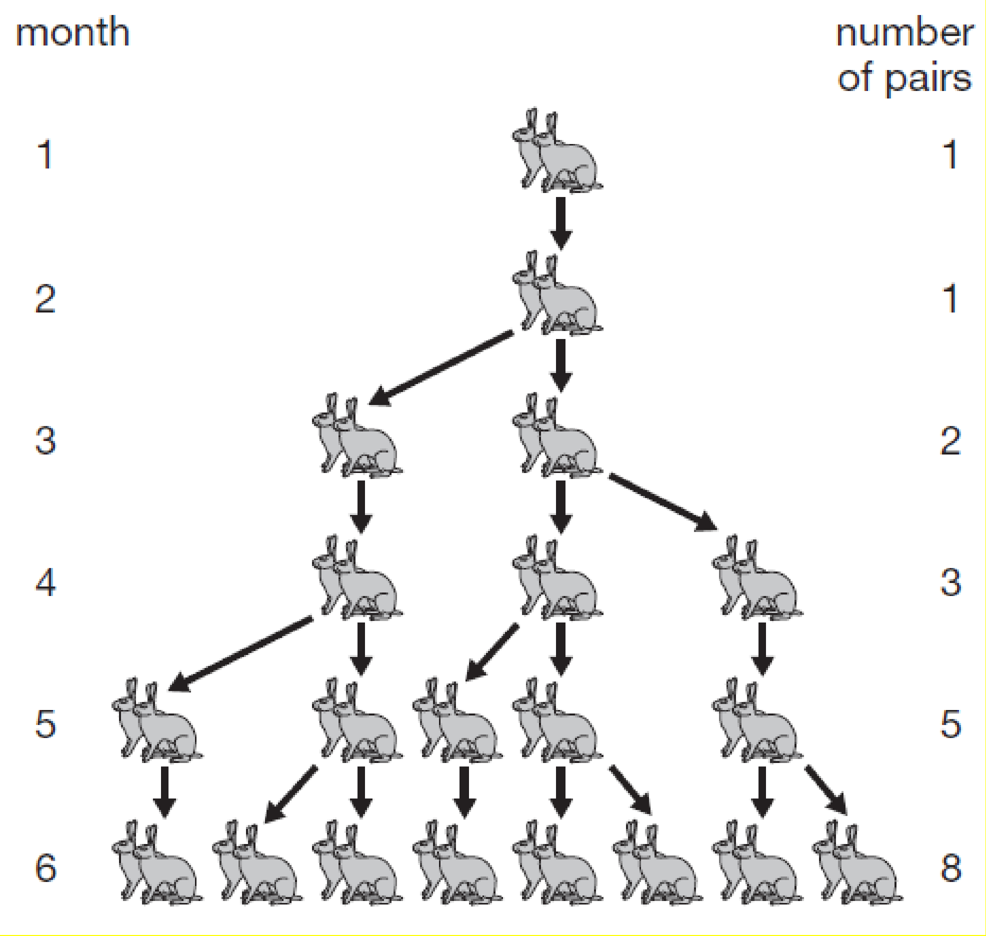
which has remarkable and pleasing properties: for example, to find its square we add 1 ($\varphi^2 = 2.618\ldots$), and to find its reciprocal we subtract 1 ($1/\varphi = 0.618\ldots$).

A rectangle whose sides are in the ratio φ to 1 is often considered to have the most pleasing shape — neither too thin, nor too fat. The following picture shows how the Fibonacci numbers can be arranged so as to give rise to a spiral pattern; further rectangles can be added at will.



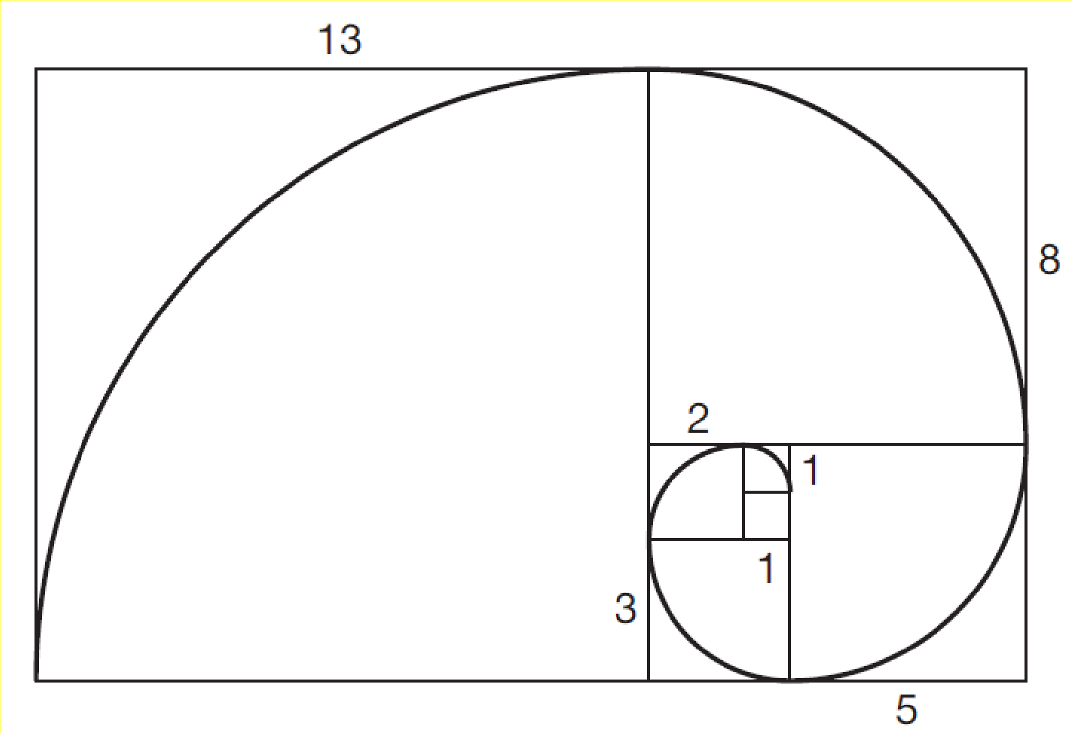
Similar spirals occur through nature — on a nautilus shell and in the pattern of seeds in a sunflower — for example, the number of seeds in such a spiral pattern is often 34, 55 or 89, all of which are Fibonacci numbers.

A farmer has a pair of baby rabbits. Rabbits take two months to reach maturity and then give birth to another pair each month. How many pairs of rabbits are there after a year?



Fibonacci sequence: 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, …,

Fibonacci sequence:
1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, …
Golden number: 1.618...



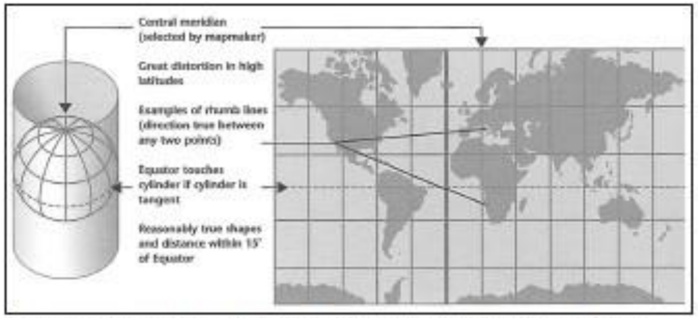
MERCATOR

The Flemish mapmaker and cartographer Gerardus Mercator (1512–1594) is mainly remembered for the *Mercator projection*, which proved to be extremely useful for navigators. This was a projection of the spherical earth on to a flat sheet of paper so that the lines of latitude and longitude, as well as the paths of constant compass bearings, were represented by straight lines. Mercator also coined the word 'atlas' for a collection of maps.

A major concern in the 16th century, an active period for voyages of trade, discovery and exploration, was to develop mathematical methods and maps to aid navigation.

The basic problem was that if you were on a ship in the middle of the ocean, how could you tell where you were, and in which direction you should sail to get to your destination? You could find your latitude by using astronomical instruments to locate the sun and stars. It was more problematic, however, to find your longitude, and a satisfactory method was not available until the end of the 18th century.

Mercator (left) and Jodocus Hondius, who published Mercator's work, on the title page of an edition of the *Mercator–Hondius Atlas*, surrounded by the tools of the cartographer.



The Mercator map projection is a cylindrical one that distorts the size and shape of areas far from the equator.

Using a magnetic compass, mariners could steer a line of constant compass bearing (a *rhumb line*); such a path crosses all lines of longitude at the same angle. However, as the 16th-century Portuguese mathematician and cosmographer Pedro Nunes discovered, a rhumb line spirals towards the pole.

