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**Mathematics and Diplomacy:   
*Leibniz (1646-1716) and the curve of quickest descent***

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**Abstract**

Not only is mathematics challenging. Mathematicians are also often challenging each other. The search for the quickest slide between two points in a vertical plane is such a challenge. It was launched 1695 by Johann Bernoulli and became famous as the Brachysto-chrone (shortest-time) problem. The launch in a journal article, repeated in Bernoulli's New Year wish for 1696, resulted in a long-lasting quarrel between Johann and his elder brother Jacob Bernoulli. Other mathematicians, among whom Isaac Newton, got involved. This year’s tercentenary of the death of Gottfried Leibniz puts Leibniz in the limelight. He deserves this in his own right, because his involvement reveals interesting mathematics as well as friendly diplomacy. With his letters and publications about the Brachystochrone Leibniz hoped to reconcile the two Bernoulli brothers, the first students of his new calculus, whom he valued highly.

# **Mathematics and diplomacy: Leibniz (1646-1716) and the curve of quickest descent.**

He is a young and ambitious intellectual, 25 years old. At the age of 20 he had finished his law studies with a doctorate. He tried a job in the free city of Nuremberg, and then he moved on, along the river Rhine aiming for “Holland and further”. In passing the city of Mainz he drew the attention of the roman-catholic bishop of Mainz, who hired him for a project to reform local law.

“He” is Leibniz, Gottfried Wilhelm Leibniz. He features here today because of the tercentenary of his death. Born 1646, he died on 14th November 1716.

In what world does young Leibniz live and work in the year 1671, the year in which he became 25? In 1671 there was no Germany. Instead, the area we now know as Germany consisted of a collection of smaller and larger states and independent cities. Among the more than 100 states there were catholic dioceses, as in the case of Mainz, Cologne and Trier, there were dukedoms, as in the case of Hanover where Leibniz later would work for 40 years, there were states ruled by counts and margraves. Next to these there were about 50 independent cities, among which Hamburg, Strasbourg, Nuremberg and Ulm. States and cities were united in the Holy Roman Empire, governed by the Emperor, in 1671 by Emperor Leopold I, who was based in Vienna. The position of the Emperor was not hereditary, the Emperor was chosen by the so-called Electors. In 1671 there were eight Electors: the bishops of Mainz, Cologne and Trier and the princes of five of the more important states, such as Saxony and Bavaria.

On the western border of the Empire was the kingdom of France. And the King in Paris was focused on expansion, much different from the rather secluded Emperor in Vienna. For Louis XIV, the Sun King, borders were to be crossed and redrawn. And on the south-eastern border of the Holy Roman Empire was the threat of an invasion by the Ottomans Turks. We should keep in mind that, on the eastern and southern borders, the Empire was much larger than current Germany. The kingdom of Bohemia belonged to it, and towards the south it ran on to the Adriatic Sea.

In 1671 Leibniz has the ambition to make a new move. He had some experience with making moves already. Five years earlier he had left the University of Leipzig because the faculty crossed him in his plan to obtain the law doctorate. On this occasion he wrote, I quote:

When I noticed the intrigue of my opponents, I changed my plans and decided to travel around and to study mathematics. For in my opinion a young man should not, as if nailed down, stay at one and the same place, and for a long time already it was my burning desire to increase my reputation in the sciences and to get to know the world better.

One year later, in 1667, Leibniz received his law doctorate in Altdorf, a small town south-east of Nuremberg. The university directly offered him a law professorship, but he waived the offer and moved to Nuremberg and by the end of 1667 to Frankfurt and Mainz, where we first met him in 1671. 1671, a year of tension in Europe, and a year of internal tension in Leibniz, who wanted to do more in life than revising the local laws in the diocese of Mainz.

In October 1671 he writes to the Duke of Hanover Johann Friedrich (maybe better known as John Frederick, uncle of the later English King George 1st). He subserviently offers his services to the Duke, and as if he wants to convince the Duke of his qualities, he presents his Curriculum Vitae, indicating the domains in which he has been working: philosophy, mathematics and mechanics, optics, navigation, hydrostatics, pneumatics, moral philosophy and law and finally theology. Under mathematics he mentions his calculating machine, also a geometrical instrument and under philosophy a treatise about combinatorics. This goes on page after page, and only then, on the final page, he decides to show his true colours and sketches his arguments and plans for a stay in Paris. It is important to keep an eye on the French, he says. I quote

It is clear that such big French army units must eventually break out, and that, when they break out in Europe, they will be responsible for a long and universal war and will woefully ruin many 100000 people. Also clear that not only the Catholics but the Christians would like to see their efforts against our traditional enemy in the Levant.

This is how Leibniz introduced his famous *Consilium Egyptiacum*, a proposal to be made to the Sun King for an expedition to Egypt. The plan came too late to divert Louis XIV from his expansion in Europe. Although Leibniz’s coming to Paris and his availability to discuss a ‘Small project’ with Louis XIV was announced to the King by the end of January 1672 and although the French foreign minister sent a welcoming reply two weeks later [February 12th], when Leibniz arrived in Paris by the end of March the war had already broken out. It started with the alliance between the English and French King against the Dutch.

But at the time of writing to the Duke of Hanover, in October 1671, this was still covered in the far and dark future. Leibniz had made his point that his presence in Paris was important for the Duke. “I do not have many acquaintances in Paris”, he writes,

but I was already recommended to Minister Colbert, and via the mathematician Antoine Arnauld I can reach his cousin Mr. de Pomponne, who is the foreign minister of Louis XIV. When I am in places unknown to me, and if I am given the opportunity to explain myself, I can generally get access to people, even without recommendation. Also many of the *Curiosi*, the investigators of nature, who did not know me when I first wrote to them, kindly answer my letters.

There follows a list of names: gentlemen from the *Académie Royale*  and the Royal Society, Athanasius Kircher, Otto von Gericke, Oldenburg and Wallis in England, Ferrand and De Carcavy, the librarian of Louis XIV in France, and more of them. “I do not even mention the chemists and mechanists”, Leibniz adds.

Leibniz also has a personal interest, I quote:

My highest wish is how I may obtain such peaceful circumstances that I may employ the little talent that God has bestowed on me for the perfection of the sciences, and for that I do not see better conditions than there are now in France. There His Majesty has issued a Royal resolution to encourage with a pension people of whom one can expect something. In which I hope to righteously succeed, since I am confident of those results that I can easily present, such as: my calculating machine, my optical inventions, my justification of the possible mysteries of the Eucharist, and finally my new counsel for a Holy War.

This, however, will cost money, for travel, for the necessary documents and for the presentation of his calculating machine and his optics. And a recommendation with a high status will also help. A recommendation from the Duke will be most welcome, since his Highness is very well respected in France, says Leibniz. And he finishes his letter with [I quote]:

This is all that I wish from Your Princely Highness. I hope that this does not inconvenience you, but that the honour of the Lord and our common welfare may increase by it, and also your own glory. And that you maybe will have the pleasure of some welcome and useful inventions, for the development of which I am only lacking the measure of the desired peace and quiet mentioned earlier.

