Fractal curves: from the esoteric to the ubiquitous

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“An excellent mathematics result, but it is far to early to be so one-sided.” P. W. Rundle (Headmaster)

[End of first year report at secondary school.]

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Caveat emptor!

Mathematics of smooth curves

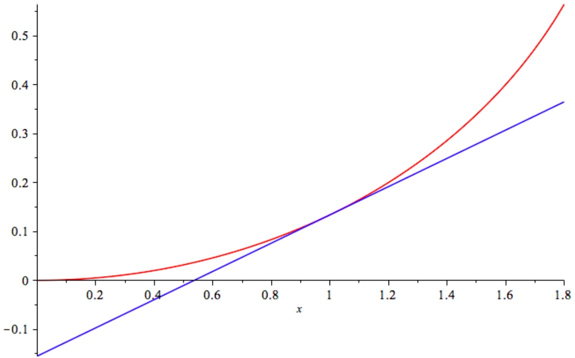
• With the development of the calculus by Newton and Leibnitz much of the mathematics and its applications in the 18th and 19th centuries related to curves or formulae to which the calculus could be applied:

- Differential equations
- Fourier series
- Geometry of smooth surfaces

- Planetary motion
- Wave motion
- Potential theory - gravitation - electromagnetic theory.

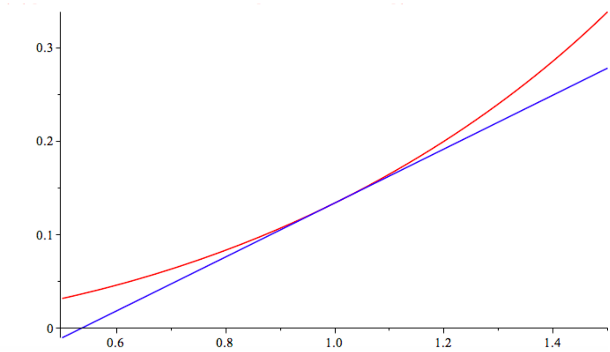
The calculus was suited to the analysis of smooth curves and 'nice' formulae.

Smooth curves and tangents



A curve is thought of as smooth if it has a *tangent* at each of its points, or if under increased magnification the curve looks almost like a straight line.

Smooth curves and tangents



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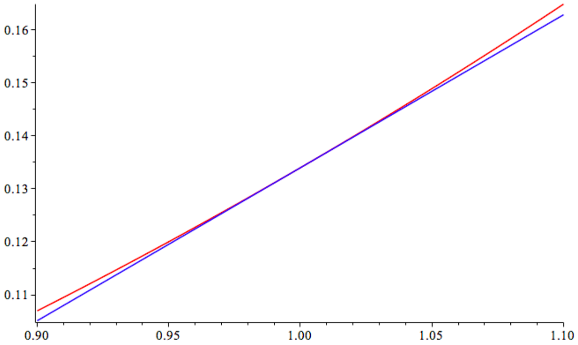

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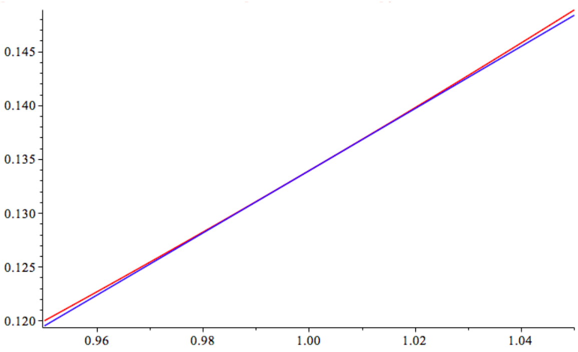

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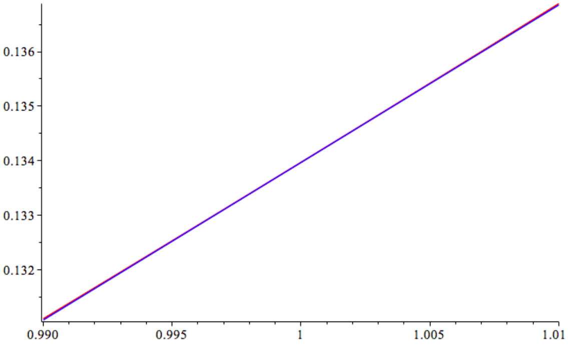

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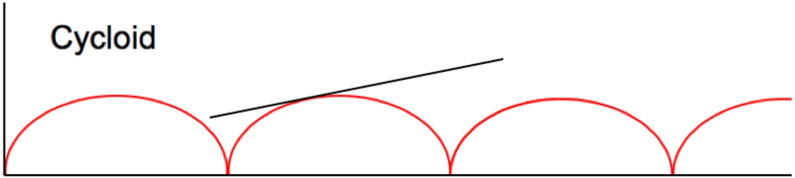
Mathematics of smooth curves

• It was generally assumed without question in the 18th and 19th centuries that to be of any interest curves should be smooth, with a unique tangent line just touching the curve at each of its points, or at least with the exception of a small number of points.

Example:



Non-smooth curves

However, not all curves need be smooth.

We will look at four *non-smooth* curves which were introduced between 1870 and 1915 that were *isolated examples* constructed to have very specific properties to provide examples or counter-examples for certain mathematical ideas.

The Weierstrass curve (1872)

"Until very recently it was universally assumed that a continuous curve always had a tangent except at isolated points. To my knowledge, even in the writings of Gauss, Cauchy and Dirichlet there is no remark from which one can infer that these mathematicians held any other opinion.

Karl Weierstrass
1815-1897

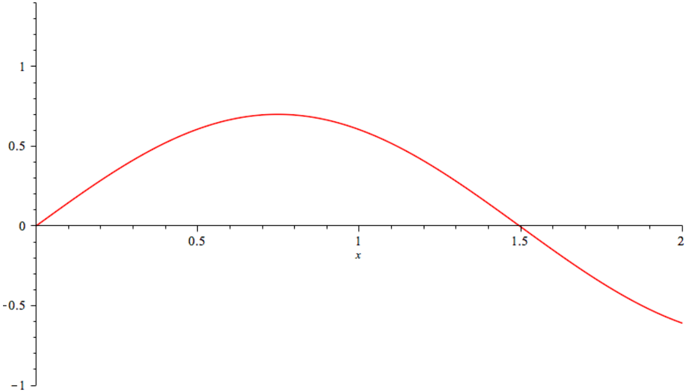
The Weierstrass curve (1872)

"Until very recently it was universally assumed that a continuous curve always had a tangent except at isolated points. To my knowledge, even in the writings of Gauss, Cauchy and Dirichlet there is no remark from which one can infer that these mathematicians held any other opinion. [...] One can, however, easily construct continuous curves for which it is possible to show by the simplest of means that they do not have a tangent at any point."