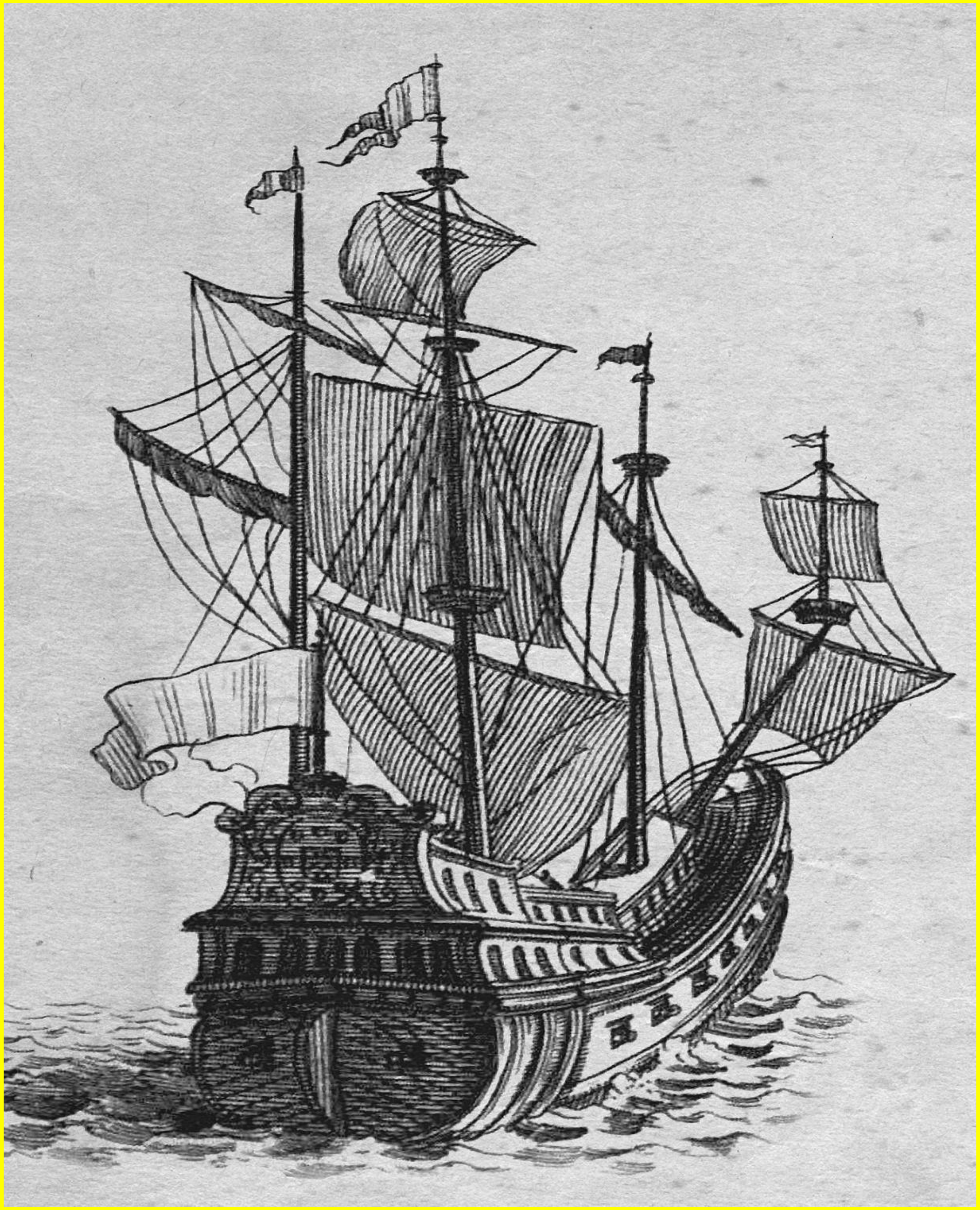
MERCATOR'S PROJECTION

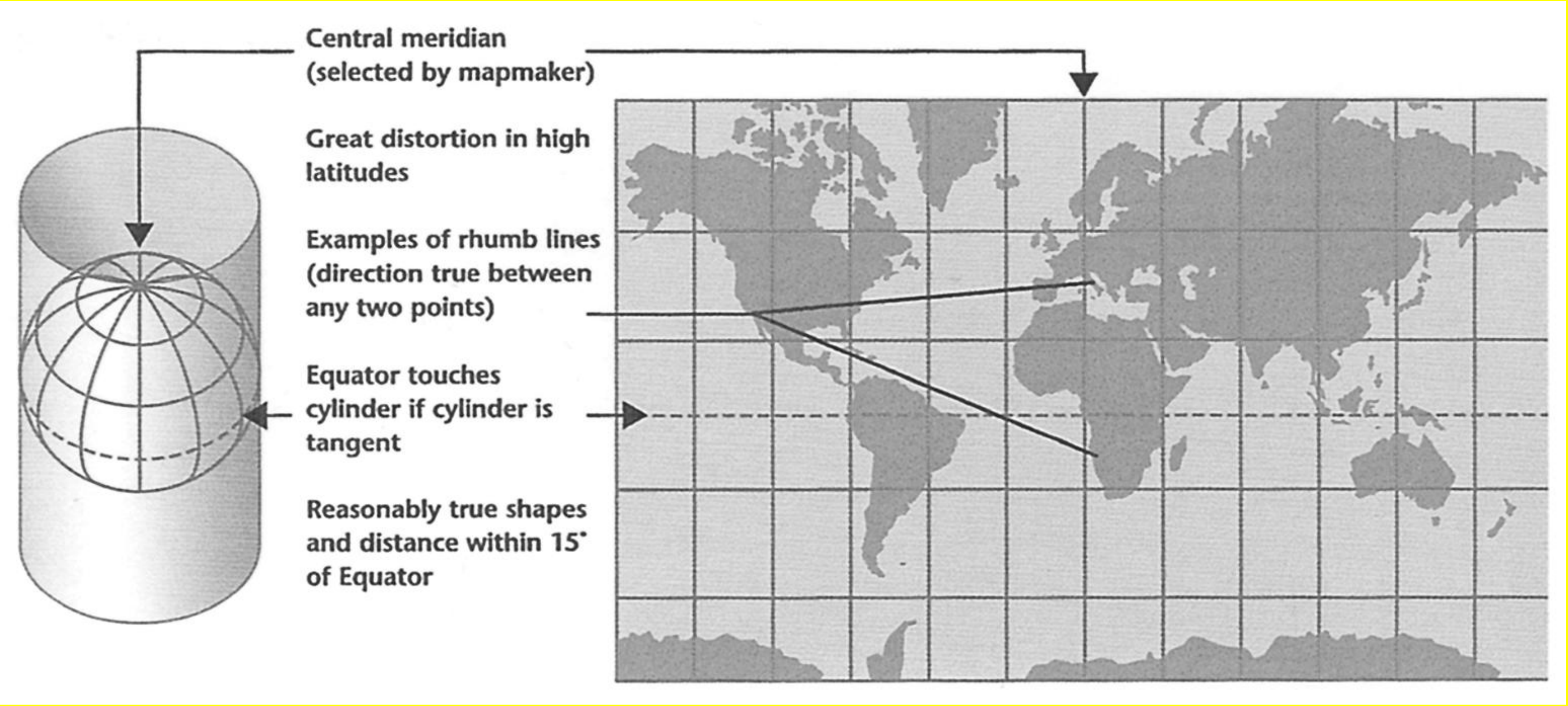
The advantages of Mercator's projection were that it represented lines of latitude and longitude as straight lines meeting at right angles, and that it also represented rhumb lines as straight lines on the map. If a navigator knew the latitudes and longitudes of the ship's current position and the destination, then the line joining the two places could be found on the map. This enabled the appropriate constant compass bearing to be determined, but would not give the shortest distance to the destination.

Mercator's projection marked a step forward for navigation

Mercator obtained his projection by projecting the sphere on to a cylinder which was then unrolled and stretched vertically so that the rhumb lines became straight; the amount of stretching varies with different latitudes, and increases the further north one goes. This has the consequence of exaggerating areas that are far from the equator: for example, Alaska appears as large as Brazil while Brazil is actually five times bigger, and Finland has a greater north-south extent than India which is incorrect.

Mercator did not present the mathematical basis for his projection. This was first given by Edward Wright in his influential *Certaine Errors in Navigation* of 1599. Wright also gave accurate mathematical tables for its construction. But it was Thomas Harriot (1560–1621) who eventually discovered the fundamental mathematical formula underlying Mercator's projection.




Central meridian
(selected by mapmaker)
Great distortion in high
latitudes
Examples of rhumb lines
(direction true between
any two points)
Equator touches
cylinder if cylinder is
tangent
Reasonably true shapes
and distance within 15°
of Equator


GERARDUS MERCATOR NATUS
RUPELMUNDÆ III NON. MARTII ANNO
CIƆIƆXII: VIXIT ANN. LXXXII. M. VIII. D.
XXVI: DENATUS IV NON. DECEMBRIS
ANNO CIƆIƆXCIV.
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PAGO FLANDRIÆ DICTO WACKENE XVI
KALEND. NOVEMBRIS ANNO CIƆIƆLXIII:
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US XIV KAL. MARTII ANNO CIƆIƆCXII.

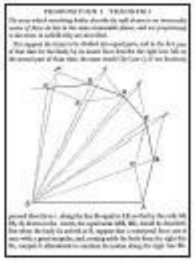
CHAPTER 3

AWAKENING AND ENLIGHTENMENT

The 17th and 18th centuries witnessed the beginnings of modern mathematics. New areas of the subject came into being – notably, analytic geometry and the calculus – while others, such as number theory, were reborn or took on a new lease of life. Fundamental problems, such as that of determining the orbits of the heavenly bodies, were solved or investigated with novel techniques.

It was the age of Newton in England, Descartes and Pascal in France, and Leibniz in Germany, followed by a succession of Continental 'greats': the Bernoulli brothers, Euler, Lagrange and Laplace.

It was also the age of gatherings – the formation of national scientific societies, such as London's Royal Society and the Academy of Sciences in Paris, and the founding of scholarly institutions such as the St Petersburg Academy and the Academy of Sciences in Berlin.



The area swept out by a moving body: Newton's use of geometry (right) contrasts with Laplace's analytical approach (far right)

CALCULUS AND DISCOVERY

Initially, the problems that mathematicians solved were geometrical, as were their answers, although the techniques they used (including the calculus) were not necessarily of this kind, being seen as methods of proceeding from a geometrical problem to a geometrical answer. The 18th century then led to a new conception of mathematics, with its most striking characteristic being its algebraic appearance.

The objects of mathematics were now described by formulas with symbols for variables and constants. A main reason for doing so was that the machinery of the calculus could then be applied both to them and to practical situations. This hastened the development of new mathematical descriptions and techniques, such as in the emerging area of *differential equations*.

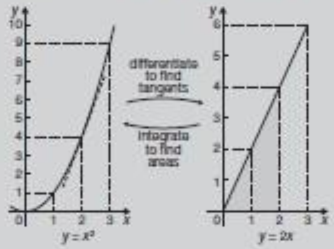
> If we project the body m, on the plane of x and y, the differential $(xdy - ydx)/2$, will represent the area which the radius vector, drawn from the origin of the coordinates to the projection of m, describes in the time dt, consequently the sum of the areas, multiplied respectively by the masses of the bodies, is proportional to the element of time, from which it follows that in a finite time, it is proportional to the time. It is this which constitutes the principle of the conservation of areas.

WHAT IS THE CALCULUS?

The calculus is made up from two seemingly unrelated strands, now called *differentiation* and *integration*. Differentiation is concerned with how fast things move or change, and is used in the finding of velocities and tangents to curves. Integration is used to find areas of shapes in two-dimensional space or volumes in three dimensions.

As the 17th century progressed, it was gradually realized that these two strands are intimately related. As both Newton and Leibniz explained, they are inverse processes — if we follow either by the other, we return to our starting point.

However, Newton and Leibniz had different motivations, with Newton focusing on motion and Leibniz concerned with tangents and areas.



Differentiation and integration

This shift towards the algebraic type of description also led to a good way of discovering new objects. Books were written in the algebraic style, and mathematicians formulated, thought about and solved problems in this way. Increasingly, algebra came to be seen as a logical language suitable for the investigation of all the sciences.

Mechanics and astronomy were the main areas of practical investigation. They both applied the calculus to functions of more than one variable, such as

$$u(x, y) = x^6 + x^3y^2 + y^4;$$

here, $u(x, y)$ can be thought of as the height of a surface above the point with coordinates (x, y) in the plane.

The equations that arose were called *partial differential equations*, because they involved 'partial differentiation'. The partial derivative $\partial u/\partial x$ is the rate of change of u in the x-direction, while the partial derivative $\partial u/\partial y$ is the rate of change of u in the y-direction.

Louis XIV visits the Paris Academy of Sciences, 1671

NAPIER AND BRIGGS

In 1614 John Napier (1550–1617), 8th Laird of Merchiston (near Edinburgh), introduced logarithms as an aid to mathematical calculation, designed to replace lengthy computations involving multiplications and divisions by simpler ones using additions and subtractions. Being awkward to use, they were soon supplanted by others due to Henry Briggs (1561–1630), and their use proved an enormous boon to navigators and astronomers.

Early ideas of logarithms had appeared around the year 1500. Nicolas Chuquet and Michael Stifel listed the first few powers of 2 and noticed that to multiply two of them one simply adds their exponents — so, to multiply 16 and 128 we calculate:

$16 \times 128 = 2^4 \times 2^7 = 2^{4+7} = 2^{11} = 2048,$

and write $\log_2 2048 = 11$.

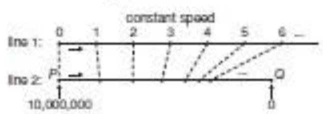
John Napier

The title page of Napier's logarithms

NAPIER'S LOGARITHMS

The idea was not developed until Napier produced his *Mirifici Logarithmorum Canonis Descriptio* (A Description of the Admirable Table of Logarithms). This contains extensive tables of logarithms of the sines and tangents of all the angles from 0 to 90 degrees, in steps of 1 minute; Napier's use of these logarithms arose because he intended them to be used as an aid to calculation by navigators and astronomers.

Napier's logarithms are not the ones we use now. He then considered two points moving along straight lines. The first travels at constant speed for ever, while the second, representing its logarithm, moves from P along a finite line PQ in such a way that its speed at each point is proportional to the distance it still has to travel. In order to avoid the use of fractions he multiplied all his numbers by ten million.

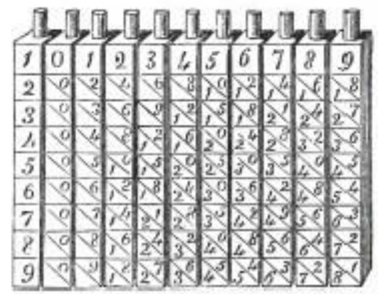


It follows from Napier's definition that the logarithm of 10,000,000 is 0. It can also be shown that, with his definition,

$\log ab = \log a + \log b - \log 1,$

for any numbers a and b; here, $\log 1$ has the cumbersome value of 161,180,956, which has to be subtracted in any calculation.

Napier also constructed from ivory a set of rods with numbers marked on them (now called *Napier's bones* or *rods*), which could be used to multiply numbers mechanically.



Napier's rods

HENRY BRIGGS

Shortly after their invention, Henry Briggs, first Gresham Professor of Geometry in London, heard about logarithms and enthused:

> *[John Napier] set my Head and hands a Work with his new and remarkable logarithms.*
> *I never saw a Book which pleased me better or made me more wonder.*

Briggs realized that Napier's logarithms were cumbersome, and felt that they could be redefined so as to avoid having to subtract the term log 1:

> *I myself, when expounding this doctrine to my auditors in Gresham College, remarked that it would be much more convenient that 0 should be kept for the logarithm of the whole sine.*

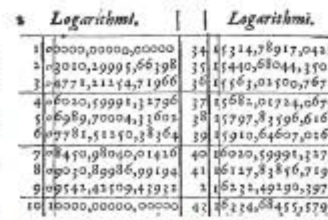
A related difficulty was that multiplication by 10 involved the addition of log 10 = 23,025,842.

Briggs twice visited Edinburgh to stay with Napier and sort out the difficulties. On returning to London, he devised a new form of logarithms, his logs to base 10, written $\log_{10}$, in which $\log_{10} 1 = 0$ and $\log_{10} 10 = 1$: to multiply two numbers one then simply adds their logarithms:

$\log_{10} ab = \log_{10} a + \log_{10} b;$

in general, if $y = 10^x$, then $\log_{10} y = x$. In 1617 he published these in a small printed pamphlet, *Logarithmorum Chilias Prima* (The First Thousand Logarithms).