For a mathematician this is a stunning letter. I knew that Leibniz was also considered to be a philosopher, but for me he was primarily a mathematician. As a freshman in mathematics I was much amazed by the so-called Leibniz series:

I found this truly astonishing. You divide one by the successive odd numbers, subtract and add them, you do this to infinity, and the sum is a quarter of the magic number, the ratio between the circumference of a circle and its diameter. And even more astonishing, I could prove this. Even if it is impossible to reach infinity during my lifetime, I was assured that the alternate sum and difference of the odd numbers produced the circumference of a circle of diameter one quarter. Even more powerful, I could help others to understand that this was true.

For me, Leibniz was a mathematician, and not one of the lesser ones. And this impression was only reinforced when I took a course about the history of the calculus. Leibniz was one of the two inventors of differentiation, the other and earlier of course being Isaac Newton. Leibniz was the first to publish his work, in a journal article that appeared in 1684. Leibniz renewed mathematics, by describing a new method to determine tangents to curves, and to calculate its maxima and minima. These words were also the start of the title of Leibniz’s article.

If we now take stock, then we see that Leibniz was a doctor of law by education, but also a mathematician, scientist and technician by interest. The ambition of the diplomat is clear as well: Leibniz proposes to act for the Duke at the court of the Sun King. During the lecture I will augment and update the various aspects of his profile.

We return to October 1671. The Duke of Hanover approves, and Leibniz stays in Paris from March 1672 until November 1676. Why was Paris so important for Leibniz? Some highlights:

* Paris was a real centre. The Holy Roman Empire had many important states and towns, which meant that it had no centre. Vienna was too eccentric, Cologne too catholic, Hamburg too far North, Berlin too Prussian. Frankfurt was where the Parliament of the Empire convened and where the Book Fair was, but it was a geographical, rather than a cultural centre. In France everything and everybody concentrated in Paris.
* One of the features of Paris that much attracted Leibniz was the new *Académie Royale des Sciences*. It was founded 1666 by the Sun King and his minister Colbert. They had invited the Dutch mathematician and physicist Christiaan Huygens to come over from The Hague and become its first president. Part of the preparation for his stay in Paris consisted of Leibniz making contact with some of the Academy members. Since June 1671 he had exchanged letters with the mathematician and academy member Carcavy, who was also the librarian of Louis XIV. Leibniz had written to him about his calculating machine, and Carcavy and Gallois, the editor of the important *Journal des Sçavans*, suggested that Leibniz should come and demonstrate the machine to the *Académie*.
* Paris was also a stepping stone towards England. Leibniz visited London twice, in January/February 1673 and in October 1676. On his first visit he accompanied an envoy from Mainz on a diplomatic mission to England. France and England were still fighting the Dutch, and the diplomat Van Schönborn came to the English court to propose a peace conference in Cologne, as he had done in Paris towards the Sun King. Leibniz’s role was to assist the envoy, especially in writing diplomatic notes.

As a scientist, Leibniz came to London well prepared. Just as the Paris *Académie* attracted him, it was the Royal Society that drew him to London. In 1670 he started a correspondence with Henry Oldenburg, the secretary of the Society. Oldenburg was a man of great reputation, and Leibniz was an anonymous doctor of law. And that is precisely how he opened his letter:

WorthySir,Pardon the fact that I, an unknown person, write to one who is not unknown; for to what man who has heard of the Royal Society can you be unknown? And who has not heard of the Society, if he is in any way drawn to an interest in true learning, the true learning I mean which under your guidance particularly is withdrawing from the groves of the critics into the fastnesses of nature.

In his letter to Oldenburg Leibniz goes on to complain about the political situation in the Holy Roman Empire, where many nations are internally competing. This makes a combined effort into the foundation of an Academy difficult. In London Leibniz attended several meetings of the Royal Society, and he applied for membership. He was appointed a fellow on April 19th, by which time he was already back in Paris.

His second visit to London was in October 1676 on his way to Hanover, before he started his work for Johann Friedrich, the Duke of Hanover. In London Leibniz again visited Henry Oldenburg and he met with John Collins, who gave him access to papers by Newton and Gregory. On his return to the continent Leibniz boarded a boat in London 29 October; he was in Gravesend 31 October, and then had to wait in Sheerness from 5 until 10 November. He arrived in Rotterdam 12 November. He then made a tour through Holland, and met with scientists and philosophers, among whom the mathematician and Amsterdam Burgomaster Hudde, philosopher Spinoza and microscopist Van Leeuwenhoek. He was not happy with his economy class transportation, hear his comments:

The boat [from London to Rotterdam], on which I slept for about 10 nights … was an attack on my health, as were the night boats on which I crossed Holland to and fro. I took them in order to have a place for the night and to gain time. On top of that came the wet and cold air in Holland, and the change of food. When I returned to Amsterdam, where I was based, I felt unwell, without any appetite and with a little fever and weakness.

So, if you want to experience Holland, do not take the night boats and maybe choose a better month than November.

* Let me conclude this review of why his stay in Paris was important for Leibniz with the wish that he expressed to the Duke of Hanover: Leibniz wanted “peace and quiet” so that he could study and further develop his inventions. And that worked out very well: during his four years stay in Paris Leibniz made considerable progress, not in the least in the mathematical domain. I will give one example, which now belongs to the basics of a 1st year calculus course, but which was considered as a big result when Leibniz presented it.

Take numbers

and so on …

I guess that you notice how the numbers are constructed. The third number has 3 times 4 in the denominator, and this is general: the n-th number has n times (n+1) in the denominator.

Now try and add these numbers together up to infinity. What would be the sum? In Paris, Christiaan Huygens proposed this question to Leibniz. Leibniz made the addition with the help of a principle that would also later be central in his differential and integral calculus: he realised that he could add by taking differences. It works like this:

and so on.

It is clear that the sum on the left side is what Leibniz was looking for, and on the right side subsequent terms cancel each other, except for the for the first, which is 1. So, the sum of the series is equal to 1.

The most important period in Paris was the end of October 1675. Manuscript notes from that period show how Leibniz invented the formalism and rules for differentiation and integration. Within three weeks mathematics was enriched with the sign for integration and the -notation for differentiation. It was 10 more years before Leibniz published these results.

**The years in Paris summarized**

Leibniz goes through major developments in his Paris years from 1672 to 1676. He grows as a diplomat, which is also noticed by the European sovereigns, since he receives invitations from several courts, from the Danish King for example, to come and work for them. In 1675 he accepts the offer from the Duke of Hanover. Leibniz extends his network widely, thanks to the many visitors to Paris and his own travels. But his main development is in the mathematical and philosophical domain. Much stimulated by the discussions with French and foreign scientists and supported by the resources in Paris, such as the Royal Library, he makes substantial progress, especially in conceiving his differential and integral calculus.