Über continuirliche Functionen eines reellen Arguments, die für keinen Werth des Letzteren einen bestimmten Differentialquotienten besitzen (1872)

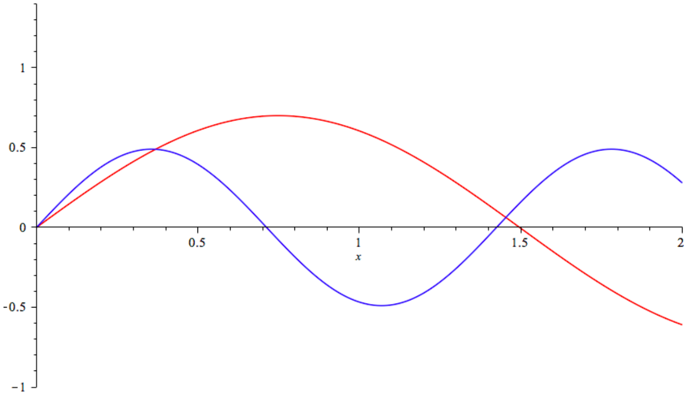
Karl Weierstrass
1815-1897

The Weierstrass curve (1872)



$0.7 \sin(2.1x)$

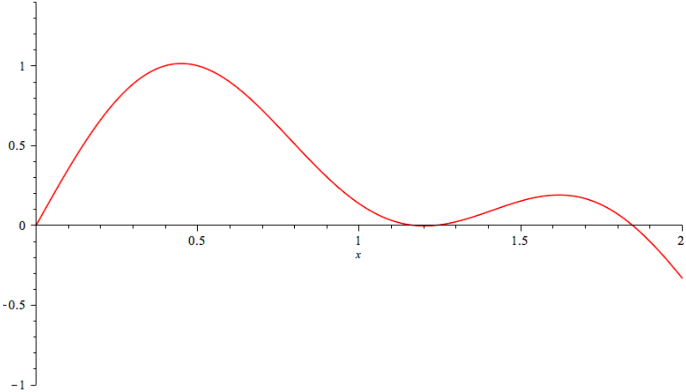
The Weierstrass curve



$0.7 \sin(2.1x)$

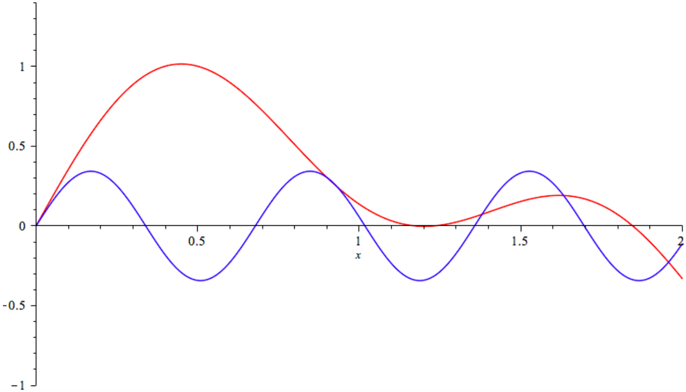
$0.7^2 \sin(2.1^2 x)$

The Weierstrass curve



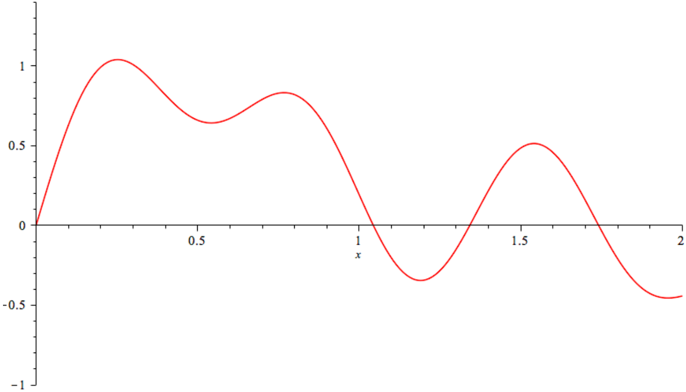
$$0.7\sin(2.1x) + 0.7^2\sin(2.1^2x)$$

The Weierstrass curve



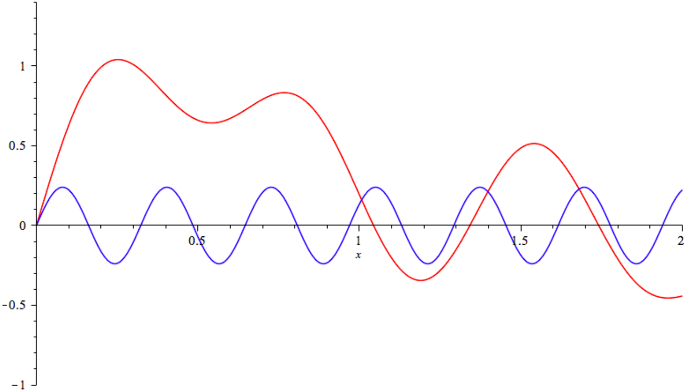
$0.7\ \sin(2.1x)\ +\ 0.7^2\ \sin(2.1^2x)$

$0.7^3\ \sin(2.1^3x)$



$$0.7 \sin(2.1x) + 0.7^2 \sin(2.1^2 x) + 0.7^3 \sin(2.1^3 x)$$

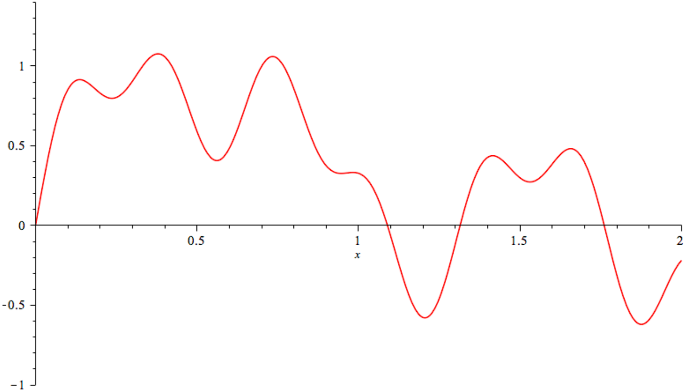
The Weierstrass curve



$0.7\sin(2.1x) + 0.7^2\sin(2.1^2x) + 0.7^3\sin(2.1^3x)$

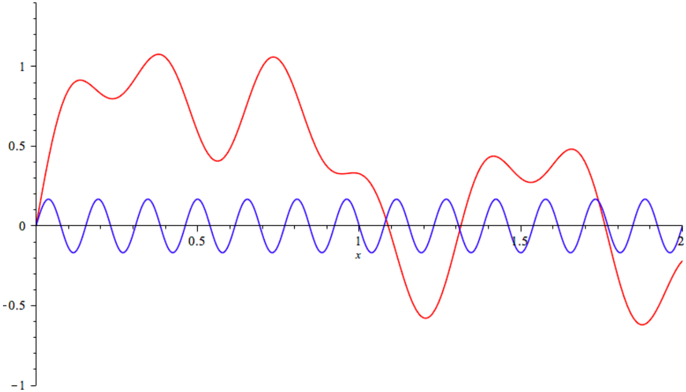
$0.7^4\sin(2.1^4x)$

The Weierstrass curve



$$0.7\sin(2.1x) + 0.7^2\sin(2.1^2x) + 0.7^3\sin(2.1^3x) + 0.7^4\sin(2.1^4x)$$

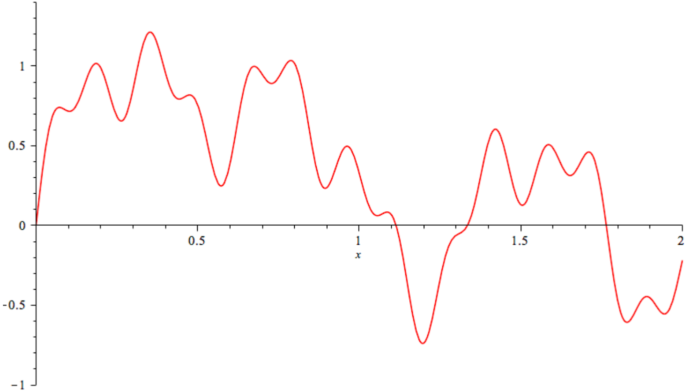
The Weierstrass curve



$0.7\,\sin(2.1x) + 0.7^2\,\sin(2.1^2x) + 0.7^3\,\sin(2.1^3x)$

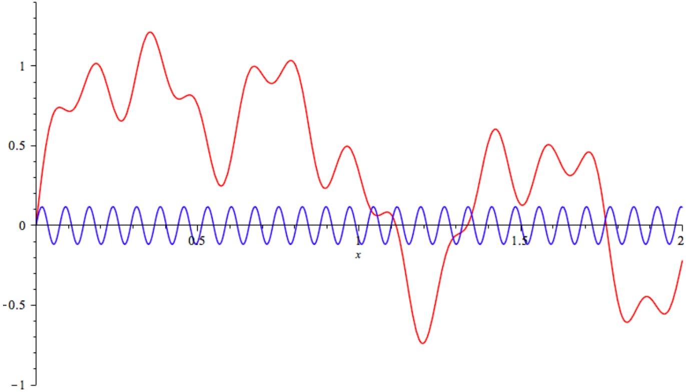
$+\,0.7^4\,\sin(2.1^4x)$ $\quad 0.7^5\,\sin(2.1^5x)$