In 1624, after he had left London to become the first Savilian Professor of Geometry in Oxford, Briggs followed this with an extensive collection of logarithms to base 10 of the integers from 1 to 20,000 and 90,000 to 100,000, all calculated by hand to fourteen decimal places. The gap in his tables (from 20,000 to 90,000) was soon filled in, by the Dutch mathematician Adriaan Vlacq in 1628.



	Logarithmi.		Logarithmi.
1	00000,00000,00000	34	15314,78917,04226
2	03010,29995,66398	35	15440,68044,35028
3	04771,21254,71966	36	15563,02500,76729
4	06020,59991,32796	37	15682,01724,06700
5	06989,70004,33602	38	15797,83596,61681
6	07781,51250,38364	39	15910,64607,02650
7	08450,98040,01426	40	16020,59991,32796
8	09030,89986,99194	41	16127,83856,71974
9	09542,42509,43933	42	16232,49290,39790
10	10000,00000,00000	43	16334,68455,57959

Some of Henry Briggs's logarithms

The invention of logarithms quickly led to the development of mathematical instruments based on a logarithmic scale. Most notable among these was the *slide rule*, versions of which first appeared around 1630 and were widely used for over 300 years until the advent of the pocket calculator in the 1970s.

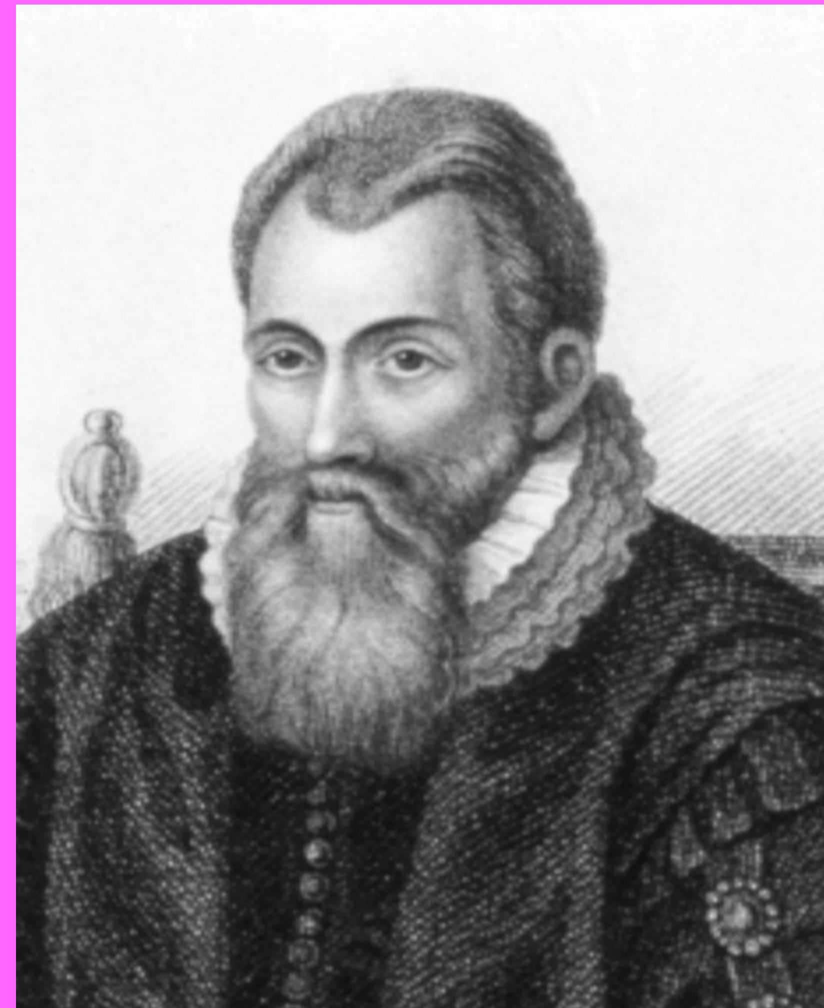
MIRIFICI

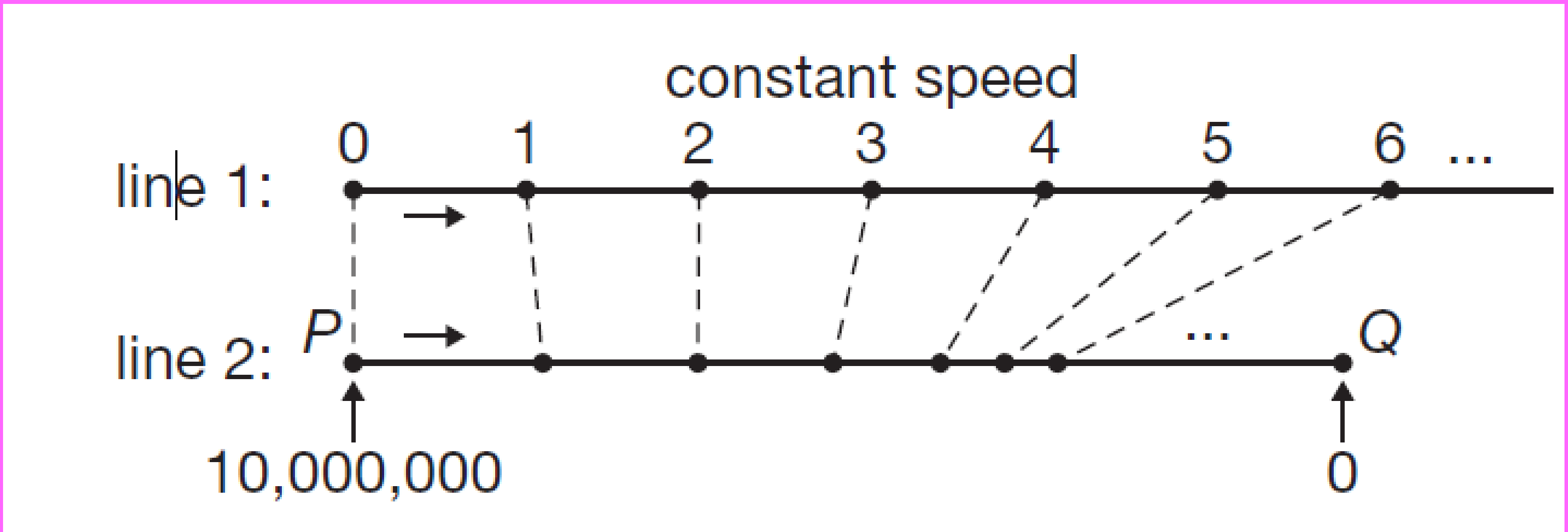
Logarithmorum
Canonis descriptio,

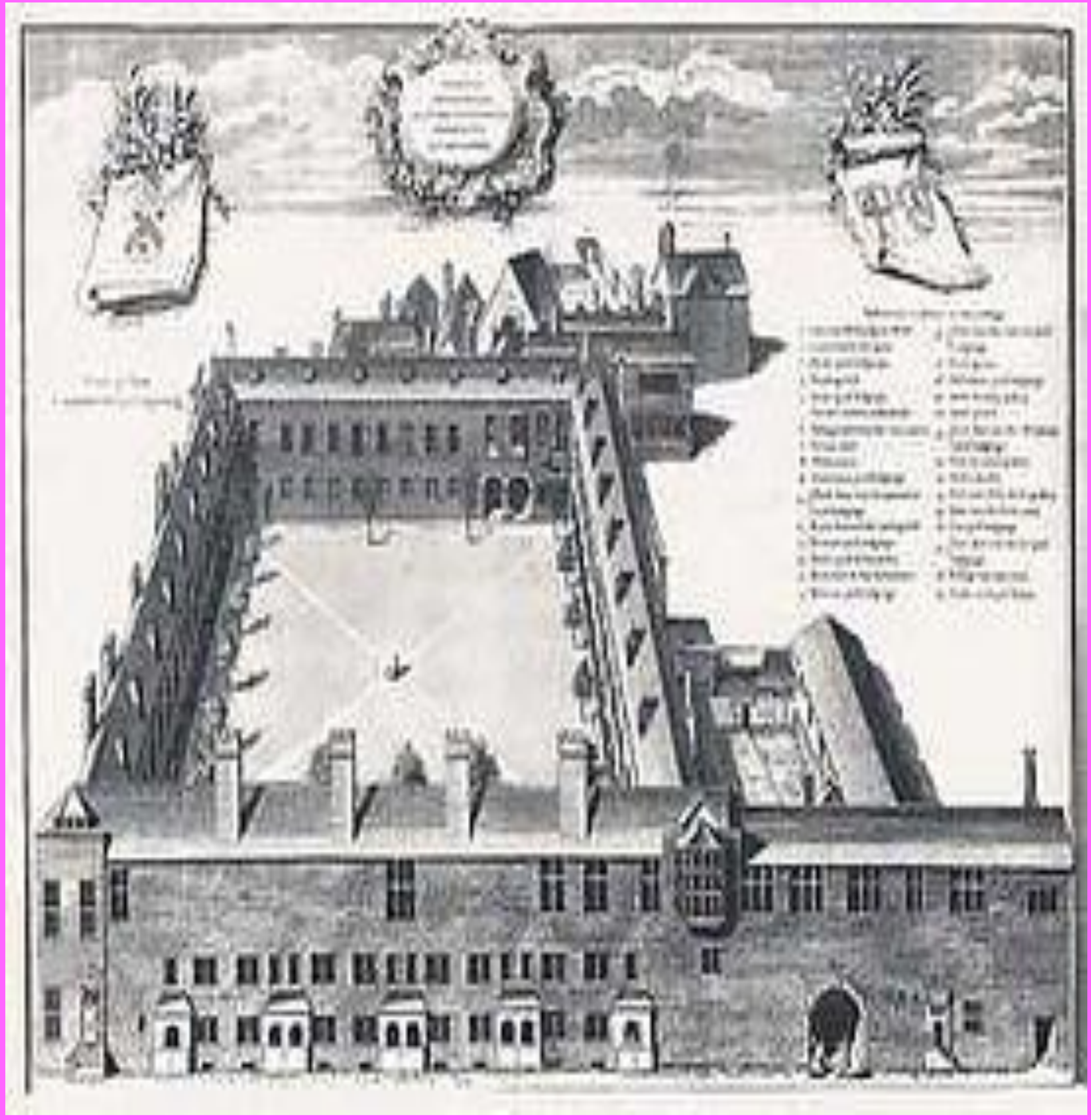
Ejusque usus, in utraque
Trigonometria; ut etiam in
omni Logistica Mathematica,
Amplissimi, Facillimi, &
expeditissimi explicatio.

Authore ac Inventore,
IOANNE NEPERO,
Barone Merchistonii,
&c. Scoto.