We now leave Leibniz for a while in Hanover, where he arrived mid-December 1676. And we jump 11 years forward, to 15 December 1687 (old style). We jump through time, and also through space, arriving in Basel, Switzerland. The recently appointed professor of mathematics of Basel University, Jacob Bernoulli decides to write a letter to his great hero Gottfried Wilhelm Leibniz. It is his first letter but Leibniz has seen the name Bernoulli already in the *Journal des Sçavans*, published in Paris, and the *Acta Eruditorum* from Leipzig. Both Leibniz and Bernoulli had published in these journals. This sounds common nowadays, but we should realise that these gentlemen belonged to the very first generation of scientists who could use the scientific journal as a medium. These two journals were established in 1665 and 1682. The *Philosophical Transactions* of the Royal Society also dates from 1665, the same year as the *Journal des Sçavans*.

1 Jacob Bernoulli, copy of a painting by his brother Nicolaus Bernoulli

Bernoulli sits down and introduces himself to Leibniz:

Most famous and worthy Sir,

My mind stimulates me already for a longer while to address your Muses. But the multitude of most important occupations, which I knew were pressing on you, and my own consciousness of my limited ability, and the reputation of your name have until now blocked my pen. For not without reason I feared that you would count me among the number of those who try –because they have no other means—to raise their fame by approaching great men. For if this insanity is strange to someone, then it must certainly be most strange to me, who would rather stay unknown, particularly to You, than making myself known. Therefore I want you to be convinced, Worthy Sir, that nothing else has driven me to write to you than the ardour by which I glow for matters mathematical, and an insatiable desire to make progress in these. Because I was unable up to now to find satisfaction from elsewhere, I who live far distant from the mathematical community and almost without resources, I was forced to yet disturb your oracle, whose sublime utterances, if I may say so, often stun me into admiration. All are stunned just as I am, even if they are not really at home in the noble sciences.

Bernoulli then refers to the exceptional humanity of Leibniz and Leibniz’s willingness to help newcomers, and then he starts business. He wants to consult Leibniz about some details in a recent article in the *Acta Eruditorum*, about the resistance of solid bodies against bending. Bernoulli would later produce important results on this topic.

What Bernoulli does not know, is that Leibniz is away from Hanover, and that he will receive the letter only in the late summer of 1690. Leibniz is on a long journey, from October 1687 to June 1690, on which he visits libraries and archives, makes new acquaintances and meets old ones. He is going South with his own coach to Vienna and from there with regular post-coaches to Graz, Venice and finally to Rome. On his return he passes the cities of Florence, Bologna and Modena. From there he crosses the Alps via the Brenner Pass to Innsbruck, then for the second time to Vienna, and finally via Prague, Dresden and Leipzig back to Hanover. Leibniz had several motives for making this complicated and sometimes even dangerous journey. The most important motive is: work.

In his new position, as a counsellor of the Duke of Hanover, Leibniz has to support the political ambitions of the Duke. In 1679 Ernst August, at that time the Roman Catholic Bishop of Osnabrück, had succeeded his elder brother Johann Friedrich as the Duke of Hanover. In the international balance of power of the time the historical connections of a state were very important. The Sun King promoted this principle and put historians at work to support his claims with historical documents. He used it, for example, to emphasize that Alsace had always been dependent on France for selling their products, which meant that Alsace should belong to France. And Louis XIV sent his troops to underline his position. One calls this the War of the Reunions, reunion of France with new territory, which according to Louis historically belonged to France.

Leibniz’s employer, the sovereign in the duchy of Braunschweig-Lüneburg, understood that he, in order to counter this argument of the French, had to produce a history of his house and its possessions. This is where the greater part of Leibniz’s energy went until the end of his life: into writing a history of the Guelf house, the oldest predecessors of the Duke’s family. The sources for this history were scattered through Europe, and the only way to know the contents was to go there and make copies. On his three years journey to Italy Leibniz copied innumerable sources for his history of the Guelfs.

Another important motive for the journey also concerns “reunion”, in this case the possible reunion of the Roman Catholic and the protestant churches. Leibniz’s employer Duke Ernst August, who is Roman Catholic, is much interested in the idea of joining the churches again. Leopold I, the Emperor in Vienna, also a Roman Catholic, has a special diplomat who crosses the empire to discuss a possible reunion with the leaders of the states in the Empire. The diplomat, Bishop Cristobal de Rojas y Spinola, visits Hanover for negotiations more than once, and each time Leibniz is involved. Leibniz, who is a Lutheran, studied since the late 1660’s the dogmatic differences between the churches, especially their dogmatic status of the Eucharist, and he always thought that he could reconcile the Catholic and protestant positions. But Leibniz also has a personal reason to maintain the contact with the bishop, since Rojas has direct access to the Emperor, and Leibniz still hopes to establish a direct contact with the Emperor. The summit of Leibniz’s first stay in Vienna, from May 1688 to February 1689, was that the Emperor gave him an audience, and that he was allowed to present his plans. The reunification of the churches was on the well prepared agenda for this audience.

The Italian trip would deserve a whole lecture. There were so many other things that Leibniz did on his way. In Hanover he had administered the mining activities in the Harz. In the 1680’s this took much of his time. So, on his journey he often visited mines and studied the mining techniques. In Hanover Leibniz was also the librarian of the Duke. On his journey he visited libraries and fellow-librarians. He spoke with the Emperor about the Imperial Library, and if he had been or become Roman Catholic, he could have accepted the prestigious position of custodian of the Vatican Library. Leibniz also happened to be in Rome in August 1689, when Pope Innocent XI died. When in October the new Pope Alexander VIII was chosen, who had the reputation of being enlightened, Leibniz lobbied for lifting the ban on Copernicanism. On his trip he met diplomats and scientists, in Italy for example the anatomist Marcello Malpighi and mathematicians Adrien Auzout and Vincenzo Viviani. And finally, the trip gave Leibniz the possibility of making some touristic outings, for example in the first half of May 1689 to Naples and Mount Vesuvius. Between Graz and Venice he even came through a place called Leibnitz. But maybe he did not even notice this himself. And when, on his return to the North, he came through Modena Leibniz also acted as a marriage broker, at the request of Sophie, Duchess of Hanover. Charlotte Felicitas, daughter of her late brother in law Duke Johann Friedrich eventually married with the Italian Duke Rinaldo III. In 1695 she became the Duchess of Modena.