The Weierstrass curve



$$0.7\sin(2.1x) + 0.7^2\sin(2.1^2x) + 0.7^3\sin(2.1^3x) + 0.7^4\sin(2.1^4x) + 0.7^5\sin(2.1^5x)$$

The Weierstrass curve



$$0.7\sin(2.1x) + 0.7^2\sin(2.1^2x) + 0.7^3\sin(2.1^3x)$$
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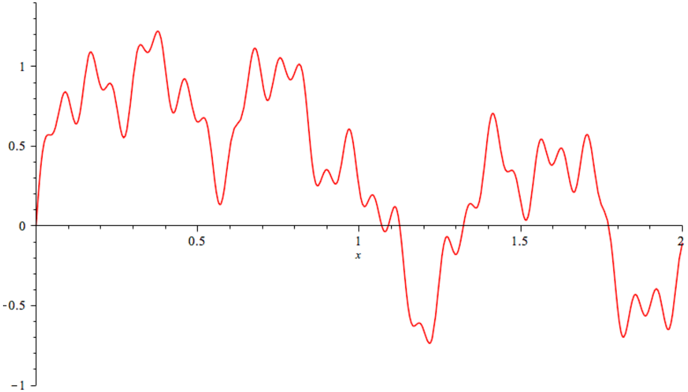
The Weierstrass curve



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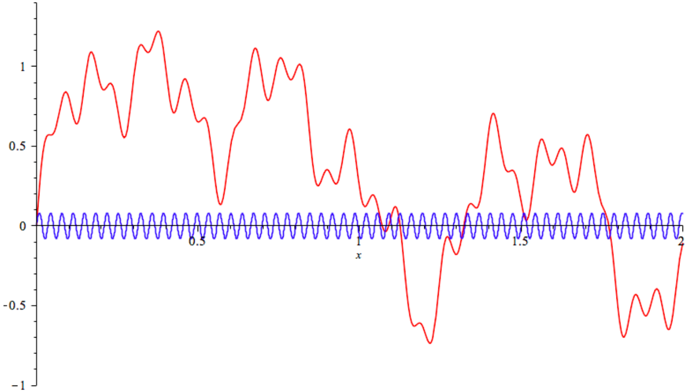
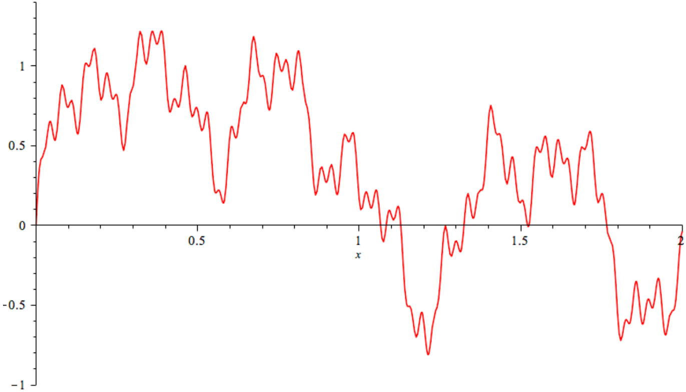


$$0.7\sin(2.1x) + 0.7^2\sin(2.1^2x) + 0.7^3\sin(2.1^3x)$$
$$+ 0.7^4\sin(2.1^4x) + \cdots + 0.7^6\sin(2.1^6x) \quad 0.7^7\sin(2.1^7x)$$

The Weierstrass curve



$$0.7\sin(2.1x) + 0.7^2\sin(2.1^2x) + 0.7^3\sin(2.1^3x) + 0.7^4\sin(2.1^4x) + \cdots + 0.7^7\sin(2.1^7x)$$

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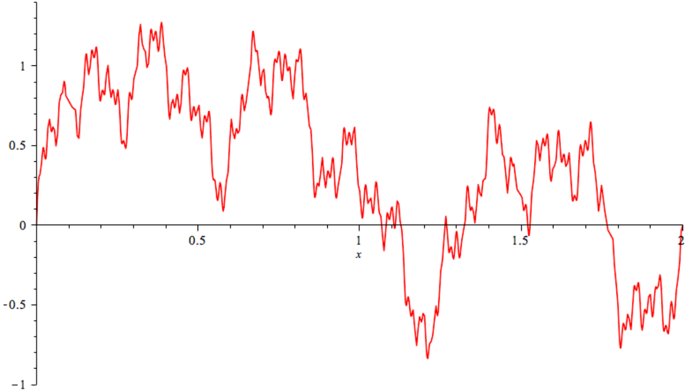
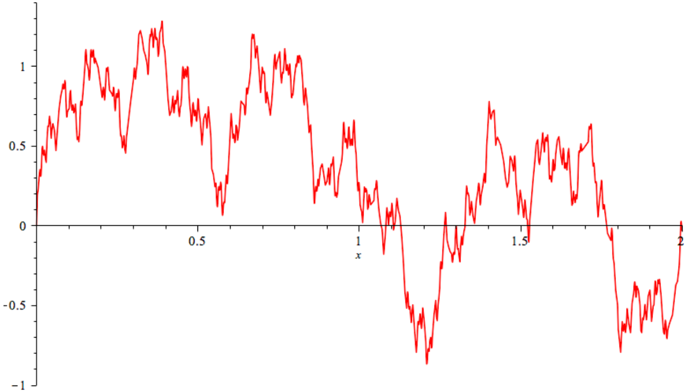


$$0.7\sin(2.1x) + 0.7^2\sin(2.1^2x) + 0.7^3\sin(2.1^3x)$$
$$+\,0.7^4\sin(2.1^4x) + \cdots + 0.7^8\sin(2.1^8x)$$

The Weierstrass curve



$$0.7\sin(2.1x) + 0.7^2\sin(2.1^2x) + 0.7^3\sin(2.1^3x) + 0.7^4\sin(2.1^4x) + \cdots + 0.7^9\sin(2.1^9x)$$