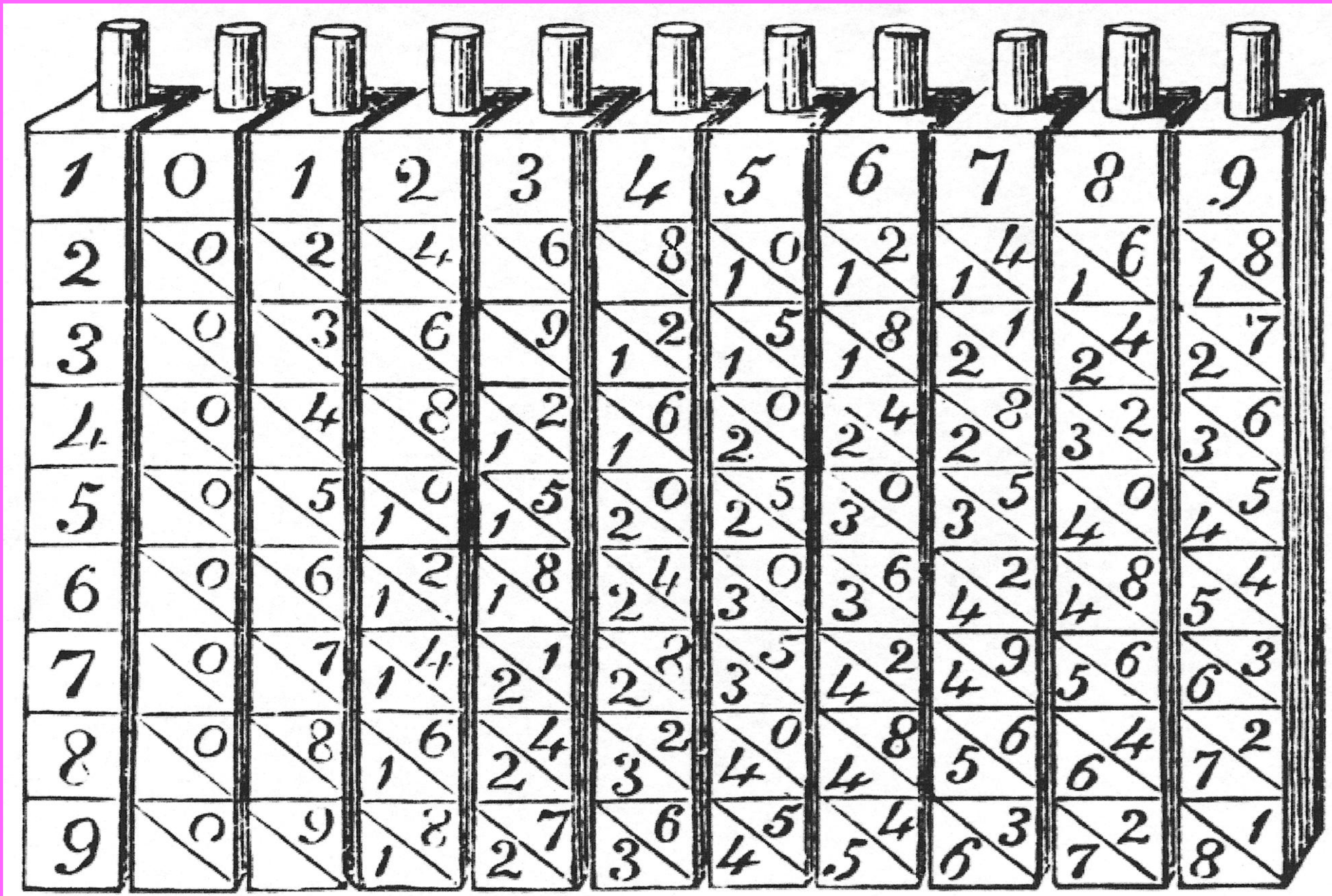
EDINBURGI,
Ex officinâ ANDREÆ HART
Bibliopôlæ, CIↃ. DC. XIV.




constant speed
0
1
2
3
4
5
6
...
line 1:
line 2:
P
...
Q
10,000,000
0



1	0	0000,00000,00000	34	1	5314,78917,04226
2	0	3010,29995,66398	35	1	5440,68044,35028
3	0	4771,21254,71966	36	1	5563,02500,76729
4	0	6020,59991,32796	37	1	5682,01724,06700
5	0	6989,70004,33602	38	1	5797,83596,61681
6	0	7781,51250,38364	39	1	5910,64607,02650
7	0	8450,98040,01426	40	1	6020,59991,32796
8	0	9030,89986,99194	41	1	6127,83856,71974
9	0	9542,42509,43932	2	1	6232,49290,39790
10	1	0000,00000,00000	43	1	6334,68455,57959



NEWTON'S SUCCESSORS

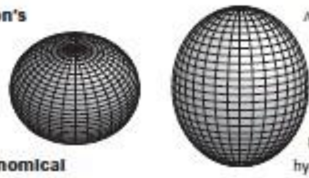
The appearance of Newton's *Principia* in 1687 caused a sensation – but scientists were puzzled by the nature of an attractive force of gravity that could apparently act over astronomical distances: to Huygens, in particular, it was an 'absurd' idea that was not capable of explaining anything. Preferable to them was some sort of mechanical theory such as that of Descartes, in which the planets are swept along by vortices like leaves in a whirlpool. But there were two main areas in which Newton's theory caused difficulties – the shape of the earth and the motion of the moon. Both were important for navigation, and in both of them Newton's views were eventually vindicated.

In 1728, the year after Newton's death, the great French author, historian and philosopher Voltaire wrote about the different world views in France and England:

> *In Paris they see the universe as composed of vortices of subtle matter; in London they see nothing of the kind. For us it is the pressure of the moon that causes the tides of the sea; for the English it is the sea that gravitates towards the moon.*

Voltaire was well placed to comment, as he had the expertise of Madame du Châtelet to inform him. She was the gifted mathematician who translated Newton's *Principia* into French. Voltaire continued:

> *In Paris you see the earth shaped like a lemon; in London it is flattened on two sides.*



A melon (oblate spheroid) and a lemon (prolate spheroid)

THE SHAPE OF THE EARTH

Under Newton's hypotheses, the rotation of the earth causes a flattening at the poles so that the earth is melon-shaped, whereas under Descartes's vortex and matter theory there is an elongation at the poles, so that the earth is lemon-shaped.

To decide on the actual shape of the earth, the Paris Academy sent two expeditions to measure the size of a degree of latitude: one to Peru, led by Charles-Marie de la Condamine in 1735, and the other to Lapland, headed by Pierre de Maupertuis in 1736. It was not until 1739 that both expeditions reported, and Maupertuis was able to confirm that Newton was right: the earth is flatter at the poles. This earned Maupertuis the nickname of 'the great flattener'.

Although this vindicated Newton's approach, it turned out that Newton had incorrectly calculated the amount of flattening because of assumptions he had made about fluid pressure, although he correctly predicted the nature of the earth's shape.

Emilie de Breteuil, Marquise du Châtelet

THE MOTION OF THE MOON

Although Newton dealt well with the motion of two bodies moving under mutual gravitational attraction, the motion of the moon depends not only on the earth but also on the sun. Even today we have no exact solution of the *three-body problem* – the problem of predicting the future positions and speeds of three bodies moving under mutual gravitational attraction.

Without the influence of the sun the motion of the moon would be an ellipse. Newton simplified the problem by assuming that the effect of the sun was to cause the moon's elliptical orbit to revolve slowly. He calculated that it would take eighteen years for the orbit to return to its original position, but observation showed that it took only nine years. As Newton wrote in later editions of the *Principia*:

> *The apse of the moon is about twice as swift.*

By the end of the 1740s Newton's theory of gravitation was under concerted investigation by those mathematicians who best understood it: d'Alembert, Clairaut and Euler. In 1747 Clairaut, who had taken part in Maupertuis's Lapland expedition, proposed to modify Newton's inverse-square law of gravitation by adding an additional term to it, while d'Alembert and Euler came up with other approaches. It seemed that Newton's law of gravitation might be wrong!

Alexis Claude Clairaut

Then, on 17 May 1749, Clairaut made a dramatic retraction:

> *I have been led to reconcile observations on the motion of the moon with the theory of attraction without supposing any other attractive force than one proportional to the inverse square of the distance.*

Clairaut had taken a new approach to the differential equations that describe the moon's motion, finding that the previous differences between theory and observation had been due to the way in which these equations were approximated.

This led to Euler publishing his theory of the moon in 1753, which enabled the astronomer Tobias Mayer to prepare a set of tables describing its motion – enabling the moon to be used as a 'celestial clock'. This led eventually to their receiving a share of the prize awarded by the British Board of Longitude for discovering a practical way of finding longitude at sea.

CHAPTER 4

THE AGE OF REVOLUTIONS

The 19th century saw the development of a mathematics profession in which people earned their living from teaching, examining and researching. The mathematical centre of gravity moved from France to Germany, while Latin gave way to national languages for publishing mathematical work. There was also a dramatic increase in the number of textbooks and journals.

Because of this increase in mathematical activity, mathematicians began to (indeed, needed to) specialize. While one would use the term *mathematician* in the 18th century, one now had *analysts*, *algebraists*, *geometers*, *number theorists*, *logicians* and *applied mathematicians*. This need for specialization was avoided only by the very greatest: Gauss, Hamilton, Riemann and Klein.

The University of Göttingen, where Gauss, Riemann and Klein worked.

In each discipline there was a revolution (as well as an evolution) in the depth, extent, and even the very existence of the discipline. But each discipline experienced a movement towards an increasingly abstract style with an increased emphasis on putting mathematics on a sound and rigorous basis and examining its foundations. We illustrate this by considering the revolutions in three areas – analysis, algebra and geometry.

FROM CALCULUS TO ANALYSIS

In the 1820s Augustin-Louis Cauchy, the most prolific mathematician of the century, rigorized the calculus by basing it on the concept of a *limit*. He then used this idea to develop the areas of real and complex analysis. This increase in rigour necessitated the formulation of a foolproof definition of the real numbers, which in turn led to a study of infinite sets by Georg Cantor and others.

Joseph Fourier's work on heat conduction also gave rise to infinite processes – in this case, infinite series – thereby stimulating Bernhard Riemann in his work on integration. Analytical techniques came to be applied to a wide range of problems – in electricity and magnetism by William Thomson (Lord Kelvin) and James Clerk Maxwell, in hydrodynamics by George Gabriel Stokes, and in probability and number theory by Pafnuty Chebyshev.

Revolutions did not happen only in mathematics: this is a miners' riot that took place in Belgium, 1868

FROM EQUATIONS TO STRUCTURES

Algebra also changed dramatically throughout the 19th century. In 1800 the subject was about solving equations, but by 1900 it had become the study of mathematical structures – sets of elements that are combined according to specified rules, called axioms.

At the beginning of the century, Gauss laid down the basics of number theory and introduced modular arithmetic, an early example of a new algebraic structure called a *group*.

A long-standing problem had to do with finding a general method for solving polynomial equations of degree 5 or more, using only arithmetical operations and the taking of roots. Niels Abel showed that there can be no such general solution, and Évariste Galois developed his ideas by examining groups of permutations of the roots of an equation.

The mystique concerning complex numbers was at last removed by William Rowan Hamilton, who defined them as pairs of real numbers with certain operations. Other algebraic structures were discovered: Hamilton introduced the algebra of quaternions, George Boole created an algebra for use in logic and probability, and Cayley studied the algebra of rectangular arrays of symbols, called *matrices*.

FROM ONE TO MANY GEOMETRIES

Over the space of one hundred years the study of geometry was completely transformed. In 1800 the only 'true' geometry had been Euclidean geometry, although there were some scattered results on spherical and projective geometry. By the end of the century, infinitely many geometries were known, while geometry had become closely linked to group theory and placed on a more rigorous foundation.

Gauss studied surfaces and their curvature, finding a relationship between curvature and the sum of the angles of a triangle on the surface, and this turned out to be related to the investigations into the parallel postulate in Euclidean geometry. Nikolai Lobachevsky and János Bolyai independently developed non-Euclidean geometry, in which the parallel postulate does not hold.

It took time, however, for the ideas on non-Euclidean geometry to become absorbed, and it was the mid-century work of Riemann that showed the importance of the new ideas and extended the work of Gauss. Through such abstract techniques, geometry was also moving out of two and three dimensions and into higher ones. Later, Felix Klein used groups to examine and classify different types of geometry.

BABBAGE AND LOVELACE

The central figure of 19th-century computing was Charles Babbage (1791–1871), who may be said to have pioneered the modern computer age with his 'difference engines' and his 'analytical engine', although his influence on subsequent generations is hard to assess. Ada, Countess of Lovelace (1816–1852), daughter of Lord Byron and a close friend of Babbage, produced a perceptive and clear commentary on the powers and potential of the analytical engine; this was essentially an introduction to what we now call programming.

A portion of the 1832 difference engine: it was to have the feature of being able to print its results, as more errors arose in printing and proof-reading than in the original calculations

THE DIFFERENCE ENGINE

Charles Babbage and John Herschel were asked by the Royal Astronomical Society to produce new astronomical tables. It was this that caused Babbage to design his calculating machine.

He wanted to mechanize the calculation of a formula such as $x^2 + x + 41$, for different values of x – this was his illustrative example. The core of his idea can be seen in the following table. In the second column are the values of this expression for $x = 0, 1, 2, \ldots, 7$, in the third column are the differences between successive terms of the second (the *first differences*) and in the fourth column are the differences between successive terms of the third column (the *second differences*); here, the second differences are all the same.

x	$x^2 + x + 41$	first differences	second differences
0	41	2	
1	43	4	2
2	47	6	2
3	53	8	2
4	61	10	2
5	71	12	2
6	83	14	2
7	97		

Note that we can reconstruct the values of the function in a steplike fashion from the shaded region containing the first term (41), the initial first difference (2) and the constant second differences (2).

This technique can be applied to any polynomial function, because continuing to take differences eventually yields constant values. Also many functions of interest which are not polynomials (like *sin*, *cos* and *log*) can be approximated by polynomials.

Charles Babbage

The construction of the difference engine ran into engineering, financial and political difficulties, and construction ended in 1833.

THE ANALYTICAL ENGINE

Babbage wondered whether his difference engine could be made to act upon the results of its own calculations, or as he put it:

The engine eating its own tail.