So far this the long explanation of why Leibniz did not answer the letter that Jacob Bernoulli wrote to him in December 1687. Now that we return to mathematics, let me make a flashback again and tell why Jacob Bernoulli was so impressed and at the same time puzzled by Leibniz. After his return from Paris, as of 1678, Leibniz started to publish about one article per year. The first two appeared in the *Journal des Sçavans* and most of the next ones in the *Acta Eruditorum*. Publication number 8, which appeared October 1684, became a milestone in mathematics. In nine pages Leibniz introduced the differential calculus. The title of the article is informative about the aims of the author. It reads: *Nova methodus pro maximis et minimis, itemque tangentibus, quae nec fractas, nec irrationales quantitates moratur, & singulare pro illis calculi genus*. This is almost English. Leibniz has elaborated the notes he made in Paris into a new ***Methodus*** for ***Maximis*** and ***Minimis*** (maxima and minima), and also for tangents, he adds. The second half of the title is a selling argument: the new method is not hindered by fractions nor by irrationals. In the 17th century several ways to determine maxima, minima and tangent for algebraic curves had already been found, but these were extremely cumbersome if variables appeared in the denominator of a fraction, or under a square root. “My method works always”, that is what Leibniz tells his readership.

It is tempting to go into detail about this article and its importance for mathematics. Yet, I will be concise and only refer aspects of the *Nova Methodus* that are important for what comes next. And next will come the Brachystochrone, which first was a problem and then, when the problem was solved, it was a curve.

Leibniz’s *Nova Methodus* of 1684 has at least three remarkable features:

1. First of all the text was a summary rather than an introduction to a new subject. Those of you who followed a course about differentiation will be amazed to see what the part from the definitions of differential unto the quotient rule looked like. This all fitted on the first page of the *Nova Methodus*, or maybe on two pages if you also count the separate page with figures. A glance from the distance is sufficient to conclude that there is some formalism and that some rules are given. But there are no explanations, nor are there proofs.
2. “This is not good maths”, seems to be the conclusion. And indeed, soon after the publication serious criticism was published by the Dutch burgomaster Bernard Nieuwentijt and the Irish bishop George Berkeley. How did Leibniz then vindicate that the absence of foundations and proofs? Well, a strong point of the *Nova Methodus* was that the theory had convincing applications. On page five of the *Nova Methodus* Leibniz writes:

Now it is better to show the use [of our calculus] in examples that can directly be understood.

He then proves the rule for the refraction of light, from one medium to another medium of different density. This rule is well known as ‘Snell’s law’. Leibniz derives Snell’s law as the solution to a minimum problem: light takes the route between the two media along which the duration of the travel is minimal. He models this with a variable and then has to differentiate a sum of two functions. The appears in a quadratic function under a square root; that is the case in both functions. And for finding the that minimizes the travel time, he solves an equation composed of fractions and square roots. That was a very hard job with the earlier methods. But with the *Nova Methodus* it runs smoothly, and Snell’s law comes out nicely.

That Leibniz’s method possessed such power, impressed his contemporaries very much.

1. The third important aspect of Leibniz’s article is the type of problems for which the *Nova Methodus* works. They are indicated in the title: it works for the determination of maxima and minima and tangents. When the curve is given by an algebraic equation in , the method enables you to quickly find the extreme values and tangents. But sometimes it happens that one does **not** know the equation of the curve, but only a property of its tangent. And one searches for the curve itself. A famous problem from the first half of the 17th century, put by Debeaune to Descartes, was such a problem, a so-called inverse tangent problem. In the final paragraph of the article Leibniz shows that the solution is a curve that requires the use of the logarithm. This final example is important in two respects: it put inverse tangent problems on the agenda, or differential equations as they are called nowadays. And it shows that the solution to such a problem can be non-algebraic.

**Leibniz in new roles**

We have met Leibniz in several new roles: traveller, mining expert, librarian and historian. His 1684 article *Nova Methodus* raised the interest of the Basel mathematician Jacob Bernoulli. And of his younger brother Johan Bernoulli, whom we shall encounter shortly.

It is the summer of 1690. Leibniz is back in Hanover, and Jacob Bernoulli waits for an answer to his letter of December 1687. These two and a half years Jacob continued his studies of Leibniz’s articles, especially of the *Nova Methodus*. His brother Johann joins him. Johann is born 1667, he is twelve years younger than Jacob. They have their interest in mathematics and physics in common, and also that they both did not want to enter the business of their father, Nicolaus Bernoulli, who was a merchant in spices. In cooperation they tried to understand the *Nova Methodus*, and they picked the first fruits from it. This joint work had started quite early. In an essay about infinite series of 1689, for example, Jacob proves that the sum of the inverses of the integers is infinite, that is:

2 Johann Bernoulli (opposite title page of Opera Omnia, vol. 1 (1742))

and he adds: “My brother was the first to understand this.” In a French eulogy of Jacob Bernoulli, published 1706, one year after his death, the history of the Bernoulli Brothers working together to understand the calculus reads as follows:

In the year 1684 geometry at once changed its face. The famous Mr Leibnitz published in the Leipziger Journal [i.e. the *Acta Eruditorum*] solutions to problems, which he had obtained with a completely new method, the secret of which he however hid. This amazed both gentlemen Bernoulli to such an extent that they worked jointly to reveal the secret of this new geometry. This secret seemed to them to lie far away from ordinary geometry. They managed in such an excellent way, that Mr Leibnitz has declared that this new art comes on their account no less than on his own.

Jacob and Johann Bernoulli in joint collaboration. This lasts some years, the period in which Jacob and Johann meet in two different roles. At the university Johann is a student who follows Jacob’s lectures, and as brothers they study Leibniz. In 1690 Johann receives the degree of licentiate in medicine, and at this point he interrupts his studies and leaves Basel for about two years. In 1691 he is in Geneva, where he teaches the calculus on a private basis, and at the end of 1691 he extends his journey to Paris. There he enters the circle of scientists around Malebranche, where he meets some mathematicians with whom he stayed in contact throughout their lives: the Marquis Guillaume-François-Antoine de l’Hospital, whose name lives on in ‘L’Hospital’s rule’, and Pierre de Varignon. With both men Johann has an intensive contact, especially with L’Hospital, who asks Bernoulli to teach him Leibniz’s New Method. Johann’s notes in preparation for these weekly lessons have been preserved, and they are the model after which L’Hospital in 1696 writes the first textbook about differentiation, his famous *Analyse des infiniment petits* (‘Analysis of the infinitely small quantities’). Apart from the personal lessons of the years 1691/2, much of L’Hospital’s textbook is based on his subsequent correspondence with Johann. Bernoulli and L’Hospital had a contract, that Bernoulli would send his new calculus results to L’Hospital and that the latter would regularly send a fee in return. That is the concise history of L’Hospital’s rule: the Marquis bought it from Bernoulli, who sent it from Basel to Paris on the 22nd of July 1694. In his book L’Hospital is completely fair and gives due credit to the creators of the new method: Leibniz and the Bernoulli Brothers, but no word about the contract.