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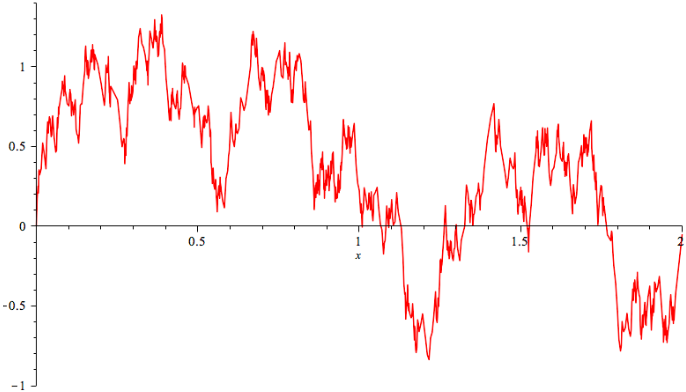
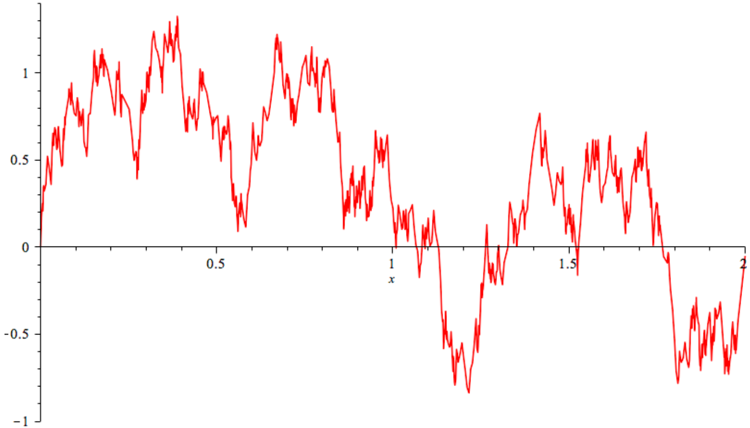
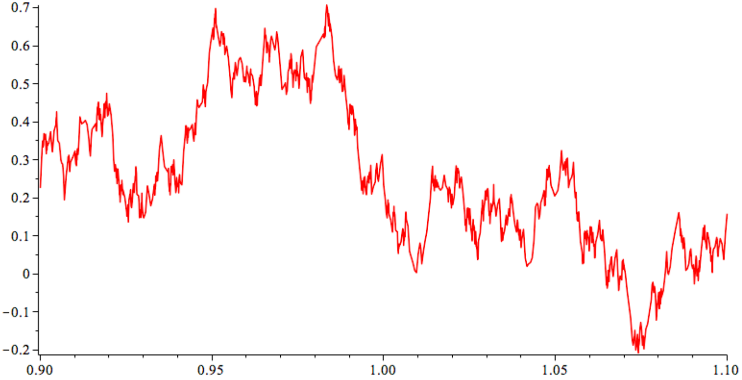


$$0.7\sin(2.1x) + 0.7^2\sin(2.1^2x) + 0.7^3\sin(2.1^3x) + 0.7^4\sin(2.1^4x) + \cdots$$

The Weierstrass curve



Weierstrass proved rigorously that the curve is continuous but at no point does it have a tangent. We can see this by zooming in on part of the curve.



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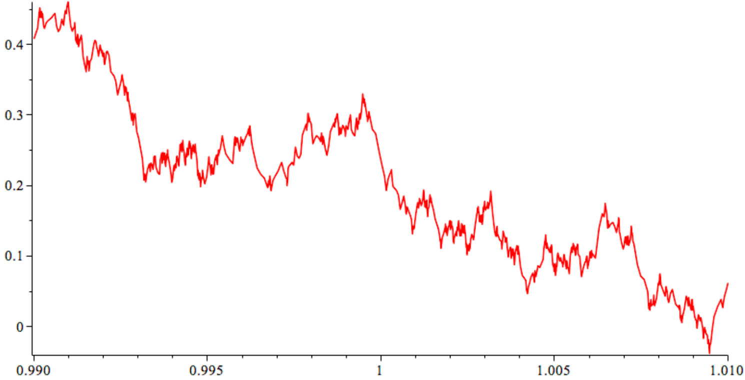

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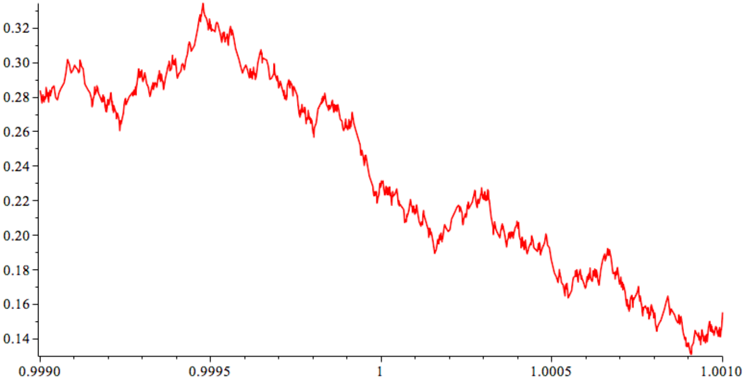

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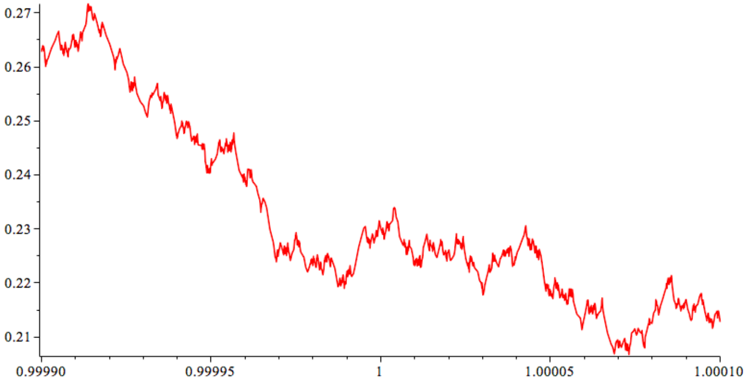

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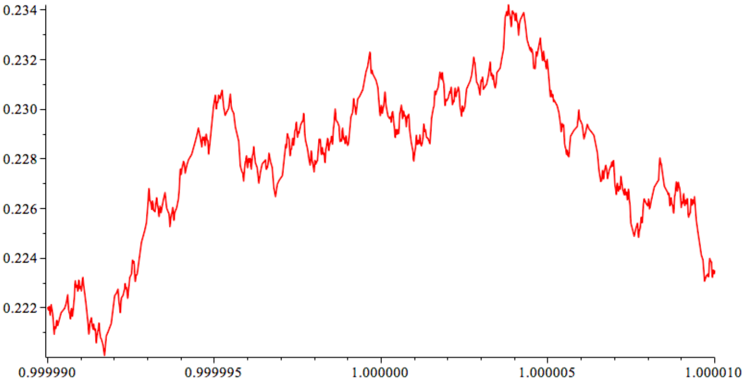

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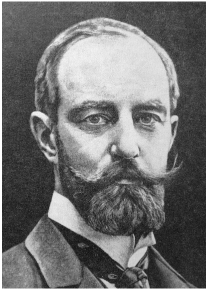
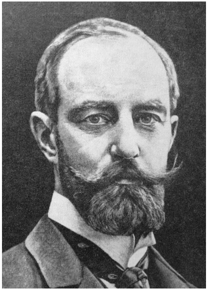


Helge von Koch
1870-1924

"Even though the example of Weierstrass has corrected the misconception [that a curve must have a tangent at most points], it seems to me that his example is not satisfactory from a geometrical point of view since the curve is defined by a formula that hides the geometrical nature of the curve and one does not see why the curve has no tangent.

The von Koch curve (1904)



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The curve I have found is defined by a construction sufficiently simple, I believe anyone should be able to see through "naive intuition" the impossibility of the existence of a tangent."