With this in mind, he designed a new engine, basing its control system on the punched cards used by Jacquard for his automatic loom.

The design for his analytical engine allowed for inputting numbers and holding them in a *store*. The instructions for the operations to be performed on the numbers would be input separately. These operations would be performed in a part of the computer, called the *mill*, and the results would be returned to the store and printed, or used as input for a further calculation, depending on the control instructions. Importantly, the operations to be performed could be made to depend on the result of an earlier calculation.

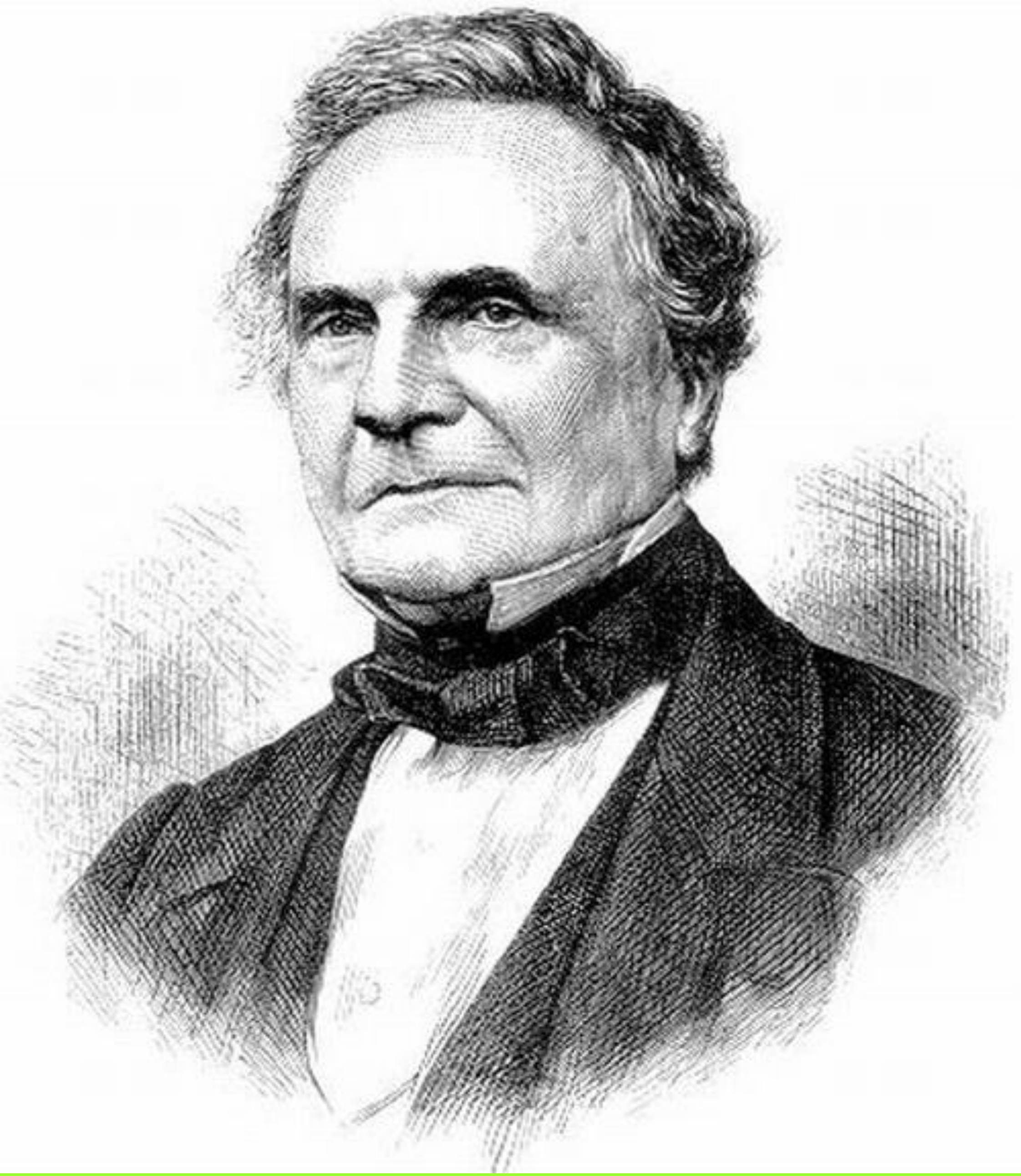
Ada, Countess of Lovelace, was encouraged in her interest in mathematics by Mary Somerville and Augustus De Morgan.

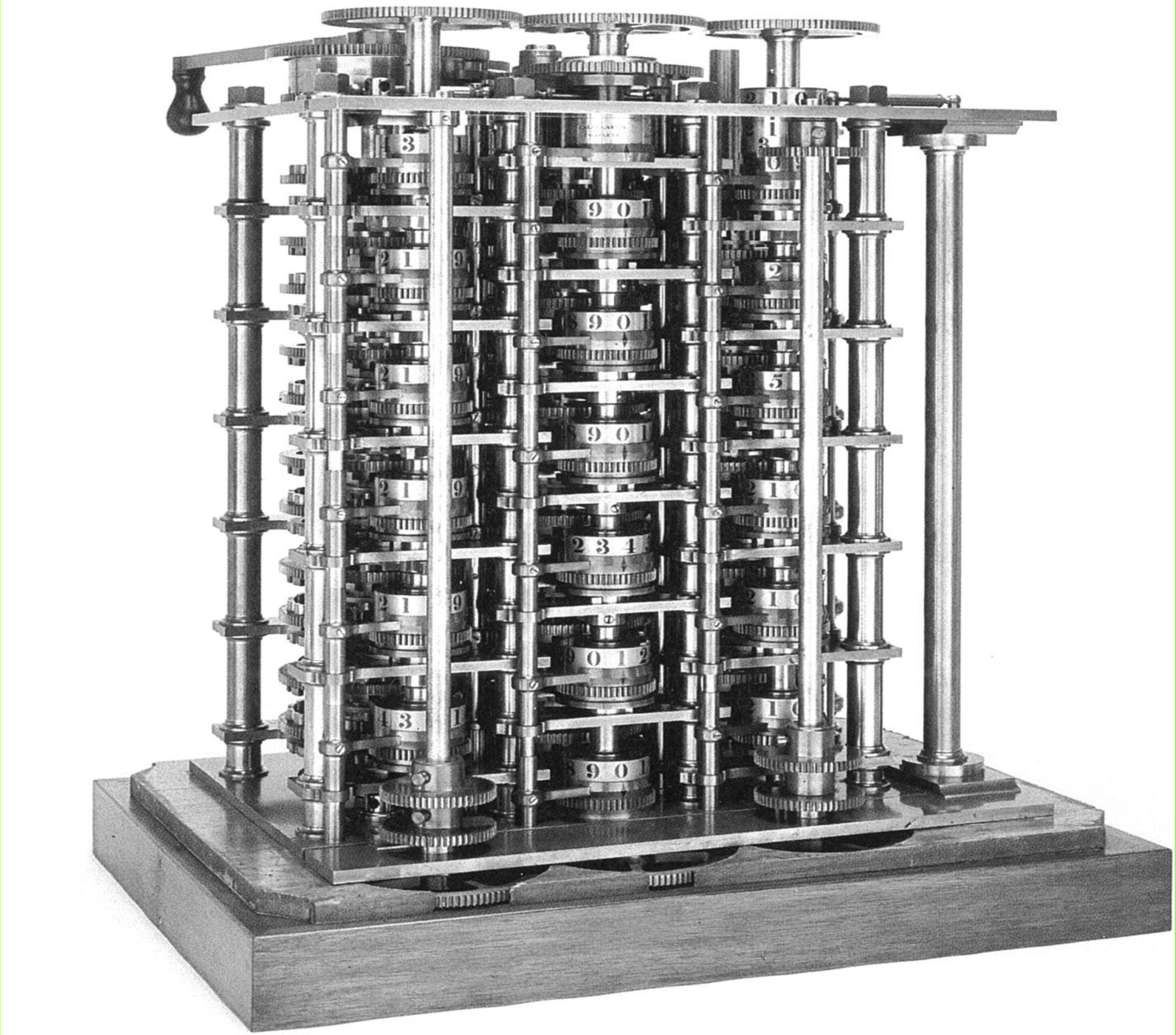
In her writings on the analytical engine, she described what it could do and how it could be instructed, and gave what is considered to be the first computer program. As she wrote:

> *The distinctive characteristic of the Analytical Engine ... is the introduction into it of the principle which Jacquard devised for regulating, by means of punched cards, the most complicated patterns in the fabrication of brocaded stuffs. It is in this that the distinction between the two engines lies. Nothing of the sort exists in the Difference Engine. We may say most aptly that the Analytical Engine weaves algebraical patterns just as the Jacquard loom weaves flowers and leaves.*

Ada, Countess of Lovelace

Although the analytical engine was never built, modern scholarship is of the view that if it had been constructed, it would have worked as Babbage intended. The name ADA is now given to a programming language developed for the United States Department of Defense.





NIGHTINGALE

Florence Nightingale (1820–1910), the 'lady with the lamp' who saved lives during the Crimean War, was also a fine statistician who collected and analysed mortality data from the Crimea and displayed them on her 'polar diagrams', a forerunner of the pie chart. Her work was strongly influenced by that of the Belgian statistician Adolphe Quetelet.

Florence Nightingale showed an early interest in mathematics – at the age of 9 she was displaying data in tabular form, and by the time she was 20 she was receiving tuition in mathematics, possibly from James Joseph Sylvester.

Nightingale regarded statistics as 'the most important science in the world' and used statistical methods to support her efforts at administrative and social reform. She was the first woman to be elected a Fellow of the Royal Statistical Society and an honorary foreign member of the American Statistical Association.

Florence Nightingale

STATISTICAL INFLUENCE

By 1852 Nightingale had established a reputation as an effective administrator and project manager. Her work on the professionalization of nursing led to her accepting the position of 'Superintendent of the female nursing establishment in the English General Military Hospitals in Turkey' for the British troops fighting in the Crimean war. She arrived in 1854 and was appalled at what she found there. In attempting to change attitudes and practices she made use of pictorial diagrams for statistical information, developing her *polar area graphs*.

The graphs have twelve sectors, one for each month, and reveal changes over the year in the deaths from wounds obtained in battle, from diseases, and from other causes. They showed dramatically the extent of the needless deaths amongst the soldiers during the Crimean war, and were used to persuade medical and other professionals that deaths could be prevented if sanitary and other reforms were made.

On her return to London in 1858, she continued to use statistics to inform and influence public health policy. She urged the collection of the same data, across different hospitals, of:

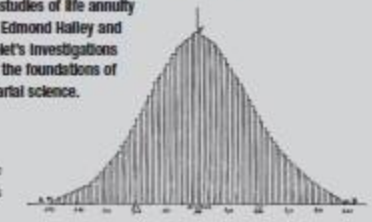
- the number of patients in hospital
- the type of treatment, broken down by age, sex and disease
- the length of stay in hospital
- the recovery rate of patients.

She argued for the inclusion in the 1861 census of questions on the number of sick people in a household, and on the standard of housing, as she realised the important relationship between health and housing. In another initiative she tried to educate members of the government in the usefulness of statistics, and influence the future by establishing the teaching of the subject in the universities.

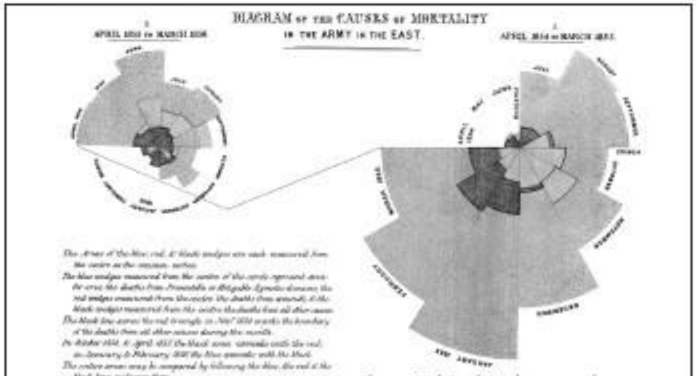
For Nightingale the collection of data was only the beginning. Her subsequent analysis and interpretation was crucial and led to medical and social improvements and political reform, all with the aim of saving lives.

ADOLPHE QUETELET

Quetelet was supervisor of statistics for Belgium, pioneering techniques for taking the national census. His desire to find the statistical characteristics of an 'average man' led to his compiling the chest measurements of 5732 Scottish soldiers and observing that the results were arranged around a mean of 40 inches, according to the normal (or Gaussian) distribution. Taken with some earlier studies of life annuity payments by Edmond Halley and others, Quetelet's investigations helped to lay the foundations of modern actuarial science.