Back in Basel, by the end of 1692 Johann is in an awkward situation. He is highly qualified in the new mathematical developments, he has given private tuition about it, both in Geneva and in Paris, but in Basel this does not earn him a twopence. Johann decides to complete his medical studies, with a doctorate on muscular movement in 1694. His application for the chair in logic at Basel University fails, so much against his liking he decides to take the job of municipal engineer. He writes to L’Hospital about the land measurements from which he had just returned (5 Febrary 1695):

now that I have given up the medical profession in favour of mathematics I have to take upon myself anything by which I can earn myself a living.

The situation is even more pressing because in April 1694 Johann got married. When, in the autumn of that year, L’Hospital invites Johann to come to his *chateau* for the wine-festival, Johann declines and writes, with the typical humour of mathematicians among each other (27 October 1694):

The reason [of my refusal] is the trouble that my wife will shortly present me with, by practising the rule of three, which I taught her in combination with multiplication, whence the two of us will shortly change into three.

On February 6th, 1695 a son Nikolaus Bernoulli is born. Johann needs to improve his position. The University of Basel had no place for him, one mathematician should do, and that mathematician was his brother Jacob.

What did all these mathematicians do, between 1690 and 1695, the years that we are now looking at? Let us see some highlights.

Leibniz answers Jacob’s letter of December 1687 on 4 October 1690. But interaction via a journal article had preceded his reply letter by more than two months. Since one of the first things Leibniz does when he is back in Hanover is to react in the *Acta* Eruditorum of July 1690 to two publications of Jacob Bernoulli of May 1690. Five years earlier Jacob had proposed to his colleagues a problem about the game of dice, and when after five years no solution had appeared, he published his own solution in the *Acta Eruditorum* of May 1690. Jacob’s second publication, in the same issue, presents his solution to a problem set in 1687 by Leibniz about the so-called isochronous curve: to find the plane curve along which a body will descend under the influence of gravity with constant vertical velocity. Solutions were given by Huygens, without proof, and by Leibniz himself in 1689. In his article of May 1690 Jacob applied Leibniz’s New Method, differential calculus, to correctly solve the problem. In his reaction to Jacob, Leibniz shows that the solution to the dice problem can be improved, but he gives great praise to Jacob’s solution of the isochronous curve.

Mr. Bernoulli, he writes,

presents his analysis according to the rules of my new calculus, which I call differential or incremental, published by me in these *Acta*. To elaborate this analysis of my problem was totally not everybody’s business, because the art of this calculus is until now known to very few people, and I know no one who has better penetrated my thoughts than Mr. B.

By the end of 1687 Mr. B had still much difficulty with the New Method, for he asked Leibniz to give him access to the Superior Geometry. In Bernoulli’s own words:

If you want to provide me with some ray of light of your method (for which I have a great desire), as far as will be possible because of your heavy occupations, you will thereby take care that I will subsequently be not merely an admirer of Your Inventions, but also a worthy expert and propagator of them.

We can draw two important conclusions from this early communication, via publications and via private correspondence:

* In three years’ time Jacob and Johann Bernoulli are able to solve their initial difficulties with the differential calculus.
* Within the scientific community of the time, challenging colleagues to solve a problem is not unusual. There were earlier examples, and between Leibniz and Jacob Bernoulli there are several examples. We saw Bernoulli in 1685 challenging his colleagues without addressing one of them directly. Leibniz set a problem in 1687, which Bernoulli solved in 1690.

This goes on and on. It is now Bernoulli’s turn. When he has solved Leibniz’s problem of the isochronous curve, he challenges Leibniz personally with the problem of the catenary: to find the shape of an idealized chain hanging under gravity. Leibniz reacts in his 1690 answer to Bernoulli in an interesting way. The problem is more intricate than my earlier one. It will need an extraordinary use of our method. Therefore, says Leibniz, I will give time to others before I publish my own solution. In this way we will get to know the best methods, and this contributes to perfect the sciences. Leibniz also sets a deadline: if no one before the end of the year indicates to have found a solution, I will publish my own result, says Leibniz, adding: Deo Volente. And finally, in the concluding paragraph of the article he addresses one group of colleagues in particular, i.e. those mathematicians, who think that the analytic geometry of Descartes and Van Schooten can solve any problem. Do try my problem, says Leibniz, algebra is not powerful enough to solve it. Wake up, and do not stake too much on acquired knowledge. Leibniz does not point in a geographical direction, but it is clear that he has one particular country in mind. He phrases this critical note in a nice and diplomatic manner.

Still no brachystochrone yet, but it will now soon descend.

**1690, the situation in a nutshell**

In 1690 Leibniz is in contact with Jacob Bernoulli. Challenges go to and fro. Leibniz chooses his from the domain of rational mechanics. He is in these years much interested in the principles of mechanics and optics, and differs more and more from the Cartesian principles. Leibniz the physicist.

Jacob Bernoulli’s challenge of the catenary, for which Leibniz specified the condition, is taken up not only by Jacob Bernoulli himself but also by Christiaan Huygens and Johann Bernoulli. It is tempting to analyse and compare the solutions, but it is even more tempting to see what happens next. Again Jacob proposes a new problem, and again Johann publishes a solution, this time in the Paris *Journal des sçavans* of 28 April 1692. The heading of the article first phrases its topic: “Solution of the problem of the curvature that a sail makes when the wind blows into it.” and then it introduces its author: “By Mr. Bernoulli, brother of the professor at Basel”. Johann tells that his brother, in his solution of the catenary problem, “took the opportunity to research several curves that Nature every day puts before our eyes, and among others that curve that represents a sail blown up by the wind, when one ignores its weight.” My brother, says Johann, arrived at a second order differential equation, which he sent to me while I was in Geneva, yet without telling me the method that he had used. He urged me to find an explicit equation of the curve, or at least to describe a pointwise construction. But I did not work on it –this is still Johann speaking– because the equation of my brother seemed very difficult, since it contained two kinds of differentials and integrals.

Johann says that he forgot about the problem until he again received a letter from Jacob, who informed him that he had transformed his equation in an equation in which only differentials appeared. Johann continues that Jacob urged him once more to complete the solution that he, Jacob, had started with and to lead it to an explicit equation. According to Johann, his brother apparently found this a hopeless effort. Then there follows a paragraph in a style that will be typical of Johann in many instances:

Thus trying the thing, I have not only found the method by which he had come to this equation; but I have completely resolved the problem: since I find, what is admirable, that the curve of the Sail is the same as that of the Chain.

So, according to Johann the curve of the sail and the shape of the hanging chain are the same. And he shows his pride. Up to this point there was no direct contact between Johann and Leibniz. There also was a break in the communication between Jacob and Leibniz, because it took Jacob five years to write again to Leibniz, after the latter had replied in 1690 to Jacob’s first letter of December 1687. Johann writes to Leibniz by the end of 1693, when he is in urgent need of a job. Johann asks Leibniz to recommend him to Duke Anton Ulrich of Wolfenbüttel. Since 1691 Leibniz on top of his duties in Hanover also was the director of the famous library in Wolfenbüttel; he was often in Wolfenbüttel. Johann praises Duke Anton Ulrich for his interest in nature and art and especially in mathematics. He hopes that the authority of Leibniz will support his case with the Duke in such a manner that it will result in a job for him. The bottom line of Johann’s first letter to Leibniz is: “My distinguished Brother gives you his sincerest greeting.”