Sur une courbe continue sans tangente obtenue par une constricioin géométrique élémentaire (1904)

Construction of von Koch's curve

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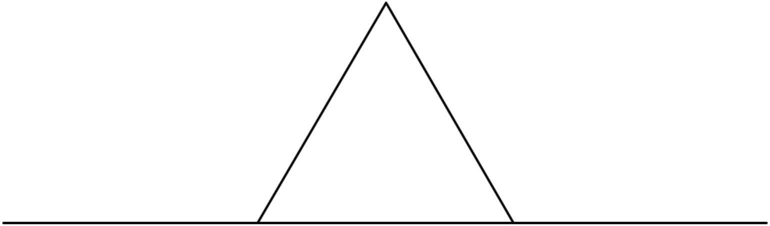

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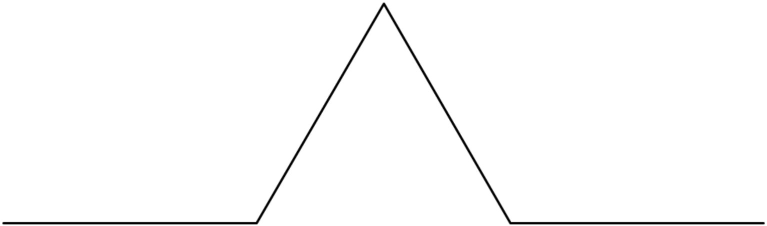

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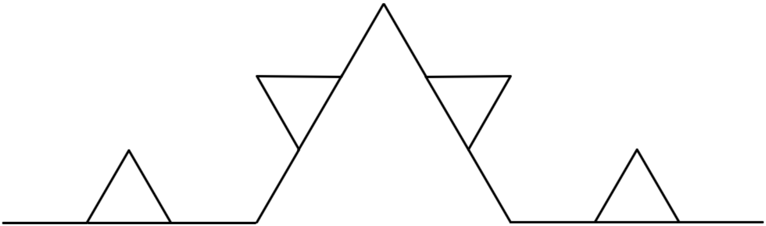

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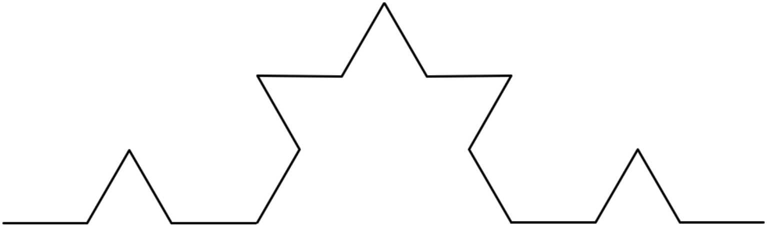

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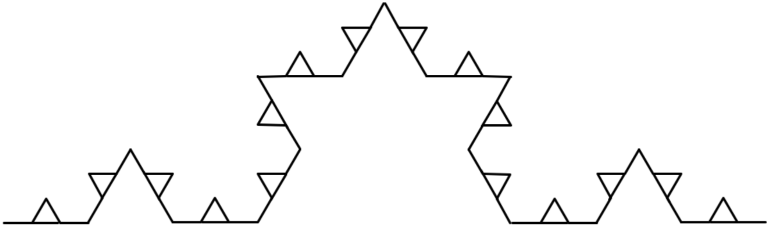

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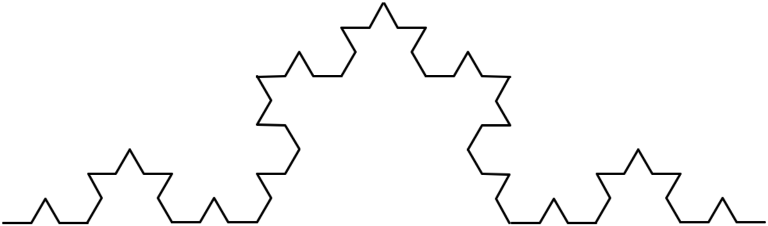

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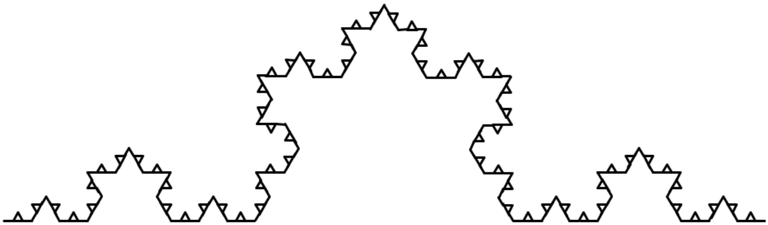

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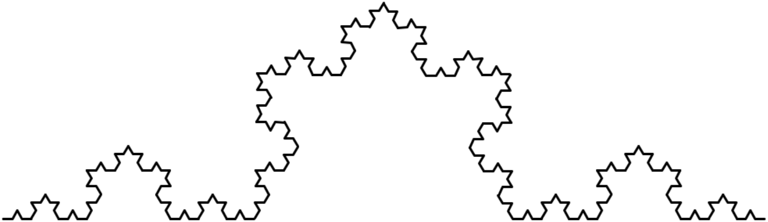

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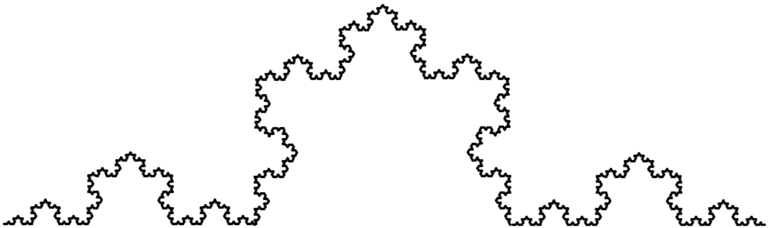
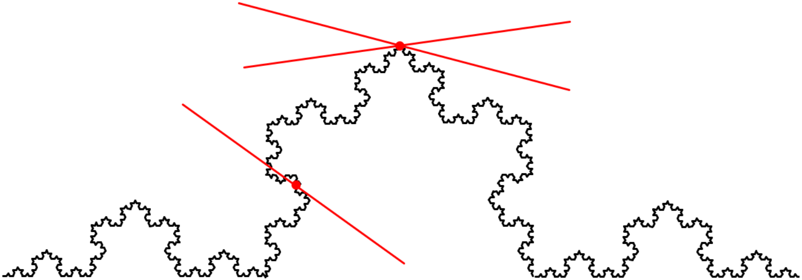
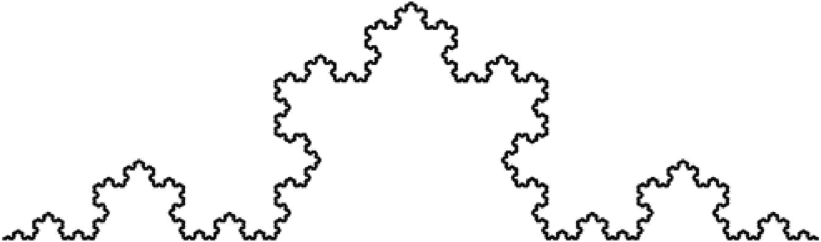


Properties of the Von Koch curve



- Far from smooth
- The curve has no tangent at any point

Properties of the Von Koch curve



- Zooming in shows detail at arbitrarily fine scales

Cantor's 'Devil's Staircase' (1883)

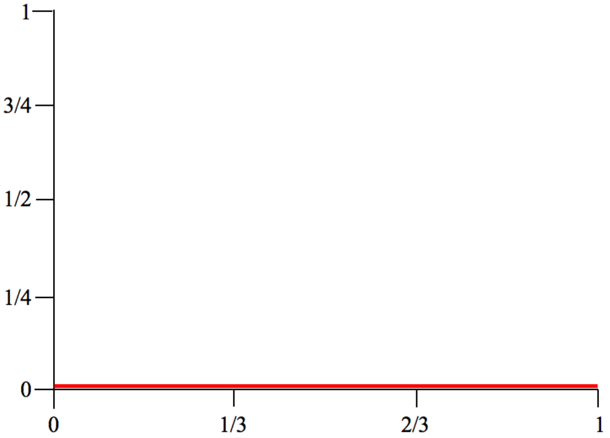
Cantor 'The Father of Set Theory' worked on rigorous properties of numbers and set theory, in particular the nature of infinity. He showed that many widely held intuitive ideas were incorrect. For example he showed 'infinity comes in many different sizes'.