Quetelet's curve, showing the distribution of some of his results



Florence Nightingale's polar area graph


Quetelet

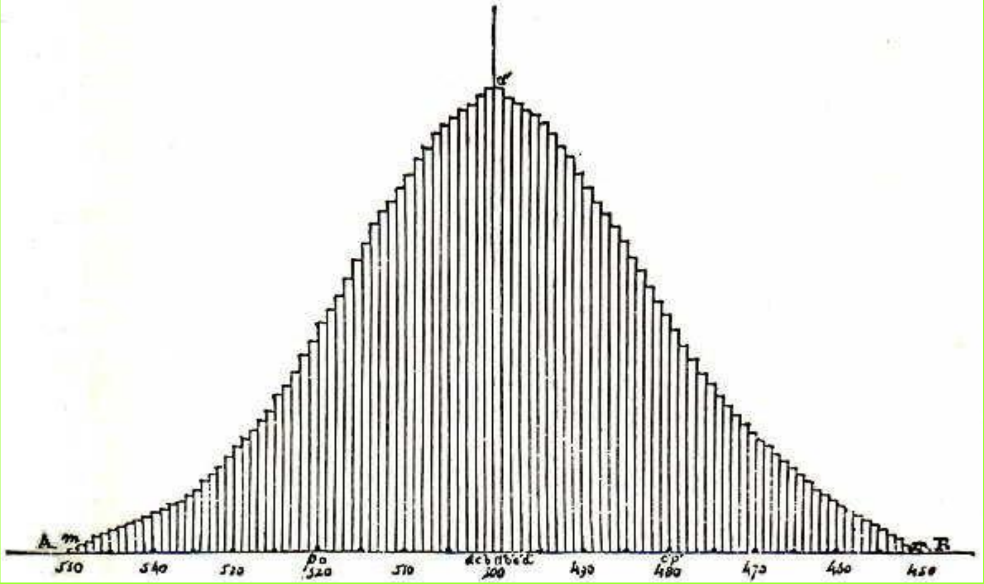
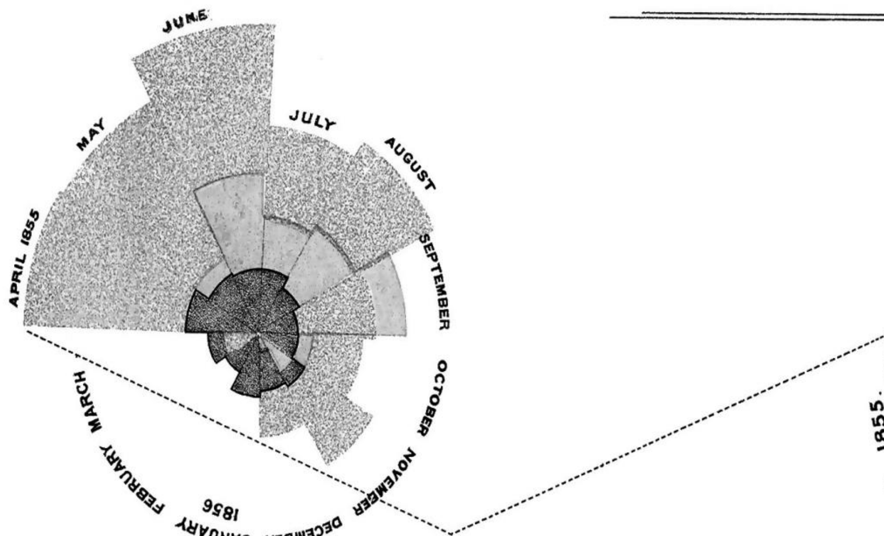
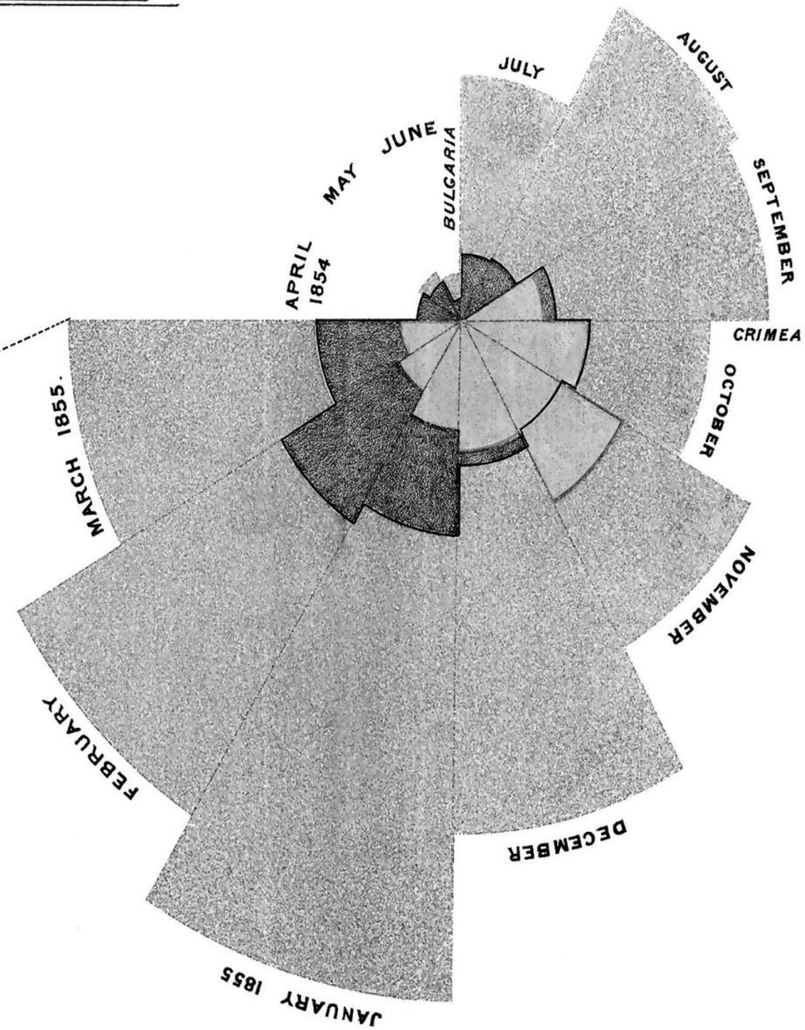

A
B

DIAGRAM of the CAUSES of MORTALITY
IN THE ARMY IN THE EAST.

2.
APRIL 1855 to MARCH 1856.



1.
APRIL 1854 to MARCH 1855.



The Areas of the blue, red, & black wedges are each measured from the centre as the common vertex.

The blue wedges measured from the centre of the circle represent area for area the deaths from Preventible or Mitigable Zymotic diseases; the red wedges measured from the centre the deaths from wounds; & the black wedges measured from the centre the deaths from all other causes.

The black line across the red triangle in Novr. 1854 marks the boundary of the deaths from all other causes during the month.

In October 1854, & April 1855, the black area coincides with the red; in January & February 1856, the blue coincides with the black.

The entire areas may be compared by following the blue, the red & the black lines enclosing them.

CHAPTER 5

THE MODERN AGE

In our final chapter we meet the mathematicians who have:

- **examined the limits of what we can prove and shown why some tasks are impossible to carry out**
- **laid the foundations of our current scientific knowledge**
- **carried out mathematical work of historical, social and political impact, and thereby changed the world we live in**
- **developed computers, both theoretical and practical, that enable us to simulate, model and prove things that we could not do otherwise, while raising questions about our identity.**

PARADOXES AND PROBLEMS

In previous chapters we have seen a developing desire to place mathematics on a sounder foundation, with the story going from the underpinning notions of the calculus to arithmetic and the theory of sets. As 20th-century mathematicians examined more carefully the nature of infinity and the problems connected with sets they met a number of problems and paradoxes. One of the most famous of these was formulated by Bertrand Russell in 1902, and necessitated a much more thorough treatment of the very foundations of set theory and of the exact nature of deductive proof.

The Princeton Institute for Advanced Study, established in 1930: 25 Nobel Laureates and 38 (out of 52) Fields Medallists have been affiliated with it

G. H. Hardy with his research student Mary Cartwright, later to be the first woman appointed President of the London Mathematical Society

Another approach was taken by David Hilbert, whose attempt to make arithmetic secure was to make it *axiomatic*, an approach that he had already used with success when dealing with the foundations of geometry. Instead of defining all the basic terms, such as point or line, he gave a set of rules (or axioms) that they had to satisfy.

Although Hilbert's approach was influential, his objectives were eventually proved to be unattainable, as demonstrated in the 1930s by Kurt Gödel and Alan Turing, who obtained a number of amazing and unexpected results about the limits of what can be proved or decided.

ABSTRACTION AND GENERALIZATION

The 19th-century trend towards increasing generalization and abstraction continued to accelerate dramatically throughout the 20th century. For example, Albert Einstein used the abstract formulations of geometry and calculus for his general theory of relativity, while algebra became an abstract and axiomatic subject, being particularly influenced by the work of Emmy Noether. Advances also continued to be made in number theory, with Hardy (and his co-workers Littlewood and Ramanujan) and Andrew Wiles making major contributions.

Meanwhile, new areas of the subject came into being, such as algebraic topology and the theory of 'Hilbert spaces', while machine computation entered the mainstream of the subject, as spectacularly illustrated by Appel and Haken in their proof of the four-colour theorem.

SPREAD AND DEVELOPMENT

The 20th century saw mathematics becoming a major profession throughout the world, with jobs in education and industry and numerous areas of specialization and application.

With mathematics developing at such a fast rate, many new journals have been created, and national and international conferences have become widespread. Most important among these meetings are the International Congresses of Mathematicians, held every four years, when the prestigious Fields medals are awarded and many thousands of mathematicians gather to learn about the latest developments in their subject.

HILBERT

On 8 August 1900, David Hilbert (1862–1943), one of the greatest mathematicians of the day, gave the most celebrated mathematical lecture of all time. For it was on this date, at the International Congress of Mathematicians in Paris, that he presented a list of unsolved problems for 20th-century mathematicians to tackle. Trying to solve these problems helped to set the mathematical agenda for the next hundred years.

David Hilbert was born in Königsberg in Eastern Prussia and received his doctorate there in 1885. After teaching in Königsberg for a few years, he was invited by Felix Klein to join the faculty at Göttingen, where he spent the rest of his life.

His mathematical range was immense – from abstract number theory and invariant theory, via the calculus of variations and the study of analysis (and so-called 'Hilbert spaces'), to potential theory and the kinetic theory of gases.

THE FOUNDATIONS OF GEOMETRY

Following Cantor's introduction of set theory and subsequent investigations by various mathematicians into the foundations of arithmetic, Hilbert became increasingly involved with the foundations of geometry.

Although Euclid's axiom system had worked well for two thousand years, it contained a number of unwarranted assumptions. Hilbert duly set about replacing it by alternative sets of axioms that were completely foolproof. His aim, in particular, was to find axiom systems that are

- *consistent*: the axioms do not lead to contradictions
- *independent*: no axiom can be deduced from the others

- *complete*: any statement that we may formulate within the system can be proved to be either true or false

In 1899 Hilbert produced his influential *Grundlagen der Geometrie* (Foundations of Geometry), in which he developed his axiom systems for Euclidean and projective geometry. Four years later he produced a second edition in which he also axiomatized non-Euclidean geometry.

Hilbert had a grand plan. He was convinced that the whole of classical mathematics could be similarly axiomatized, and with Paul Bernays he wrote a two-volume work with this purpose in mind. But as they progressed, they experienced unexpected difficulties with the details of their arguments, and it soon became apparent that Hilbert's plan was doomed to failure.

THE HILBERT PROBLEMS

Who of us would not be glad to lift the veil behind which the future lies hidden: to cast a glance at the next advances of our science and at the secrets of its development during future centuries?

So asked David Hilbert in his famous address at the Paris Congress, at which he presented his list of twenty-three unsolved problems. We have already met one of these problems, the Riemann hypothesis, which remains unsolved to this day. Here we present a few more, some of which will be discussed later in this chapter.

Problem 1: *Prove the Continuum hypothesis, that there is no set whose cardinality lies strictly between those of the integers and the real numbers.*

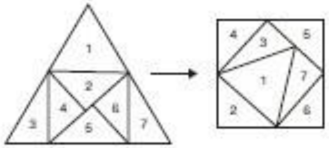
We recall that Cantor proved that infinities can have different sizes, and that the set of real numbers is strictly larger than the set of integers (or fractions). This problem asks us to prove that no infinite set is larger than the set of integers but smaller than the set of real numbers.