But Johann changes his mind about an application to Wolfenbüttel. He informs Leibniz that he instead prefers to stay in Basel. Leibniz answers March 1694:

I understand that your plans have changed and that your fathertown has laid its hand on you. […] You have treated him [the Duke in Wolfenbüttel, JvM] rightly, I hope that this also holds for Yourself, for whom I hope that all things are blessed with your family; then it is alright, even with a lower income. Especially with the undoubted prospect of greater things. Moreover, in my judgment and feeling the close contact with Your Brother alone could bind you there [i.e. in Basel] while you are towards each other a mutual support and stimulus. Certainly for me, if someone like the two of you were available to me, that would be a pleasure, much preferred above most others.

Leibniz also finishes his letter with a greeting for Johann to pass on to Jacob. But gradually Johann’s alleged pleasure of collaborating with Jacob wanes. I will now pass by some of the events in the worsening contact between Jacob and Johann, and pick up again the thread of Jacob’s challenge of the curvature of the sail. Jacob was not amused about how Johann had presented his solution in the *Journal des sçavans*. In a long article in the *Acta Eruditorum* of December 1695 Jacob gives his version, which amounts to the assertion that he, Jacob was the first to understand that the Sail is curved in the same manner as the hanging chain. He writes that he knew it, but that he informed his brother only of the equation that led him to this conclusion. In other words: Johann received the solution from him, and not the other way round as Johann had presented it in 1692.

Johann is furious that Jacob publicly calls him a liar. He is especially angry, because in 1695 he had decided to leave Basel in order to become the professor of mathematics at the University of Groningen, in the North of the Netherlands. In April 1696, when he has just received the December 1695 issue of the *Acta Eruditorum* he writes to Leibniz to unburden his heart. I let Johann speak for himself:

Some days ago I received the *Acta Eruditorum*, among which the issue of December last year; I cannot be very much surprised that the mood of my Brother is still occupied by too malicious envy; I believed certainly that my departure [from Basel, that is] had reconciled everything, and therefore I have written to him from here, directly after my arrival, as friendly as possible that I reconciled me with him all the easier, but he has not yet answered and I think that he will never answer. I really rather see the contrary happen; probably you have not yet seen in this month’s issue how viciously he has written against me, driven by I don’t know what jealousy or animosity, and how low he speaks about me. I absolutely do not owe him an answer, whence I maybe give rise to an admonition from You and Mr Mencke, but what should I answer to his nasty remarks, his nonsense, the mean follies which swarm through every writing of his. Meanwhile I would appreciate it very much –if the possibility arises– that You yourself would support my case and would vaguely hint what according to You originates from my brother and what from me, so that the Readers may judge what to think about both our minds, and that they will not directly be overtaken by his nonsense.

What, I wonder, does he ask on page 546 with his *little history*? What does it sow or mow for the Reader? Or because of what necessity or occasion does he bring this forward? Or it should be in order to extraordinarily praise his dexterity in solving problems; but he pushes me down as far as he can. Meanwhile, if the matter should be in the open (and Mr L’Hospital will be my witness), then he has not been able to solve the equation , that he had reached and he was stuck for a year or more to finalize it in any manner, just as I can show from his letters. When he then communicated it [the equation, JvM] to me during my stay in Paris, I had solved it on the spot and had seen that the curve behind this equation is the same as the catenary; and when I had shared the solution with my brother (ignoring the advice of Mr L’Hospital to keep it to myself for a while, and to inform him only that I was in the possession of a solution, in order to see if he would find it himself), he soon wrote back to me that, just before he had received my letter, he had come across the catenary curve. And he has immediately sent this invention as his own to Leipzig; now I leave it to you to judge, if it is likely that my brother precisely in that period in which my letter was under way from here to there, achieves what he could not achieve during the whole year before; and rather, if it is not probable that he plagiarized by claiming my solution for himself.

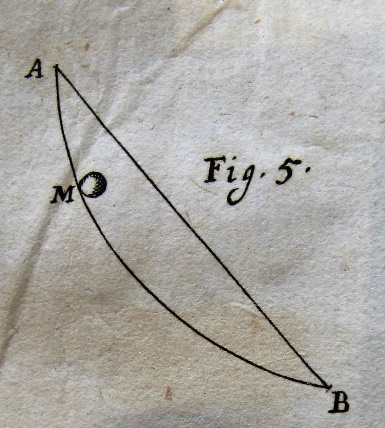
Leibniz reacts six weeks later. He is moderate and mediating and positive to the request of Johann, who had asked Leibniz to intervene. But he does not indicate how and when precisely he will carry out his promise. Neither does Leibniz discuss the accusations of Johann against Jacob. He takes a higher stand, in which he makes clear that he values both. The balance in the letter is slightly in favour of Johann. In his own words it sounds like this:

You act in a praiseworthy manner, that you have chosen to stay moderate towards your brother. I will be glad, if an occasion arises, to give a testimony (that you do not need) according to my opinion about Your candour and about the merits that are the subject of the quarrel between the two of you. Why can You not be praised for your merit? It has no damage for him, it goes rather with honour for him, since he was the first to educate you towards these studies.

Johann’s turn. He is in a dilemma, since he owes Leibniz an answer, but at the same time Jacob’s article screams for a reply. Johann writes to Leibniz in private, and to his brother and the whole mathematical community in public. His message to Leibniz is: of course I need your testimony. I am much distressed, it is about my honour, even more because it is attacked by my own brother. His public message is the challenge of the brachystochrone. In June 1696 the challenge is published in the *Acta Eruditorum*, and the challenge is also included in Johann’s “of course I need your testimony” letter to Leibniz. These are Johann’s words to Leibniz:

I attach a curious problem, for which I allow the Geometers the remainder of the current year. If no one would produce a solution within that time, I have said that I will produce mine. Because I do not know if it has already appeared in the *Acta* or will soon appear, I will repeat it here with pleasure. I ask you, if time permits you may want to devote some effort to it: Given two points *A* and *B* in a vertical plane, to determine the path *AMB*, along which the movable point *M*, starting to move from point *A* and descending because of its own weight, reaches point *B* in the shortest time. I am amazed that this problem until now has not come to anyone’s mind.

The problem does not yet have a name, it will be baptized six weeks later. What does Johann propose to the world?