His results did not find favour with many contemporaries:
"I realise that I place myself in a certain opposition to views widely held concerning the mathematical infinite and to opinions frequently defended on the nature of numbers."

In an 1883 paper Cantor showed how to define a graph which seemed to contradict fundamental properties of the calculus.

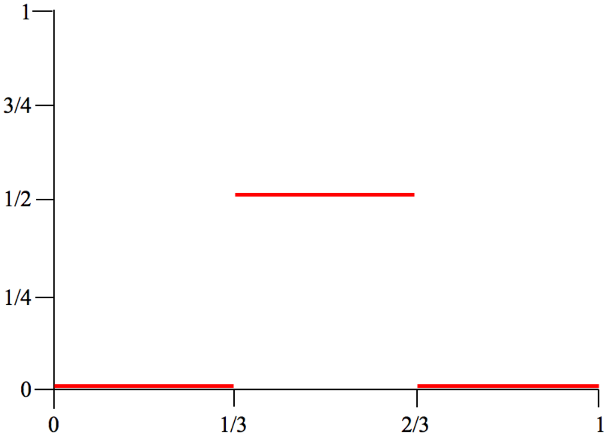
Georg Cantor
1845-1918

Cantor's 'Devil's Staircase' (1883)



The curve is constructed by repeatedly splitting intervals into three and 'lifting' the middle portion.

Cantor's 'Devil's Staircase' (1883)



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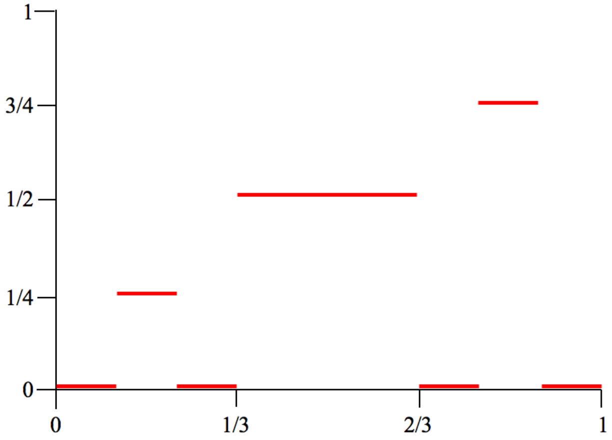

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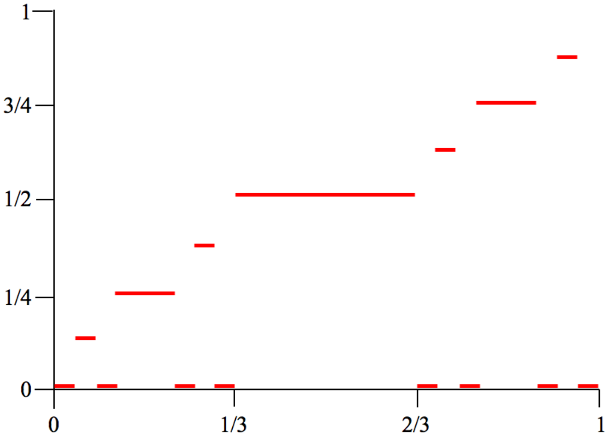

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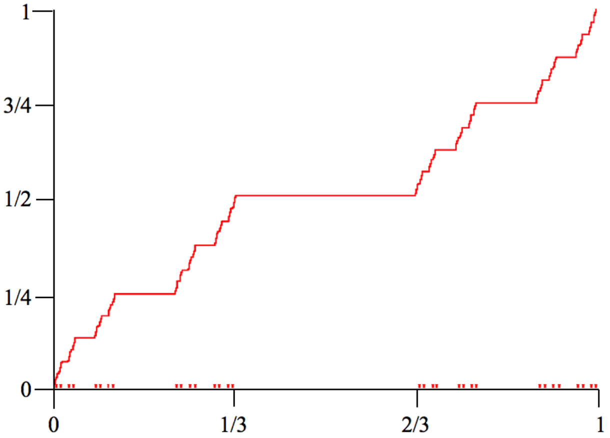

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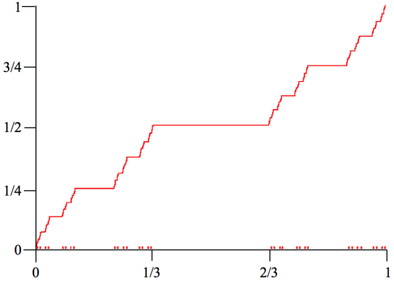
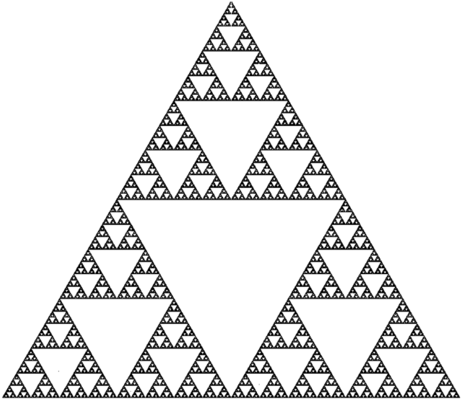


The Staircase is:

- continuous,
- flat, except above the 'Cantor set' which has length 0 so is negligible,
- the value rises from 0 to 1 though virtually never going upwards,
- The Staircase raised questions about the 'fundamental theorem of the calculus'.

The Sierpiński Curve (1915)

Sierpiński was noted for work on number theory, set theory and topology, in particular the topology of curves. He wanted to demonstrate that it is possible to define a curve to form an object such that every point is a 'branching' point, that can be approached along three non-intersecting routes. In 1915 he constructed a curve now known as the Sierpiński triangle.



Wacław Sierpiński
1882-1969

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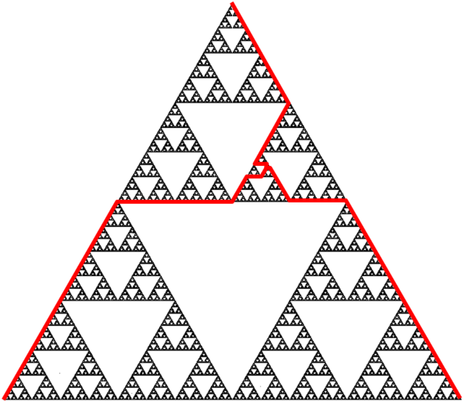