Problem 2: *Are the axioms of arithmetic consistent?*

Hilbert based his treatment of the consistency of his geometrical axioms on the assumption that arithmetic (that is, our real number system) can be similarly axiomatized. This problem asks whether this latter assumption is valid, or whether there could be, 'somewhere out there', a contradiction that we never expected.

Problem 3: *Given two polyhedra with the same volume, can we always cut the first into finitely many pieces that can then be reassembled to give the second?*

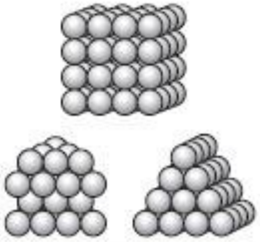
In 1833, János Bolyai proved that if two polygons have the same area, then the first can be cut into pieces that can be rearranged to give the second; the following example shows a triangle reassembled to give a square. This problem asks whether a similar result holds in three dimensions.



The answer is no. Within two years Max Dehn proved that a regular tetrahedron *cannot* be cut into pieces that can then be reassembled to give a cube with the same volume.

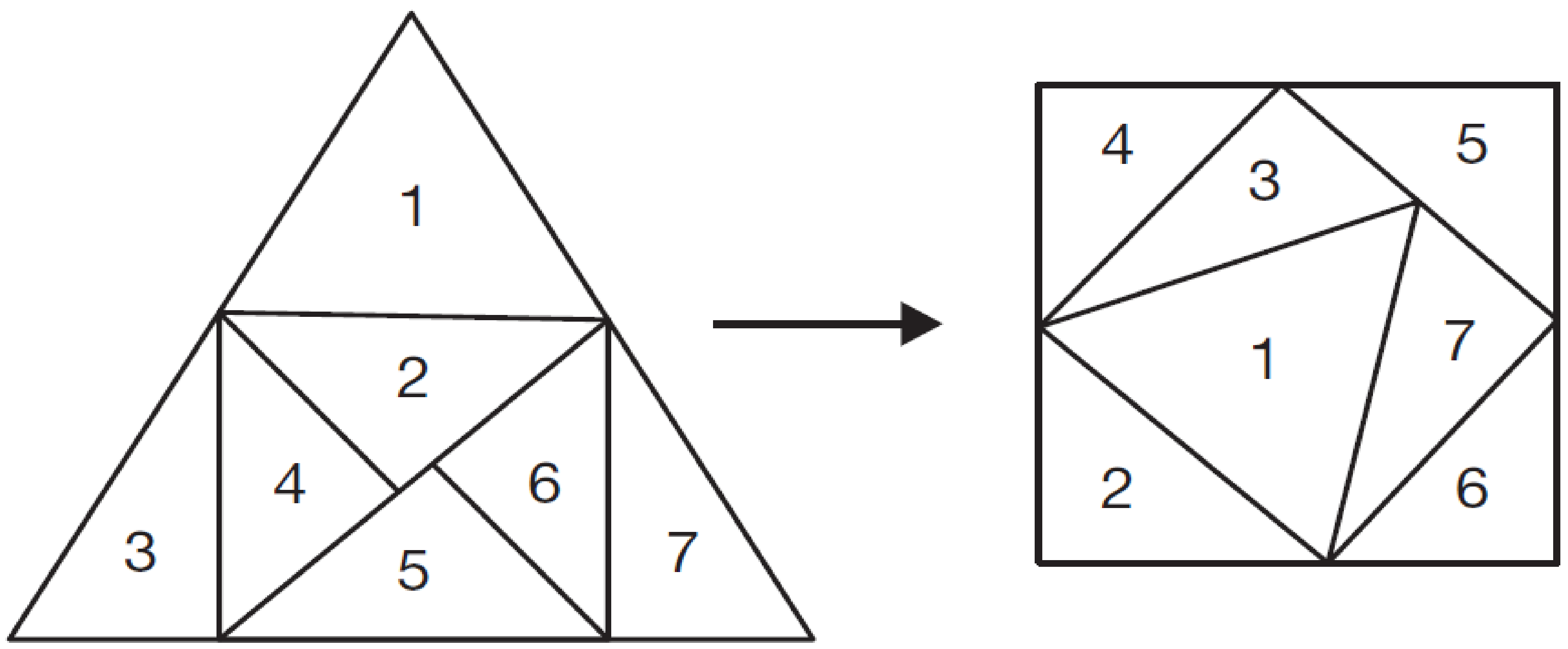
Problem 18: *What is the most efficient way to stack spheres so that the amount of empty space between them is as small as possible?*

This problem was considered by Harriot and Kepler. Two ways to stack the spheres are cubic stacking and hexagonal stacking, but neither is the most efficient. It turns out that the way your greengrocer stacks oranges is the most efficient – the proportion of empty space is about 0.26, which is less than the 0.48 and 0.40 proportions of the other two. But to prove this rigorously was horrendous: in 1998 Thomas Hales gave a computer-aided proof that involved three gigabytes of computer power.

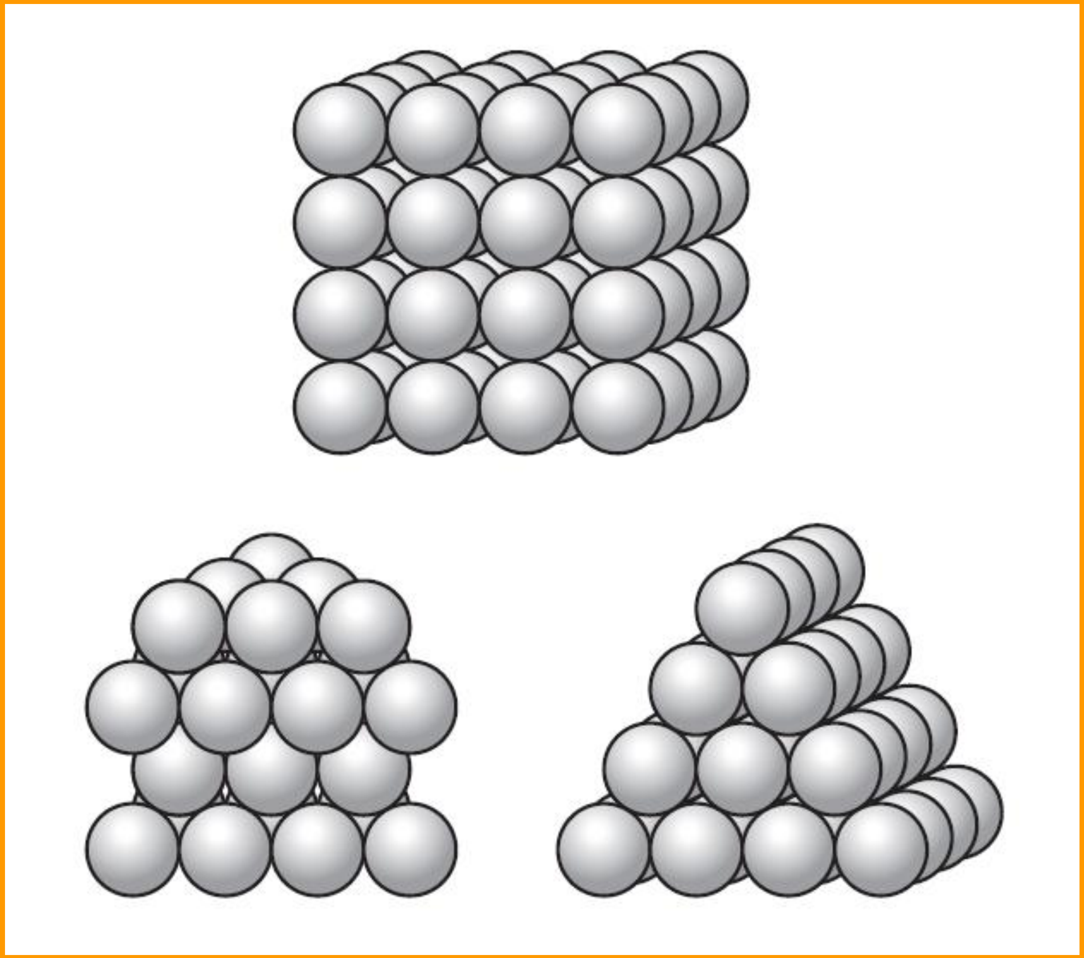


Cubic stacking, hexagonal stacking and greengrocer's stacking

Problem 3: *Given two polyhedra with the same volume, can we always cut the first into finitely many pieces that can then be reassembled to give the second?*



Problem 18: *What is the most efficient way to stack spheres so that the amount of empty space between them is as small as possible?*



EINSTEIN AND MINKOWSKI

Albert Einstein (1879–1955), an iconic figure of the 20th century, was the greatest mathematical physicist since Isaac Newton. He revolutionized physics with his theories of special and general relativity. These drew on mathematical ideas, not previously used in physics, some of which had been developed by Riemann and by Hermann Minkowski (1864–1909).

Albert Einstein — a plaque in Ulm

Einstein was born in Ulm, Southern Germany moving to Munich the next year. He was slow in learning to speak and showed little promise in his early schooling. He was admitted to Zürich polytechnic at his second attempt in 1896 to a course for mathematics and science teachers and graduated in 1900. Although one of his lecturers was Minkowski, he gained little from the formal teaching and preferred to read independently and think deeply about the fundamental ideas and assumptions of physics. After graduation he supported himself by part-time teaching until he obtained a position in the Swiss Patent Office in Bern.

In 1905 Einstein submitted his paper on special relativity to the University of Bern in support of his application for a doctorate, and it was rejected! However recognition of his work soon arrived as it became more widely known. He then held positions at the Universities of Zürich, Prague and Berlin, and announced his general theory of relativity in 1915. He was awarded the Nobel Prize in 1921 for his work on quantum theory, rather than relativity. In 1933 he went to America and from then on was based at the Institute for Advanced Study in Princeton.

EINSTEIN'S *ANNUS MIRABILIS*

In 1905, his 'year of wonders', Albert Einstein published four papers of ground-breaking importance. First he published the work that introduced quanta of energy – that light can be absorbed or emitted only in discrete amounts, a core idea of quantum theory. Next was a paper on Brownian motion, explaining the movement of small particles suspended in a stationary liquid.

His third paper, on the electrodynamics of moving bodies, introduced a new theory linking time, distance, mass and energy. It was consistent with electromagnetism, but omitted the force of gravity. This became known as the *special theory of relativity* and assumed that c, the speed of light, is constant, irrespective of where you are or how you move.

On 21 November 1905 he published *Does the Inertia of a Body Depend Upon Its Energy Content?* This contains one of the most famous equations of all, $E = mc^2$, asserting the equivalence of mass and energy.

MINKOWSKI AND SPECIAL RELATIVITY

Minkowski was born of German parents in Lithuania. In 1902 he moved to the University of Göttingen, where he became a colleague of Hilbert. He developed a new view of space and time and laid the mathematical foundations of the theory of relativity. Minkowski described his approach as follows:

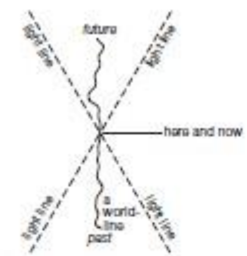
> *Henceforth space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality.*

The *kind of union* that Minkowski mentions is now known as *space-time* and is a four dimensional non-Euclidean geometry that incorporates the three dimensions of space with the one of time. It comes with a way of measuring the distance between two different points of space-time. Space and time are now no longer separate, as Newton had thought, but are intermixed. A reviewer said of his work that

> *purely mathematical considerations, including harmony and elegance of ideas, should dominate in embracing new physical facts. Mathematics, so to speak, was to be master and physical theory could be made to bow to the master.*