He takes two points *A* and *B* in a vertical plane. *A* is higher in the plane than *B*, and *A* is not directly above *B*. Then there is a point *M*, which moves along a curve from *A* to *B* because of its own weight. A good way to imagine this, is that *A* and *B* both are points on a ramp or slide, and that *M* is a marble which rolls down along the ramp. Or think about a piece of wire which connects *A* and *B*. In that case *M* can be a bead that slides along the wire from *A* to *B*. There are infinitely many different shapes of ramps or wires connecting *A* and *B* and Johann’s problem asks to find the shape of the ramp along which marble *M* reaches *B* in the shortest time. That shape, that curve is the solution to Johann’s problem.

3 The diagram in the Acta Eruditorum of June 1696

Would the straight line that connects *A* and *B* be the solution? Time for a physical experiment. We see two students of the Liceo Ancina in Fossano, Italy, a girl and a boy. The girl’s father is a carpenter and has made two ramps. Both have the same highest and lowest point *A* and *B*, but one ramp is curved and the other one is straight. Let us first look, then I will give some comments, after which we will look again.

[video [Italian students](https://youtu.be/p62IVE7GTYM): URL: <https://youtu.be/p62IVE7GTYM>]

The students perform two different experiments. The second one, which is fresher in your memory, shows that this particular curved ramp is quicker than the straight line.

In the first experiment both marbles roll on the curved ramp. If the experiment is executed precisely, which is to say that both marbles are released precisely at the same time, they will arrive at the lowest point precisely at the same time. So, the duration of the descent is always the same irrespective of how high the marble is released. Of course, this is only the case if the curved ramp has a very special shape. Only if the curve is a cycloid, it has this property that the duration of descent is independent of the starting point of the descent. The curve is called *tautochrone*, the name is composed of the Greek words for ‘same’ and ‘time’. The tautochrone property of the cycloid was discovered in 1659 by Christiaan Huygens, who published it 1673 in his famous book about the pendulum clock. In Paris Huygens gave Leibniz the book as a present. The Bernoulli brothers also knew the book.

Let us see the video again.

First the students demonstrated the tautochrone property of the cycloid. And then they showed that the descent on the cycloidal ramp takes less time than the descent on the straight ramp. Even stronger, Johann Bernoulli and some of his colleagues proved in 1696 and 1697 that the cycloid is the best possible path. Along the cycloid the movable point *M* reaches *B* from *A* in the shortest time.

What is a cycloid? I will show that with a short fragment from a video by the Open University team that created the History of Mathematics course MA 290. They stage the cycloid in the work of Roberval, a French mathematician around 1650,

[video [MA290\_cycloid](https://youtu.be/Psfkpfstj8E?list=PL1BAF8C53CA724006&t=399) ] URL: <https://youtu.be/Psfkpfstj8E?t=399>

Leibniz’s calculus was the outstanding technique for studying curves such as the cycloid. And in the hands of Jacob and Johann Bernoulli it was also the technique to solve Johann’s Brachystochrone problem. This is a good point at which to say something about the name “Brachystochrone”. May I remind you: when Johann posed the problem, in his letter to Leibniz, it did not yet have a name. There was only one week between Johann’s letter and the reply of Leibniz. Both Leibniz and the posts between Groningen and Hanover were extremely fast. Leibniz likes and praises the problem. He transforms it into a differential equation for the curve, but does not yet realize that the curve that satisfies the equation is a cycloid. He will hear that soon from Johann. Also, Leibniz proposes to call the curve “tachystoptote”: the curve of quickest fall. Five weeks later Johann presents what he thinks is a better name: *brachystochrone*. His argument is that when he realized that the solution to his problem was the cycloid, he immediately thought about Huygens, who found the tautochronism of the cycloid, or isochronism as Johann prefers to call it. And he now wants a name for his own problem that associates with isochrone, and chose brachystochrone, the curve of shortest time.

So, on 31 July 1696 the problem and curve have a name, and Johann and Leibniz know the solution: the cycloid.

By this time the problem has reached the readership of the *Acta Eruditorum*. Leibniz and Bernoulli also spread the word via correspondence. John Wallis receives it, Varignon in Paris and via him L’Hospital, Leibniz sends it for publication to correspondents in Florence and to the *Journal des sçavans*, Bernoulli distributes it in the Netherlands. At first the deadline is: the end of 1696, but then Bernoulli decides to include the problem in a pamphlet, a broadsheet printed by the Groningen university printer, which he uses as a seasonal greeting for New Year 1697. The pamphlet extends the deadline until Easter 1697. Johann sends it out widely, in England Newton and Wallis received a copy. Newton’s solution appears anonymously in the *Philosophical Transactions* of January 1697. According to the anonymous author the solution was produced within one day.

Jacob, who also receives a copy of the pamphlet on 6 February 1697 writes a furious letter to Leibniz, I quote the PS, which Jacob wrote when he was about to post the letter:

Precisely at this moment a kind of giant printed Programme runs into my hands, in which my brother for the third time urges all Geometers of the whole world, and as it seems especially me, to solve his problem, with words full of boast and bile. I know my limitations and I do not believe that I have solved it, but rather that God solved it through me in order to calm down his excessive haughtiness. Yet I bitterly regret that this man has forgotten himself to such extent that he does not remember any more by what instrument the grace of God has worked in him long ago.

Jacob received the problem in the *Acta Eruditorum*, Leibniz himself had included it in a letter to Jacob, and now Jacob receives Johann’s printed sheet, dated New Year’s Day 1697. As he wrote to Leibniz, Jacob felt like trying to solve the brachystochrone problem when Leibniz had written to him that he, Leibniz, had solved it. Jacob adds that he solved the problem quickly, but that he wanted to use the occasion to widen the view to other, more difficult matters, to be proposed in the *Acta Eruditorum*. The culture of challenges is still very much alive: if I solve your problem, you may try this more intricate problem of mine.

What comes next in this story is amazing and horrible at the same time. Jacob and Johann fight as madmen, for several years. The recent edition of their papers fills more than 400 pages in the *Streitschriften* volume of the Basel Bernoulli edition. An English equivalent of *Streitschriften* would be *Fighting Papers.* That would be too much for today. Moreover, we celebrate the tercentenary of the death of Leibniz, so let me finish by returning to the diplomat Leibniz and presenting some of his actions to moderate the fight between the brothers. That will also add to Leibniz’s biography.

First Leibniz tries to calm down the brothers via his letters. Johann Bernoulli was his main correspondent, 139 letters from Johann Bernoulli to Leibniz and 146 letters from Leibniz to Bernoulli have been preserved. The combination of frequency, length and content of the letters is amazing. The exchange with Jacob Bernoulli is small, about 20 letters in total. We hear Leibniz intervene, first in April 1697 towards Jacob:

I received the programme of Your brother, too. It seemed to me that he had addressed some words, not to You but to Mr. T[schirnhaus], but I may be wrong. And apart from that, he himself finds that you have treated him wrongly in the *Acta*. I, who have the highest esteem for both of you, would like that you are the best friends, and not that you are ill-disposed towards each other. There is no doubt, I think, that he [your brother] owes to you at any rate the basics of his studies and for the greatest part its increase too. After all he is introduced into these mysteries by his elder brother. And this is also the reason why he, when feeling attacked, has not wanted nevertheless to respond to you in a too sharp manner. If we set that aside, may I then think that the younger also gives very much to the older, and that the older must use this privilege with moderation. And if I can contribute anything to revive your mutual affection, I would spare no effort.