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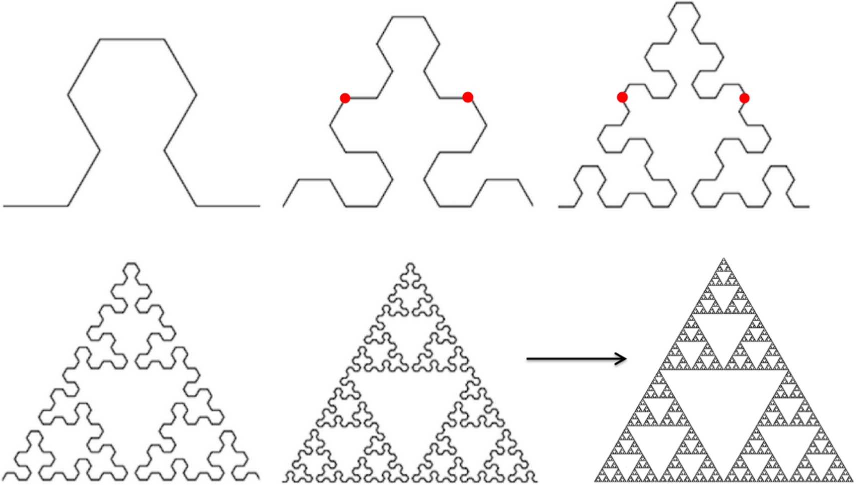
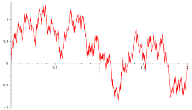
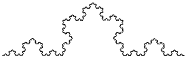
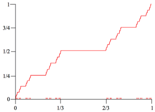
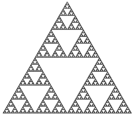


Common features of these constructions



- These four constructions, as well as various others around late 19th/ early 20th century, were ‘pathological’ curves specifically designed purely to exhibit certain mathematical properties or to show that widely held ideas needed refining, rather than because of any intrinsic elegance of the curves themselves.
- Nevertheless, they have various common features.

Common features

- A simple 'recursive' construction, i.e. repeating a process 'ad infinitum'
- 'Fine structure' - detail at arbitrarily fine scales
- Classical methods of geometry and calculus are inapplicable

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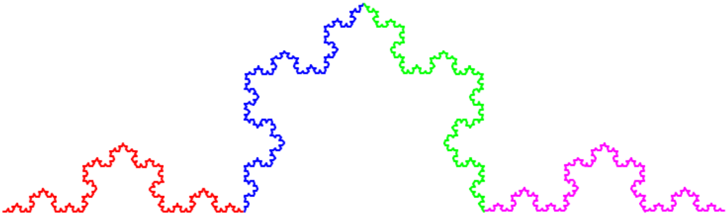
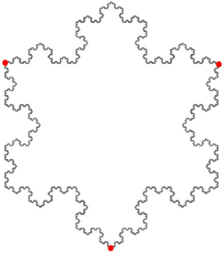


  – Made up of four $1/3$ size copies of itself

Common features

- A simple 'recursive' construction, i.e. repeating a process 'ad infinitum'
- 'Fine structure' - detail at arbitrarily fine scales
- Classical methods of geometry and calculus are inapplicable
- Self-similarity
- Natural appearance - e.g. von Koch snowflake



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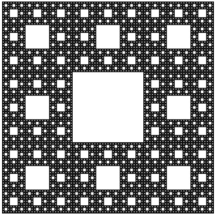
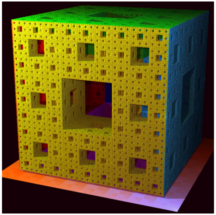
Objects with these sorts of properties are now known as

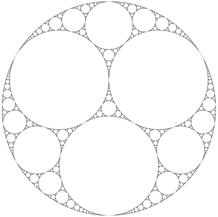
FRACTALS

1900-1960

• Other isolated examples of fractals appeared from time to time to demonstrate mathematical properties, for example:







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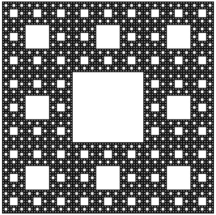
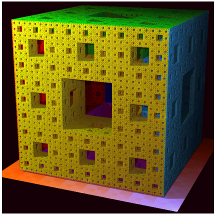
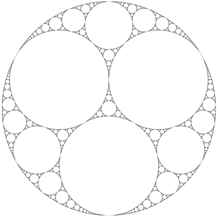




• Various random fractal curves were introduced by Paul Lévy and other probabilists in the 1940s, but only to demonstrate that curves with certain statistical properties existed.

• With computers still far to slow too draw intricate objects, there were few good drawings of fractals.

Benoit Mandelbrot

Benoit Mandelbrot
1924-2010

- 1924 - Born in Poland.
- 1936 - Moved to France with his family where he learnt maths with his uncle.
- 1945-49 Attended École Polytechnique in Paris, California Institute of Technology.
- 1949-58 - Centre National de la Recherche Scientifique 'CNRS', PhD, travelled widely.
- 1958 - Joined IBM at Yorktown Heights, Visiting Professor at Harvard.
- 1987 - Mathematical Science Department, Yale

Benoit Mandelbrot

• Wide ranging interests: Many areas of mathematics, Fluid dynamics, Random processes, Aeronautics, Financial markets, Geology, Hydrology, ...

• He observed that in these areas 'irregular' rather than 'smooth' behaviour was often the norm.

• In the mid 1960s and 1970s computers became powerful enough to produce reasonable pictures of fractals and to allow experiment with variations. Mandelbrot had a fellowship at IBM - and made the most of the facilities available. He produced the first good pictures of a whole range of fractals.
Some appeared in his groundbreaking books.

• 1975 *Les Objets Fractals: Forme Hasard et Dimension*;
 1977 (Translation) *Fractals: Form, Chance and Dimension*
• 1982 *The Fractal Geometry of Nature*

Benoit Mandelbrot

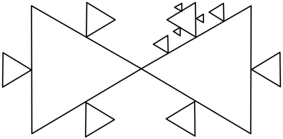
He promoted two basic principles:

• In nature, physics, biology, social science, etc., fractality is the norm rather than the exception.
"Clouds are not spheres, mountains are not cones, coastlines are not circles, and bark is not smooth, nor does lightning travel in a straight line."

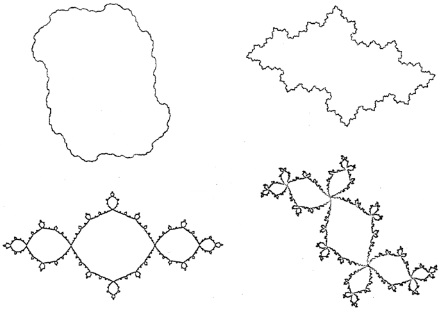
• In mathematics, irregular geometric curves and objects such as the von Koch curve or Sierpiński triangle are not isolated quirky examples, but special instances of a vast family of mathematical fractals.

- Thus fractals should be studied as mathematical objects in their own right and their mathematics should be applied to physical situations.

Example: The family of Julia sets



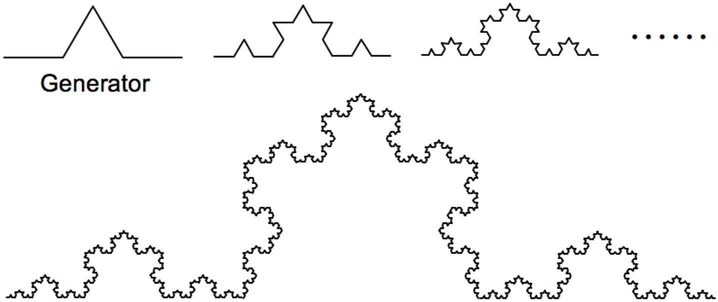
From Cremer (1924)



In 1919, Gaston Julia and Pierre Fatou had shown that a family of curves was very naturally associated with the formula $z^2 + c$ (for complex numbers z and c). But they had no idea what the curves looked like. Mandelbrot had studied their work as a student and in the late 1970s tried drawing the curves by computer. He was amazed when he saw their highly complicated fractal nature.