Hermann Minkowski

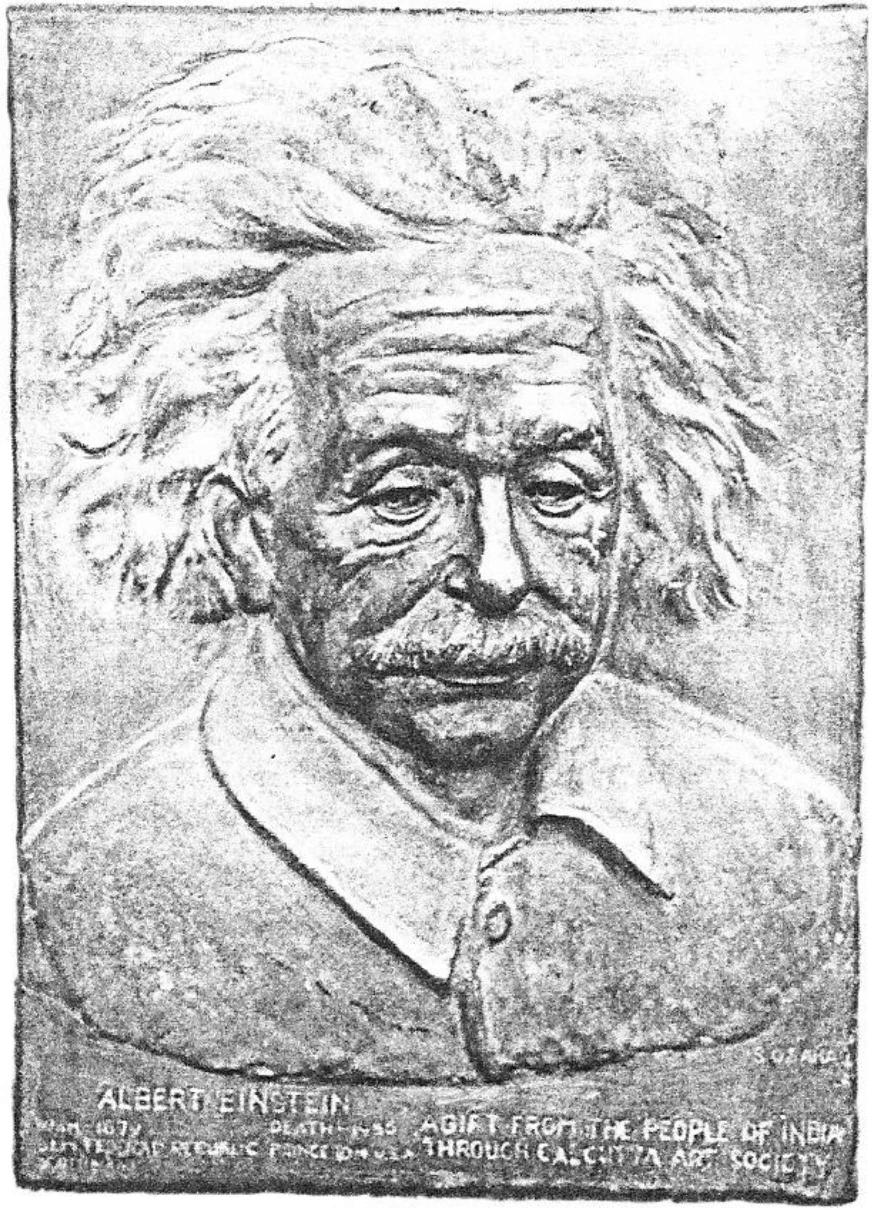
Below is a simplified diagram of space-time with only one space dimension going horizontally and with time going vertically. In Euclidean geometry the distance of each point (x, t) to the origin is $\sqrt{(x^2 + t^2)}$, but the requirements of relativity replace this in space-time with the distance $\sqrt{(x^2 - c^2t^2)}$. The minus sign implies that events in space-time, such as the one labelled 'here and now', are associated with two cones. With just one space dimension, these cones are now triangles, with one representing the future of the 'here and now' and the other its past.



GENERAL RELATIVITY

Einstein initially thought little of Minkowski's approach to space-time, but later found it invaluable, indeed essential, when he was trying to extend his theory to include gravity. His *general theory of gravity*, building also on Riemann's geometrical ideas, produced space-time that was curved as a result of the presence of mass and energy. The curvature increased near to massive bodies, and it was the curvature of space-time that controlled the motion of bodies.

The theory predicted that light rays would be bent by the curvature of space-time produced by the sun, an effect that was observed during the 1919 solar eclipse of the sun.


ALBERT EINSTEIN
A GIFT FROM THE PEOPLE OF INDIA
THROUGH CALCUTTA ART SOCIETY

FIELDS MEDALLISTS

A special feature of the International Congresses of Mathematics, held every four years, is the award of Fields medals to the most outstanding young mathematicians. For many years these were regarded as the mathematical equivalent of Nobel prizes, but recently a new prize, the Abel Prize, has been instituted and is awarded annually.

The Proceedings of the first International Congress in Zürich, 1897

As part of the 400th anniversary celebrations of Columbus's voyage to America, a 'World Congress' of Mathematicians took place at the World's Columbian Exposition in Chicago in 1893. Forty-five mathematicians attended and the opening address, on 'The present state of mathematics', was given by Felix Klein of Göttingen, one of just four participants from outside the USA.

The first official Congress was held in Zürich in 1897, where it was decided to hold such international meetings every three to five years, and it was at the next one, in Paris in 1900, that Hilbert presented his famous lecture on the future problems of mathematics. Since these early gatherings, more than a score of international congresses have been held around the world, usually every four years.

These meetings usually take place without incident, but there have been a few difficulties along the way. The 1920 and 1924 Congresses were boycotted by many mathematicians, since Germans and Austrians were excluded, while the 1982 Warsaw Congress had to be postponed for a year due to the introduction of martial law in Poland.

Two medallists have been unable to attend due to visa restrictions, while another has declined the award.

THE FIELDS MEDAL

John Charles Fields was a mathematics professor at the University of Toronto, and President of the Toronto Congress in 1924. The profit from this meeting, together with later money from his estate, provided funding for the 'International Medals for Outstanding Discoveries in Mathematics', now known as Fields Medals. First awarded in 1936, the gold medals are produced by the Royal Canadian Mint and feature Archimedes on one side and an inscription on the other.

INTERNATIONAL CONGRESSES AND FIELDS MEDAL WINNERS

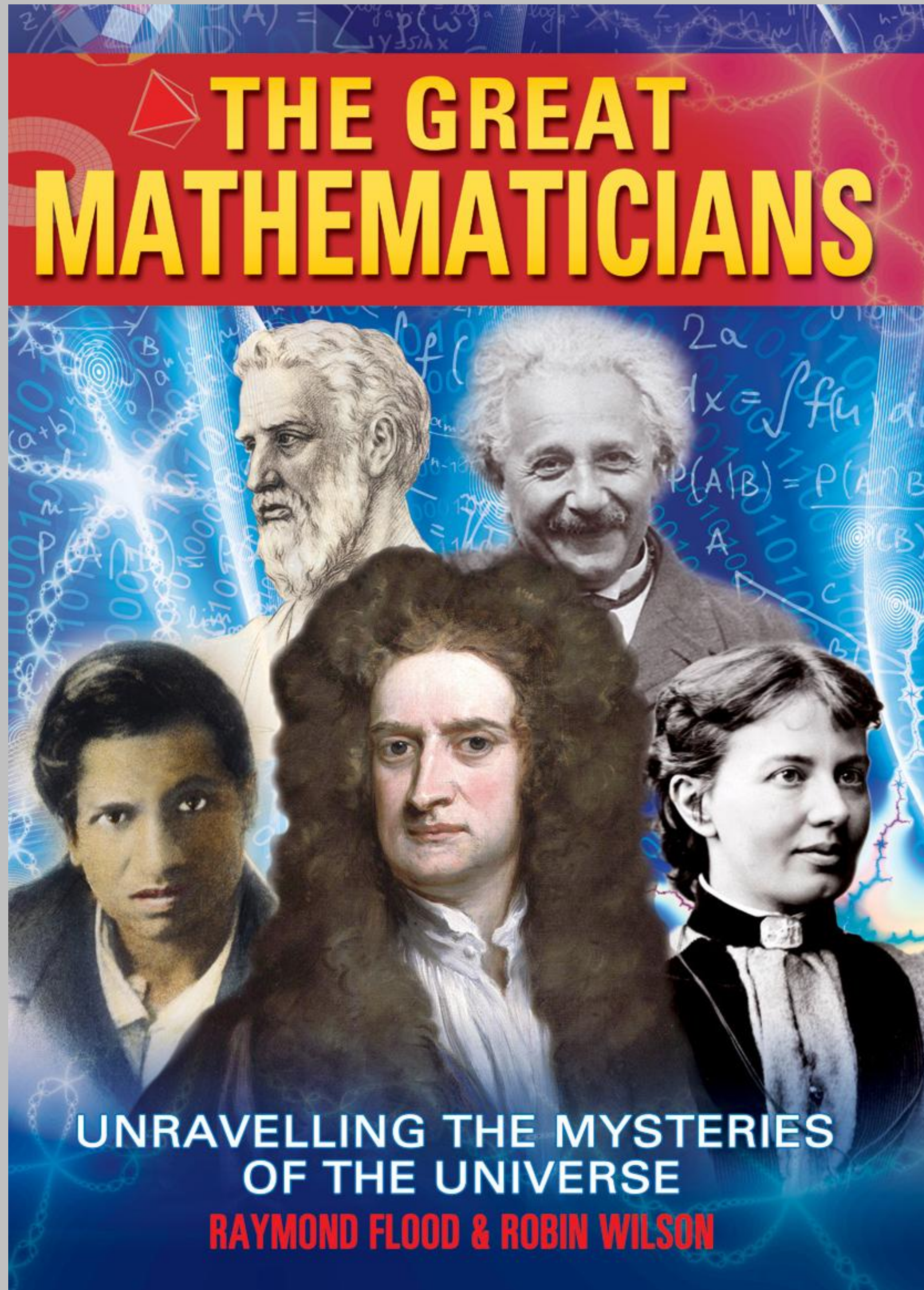
In the table we show the country with which each medallist is mainly associated.

1897: Zürich, Switzerland	–
1900: Paris, France	–
1904: Heidelberg, Germany	–
1908: Rome, Italy	–
1912: Cambridge, UK	–
1920: Strasbourg, Germany	–
1924: Toronto, Canada	–
1928: Bologna, Italy	–
1932: Zürich, Switzerland	–
1936: Oslo, Norway	Lars Ahlfors (Finland); Jesse Douglas (USA)
1950: Cambridge, USA	Laurent Schwartz (France), Atle Selberg (Norway)
1954: Amsterdam, Netherlands	Kunihiko Kodaira (Japan/USA), Jean-Pierre Serre (France)
1958: Edinburgh, UK	Klaus Roth (UK), René Thom (France)
1962: Stockholm, Sweden	Lars Hörmander (Sweden), John Milnor (USA)
1966: Moscow, USSR	Michael Atiyah (UK), Paul Cohen and Stephen Smale (USA), Alexander Grothendieck (Germany)
1970: Nice, France	Alan Baker (UK), Heisume Hironaka (Japan), Sergei Novikov (USSR), John Thompson (USA)
1974: Vancouver, Canada	Enrico Bombieri (Italy), David Mumford (USA)
1978: Helsinki, Finland	Pierre Deligne (Belgium), Grigory Margulis (USSR), Charles Fefferman and Daniel Quillen (USA)
1983: Warsaw, Poland	Alain Connes (France), William Thurston (USA), Shing-Tung Yau (China)
1986: Berkeley, USA	Simon Donaldson (UK), Gerd Faltings (Germany), Michael Freedman (USA)
1990: Kyoto, Japan	Vladimir Drinfel'd (USSR), Shigefumi Mori (Japan), Vaughan Jones (New Zealand), Edward Witten (USA)
1994: Zürich, Switzerland	Jean Bourgain (Belgium), Pierre-Louis Lions and Jean-Christophe Yoccoz (France), Efim Zelmanov (Russia)
1998: Berlin, Germany	Richard Borcherds and Timothy Gowers (UK), Maxim Kontsevich (France/Russia), Curtis McMullen (USA)
2002: Beijing, China	Laurent Lafforgue (France), Vladimir Voevodsky (Russia)
2006: Madrid, Spain	Andrei Okounkov and Grigori Perelman (Russia), Terence Tao (Australia/USA), Wendelin Werner (France)
2010: Hyderabad, India	Elon Lindenstrauss (Israel), Ngô Bao Châu (Vietnam), Stanislav Smirnov (Russia), Cédric Villani (France)

ABEL PRIZEWINNERS

In June 2002, to commemorate the bicentenary of Abel's birth, the Norwegian Academy of Science and Letters launched the **Abel Prize**, to be presented annually by the King of Norway for outstanding scientific work in the field of mathematics.

2003: Jean-Pierre Serre (France)
2004: Michael Atiyah (UK) and Isadore Singer (USA)
2005: Peter Lax (Hungary/USA)
2006: Lennart Carleson (Sweden)
2007: Srinivasa Varadhan (India/USA)
2008: John Thompson (USA) and Jacques Tits (France)
2009: Mikhail Gromov (Russia)
2010: John Tate (USA)
2011: John Milnor (USA)


THE GREAT
MATHEMATICIANS
UNRAVELLING THE MYSTERIES
OF THE UNIVERSE
RAYMOND FLOOD & ROBIN WILSON