This sounds to me diplomatic, Leibniz is kind and little critical at the same time. For Jacob it is much too critical, for the second time he does not answer and the correspondence is interrupted for another five years.

Now a quotation from one of the many letters in which Leibniz discussed the struggle with Jacob. Leibniz writes 24 June 1701. Much has happened since 1697. I will summarise the sequence of events, for a better understanding of the quote. In the *Acta Eruditorum* of May 1697 Jacob had published his correct solution of the brachystochrone problem, together with a new challenge. Jacob even put up a considerable amount of money, more than 10 percent of his annual salary, for the best solution and asked the Paris Royal Academy to organise the contest. Then a judge was to be appointed to choose the best solution of Jacob’s new challenge. There was even a fight about the choice of judge. In April 1701 Leibniz keeps the promise made to Johann years earlier and writes in the *Acta Eruditorum* a declaration in which he presents his view on the affair between the two brothers. In this sequence of hostilities Leibniz writes to Johann about his declaration in the *Acta*. This is Leibniz writing to Johann Bernoulli on 24 June 1701:

When you consider the case, you can easily understand that I could not have written more evenly about the quarrel between both of you than I have done. I have said it as it is, that I have received your solution in a timely manner, and that I have approved of it. But I had yet to add, what is also true, that I have not been able to pay such attention to it that would have been obligatory for a judge. Obviously, lest your Bother says that with my *declaration* I want to take up a position as a judge. So, he will, with no instrument of torture, squeeze a reconsideration from me. If he will do this, he will cause a public contradiction from my side. I do not say that the two of you have not been able to master your emotions in a futile affair; but only this, that both of you have to be careful, for your own prudence, that no one can blame you for it. And this warning was absolutely necessary, in order to not make the impression that I want to keep the brotherly quarrel going (it cannot be said enough how much everybody is displeased at it).

This, so far, is Leibniz’s warning to Johann, and it is only the first half of his critical, yet friendly words.

A second approach of Leibniz and some colleagues to calm down the brothers, is to offer them a carrot, provided that they would behave properly. The carrot is a prestigious academy membership, in this case even the membership of two scientific academies. First Leibniz promoted the idea that Jacob and Johann should be appointed members of the grand old Paris Academy of Sciences, and since he was not a member himself he suggested that he himself would also be appointed. But the appointment should only be effected on the condition that they would stop their fighting. In the words of Varignon, Johann’s good friend and member of the Paris Academy, 19 February 1699:

I had the honour to tell you in my last letter that if something that I expected, would come true, you would have reason to be pleased, your brother and you and also Mr. Leibnitz. This “something” has happened, and I hope with it to reconcile you completely with Monsieur your brother. You were both received by the Academy last Saturday; but on the condition that you would cease to fight each other, that you would both drop your objections, and that you together would never have these again. I have warned you both about this article, as it also was a matter of honour for me that you would both keep to it.

Some 25 years after Leibniz had demonstrated his calculating machine in Paris to the Academy, he was now one of its members; no special conditions set on his behaviour.

Another academy membership for Johann and Jacob Bernoulli was easier for Leibniz to settle. Since 1700 Berlin had its own *Sozietät der Wissenschaften*, Society of Sciences, and Leibniz was its first president. The Berlin Society had little resources. This prevented Leibniz from having Johann Bernoulli come over to Berlin. But with his appointments of academy Leibniz also had in mind to foster the peace between Jacob and Johann. So, he invited not only Johann but also Jacob to become a member of his Society. The invitation once more broke the ice between Jacob and Leibniz. On 15th November 1702 Jacob writes his first letter in five years to Leibniz. He devotes some lines to the quarrel with Johann and especially to the role Leibniz had played. You were biased, says Jacob, and I doubted your good faith. That being said, he exclaims (on paper, that is):

Let these things now go by, and let any memory of them disappear.

(For a historian this is a painful line, since what I am doing here is reviving these things. But this in parentheses.)

Leibniz acted via his private correspondence, and via the circles in which he had influence, such as the scientific academies. He also acted in public, via his article. I mentioned already his view on the solutions of the brachystochrone problem, which he published in the *Acta* of April 1701. At the end of this lecture I will give the last word to Leibniz. So, let me now, as a summary, draw some conclusions, after which Leibniz will complete the lecture:

* first: it was an amazing experience for me to read so much Leibniz, who appeared to be so much more than the mathematician who in 1684 published the crucial *Nova Methodus*.
* second: Leibniz’s diplomacy was often explicit, when he was following the orders from the bishop in Mainz and the Dukes in Hanover. But more often his diplomacy was informal. And Leibniz certainly did not forget his own interests.
* then: this talk could have had more so-called hard mathematics in it. It would certainly have given more esteem to Jacob Bernoulli, who augmented the Leibnizian calculus in a fundamental way. The mathematical world knows Jacob better than Johann. One way to see this is to look at stamps. I doubt whether Johann reached a stamp, but Leibniz and Jacob Bernoulli can be seen on stamps, for example in the book *Stamping through mathematics* by Robin Wilson. But of these two, only Jacob features on the cover of the book.
* finally: my sympathy goes to the younger brother, who for ten years left his country and went as a foreign worker to the Netherlands, to Groningen, the city where I live. In Groningen Bernoulli is a name. Johann’s famous son Daniel is remembered on a plaque at the location where in 1700 he was born. And his father Johann has a monument at the entrance of the University campus. It is composed of two cycloidal arches and their generating circle and reminds Groningen students and faculty of the brachystochrone problem. Although most of them will not have the faintest idea what the thing represents.

Leibniz has the final word. It comes from his 1701 view on the brachystochrone problem:

Professor Jacob Bernoulli from Basel has sent me his open letter to the Groningen professor Johann Bernoulli, from which it is only too clear that his intention is to invite me in a certain way to take a particular position. I, for my part, hold both brothers in so great esteem as one can only esteem a mind that has penetrated into mathematics very deeply. And I owe both my sincerest respect, and the scientific community does so even more, that mainly the seeds they spread, partly indirectly and partly by their own inventions, resulted for my method in such a rich harvest. [……] For the rest I wish very much, yes I ask for it, that a noble pair of brothers may temper its rivalry, though most useful for the study of higher mathematics, so that people do not think worse about the sciences, when they must see that men of so big and immeasurable mental gifts, yes what is the summit, brothers, in a minor case have not been able to control their emotions.

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