Fractal curve generators

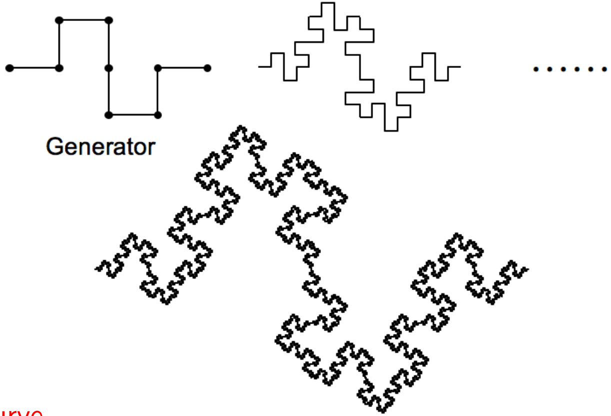
Mandelbrot promoted a way of specifying fractal curves by defining a ‘generator’ which is repeatedly substituted in itself to give the curve. (The botanist Aristid Lindenmayer proposed a similar system in 1968 for providing a mathematical description of plants.) For the von Koch curve:



von Koch curve

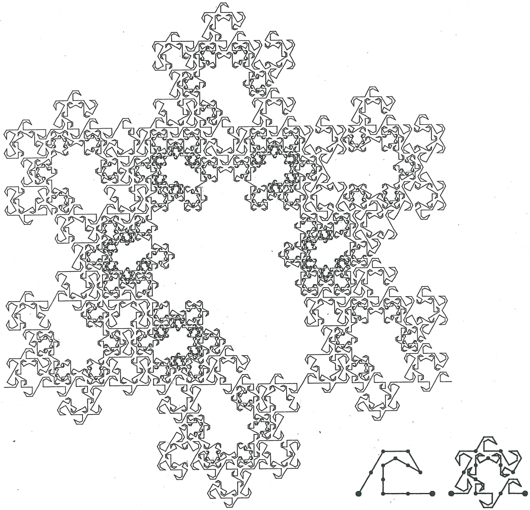
Fractal curves

Here is a 'squig curve' constructed in the same way.

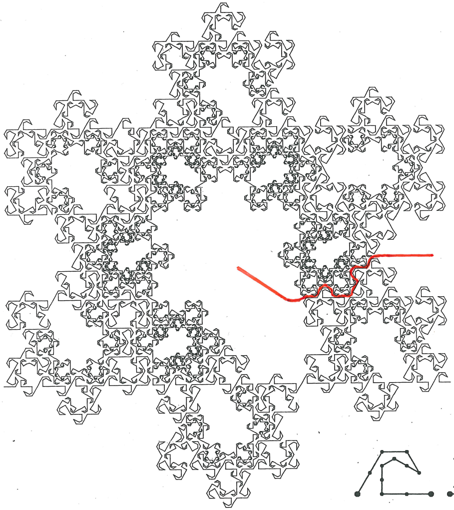


Squig curve

Fractal curves



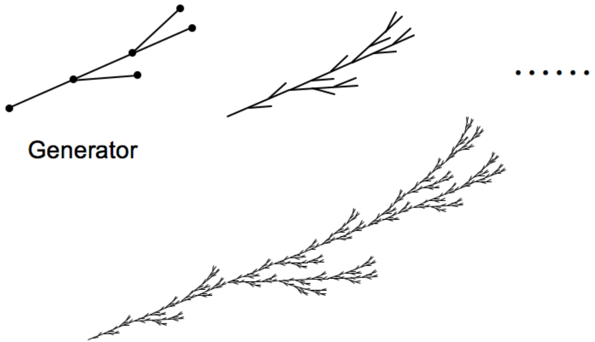
Fractal maze



Fractal maze

Other fractals

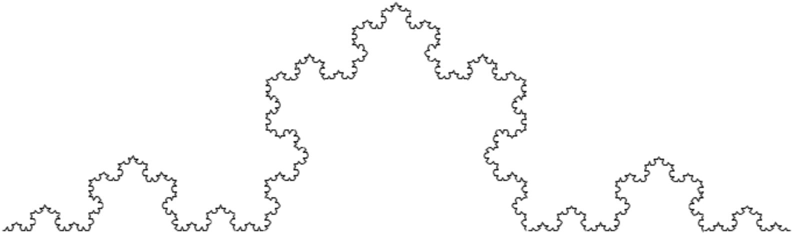
Fractals other than curves can be formed in the same way, as for this grass.



Fractal grass

Templates

Mandelbrot's 'generator' idea for constructing fractals and curves has now largely been superseded. In 1981 John Hutchinson introduced what are now termed 'iterated function systems' to define fractals. The idea is that you can specify fractal curves and other fractals by simply specifying their self-similarities. This can be done using a 'template'.



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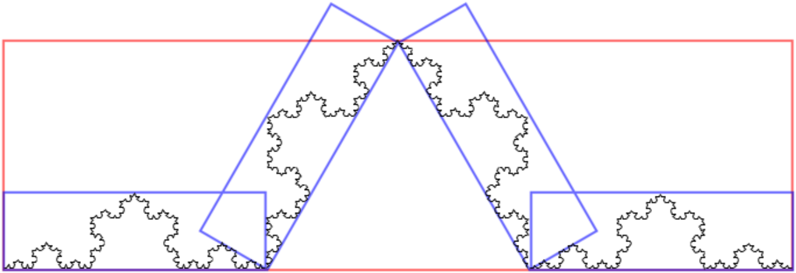

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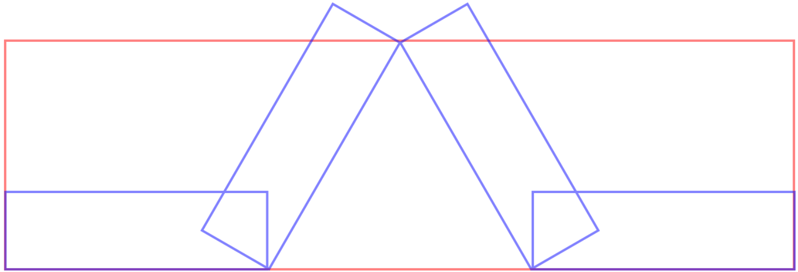

Remarkably, the template completely defines the von Koch curve, which is (essentially) the only object which is made up of copies of itself scaled to fit into the smaller rectangles.

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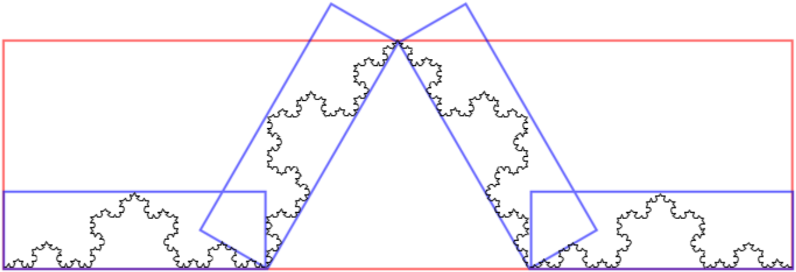

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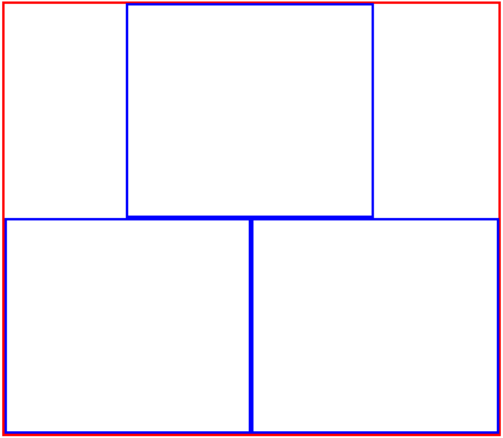

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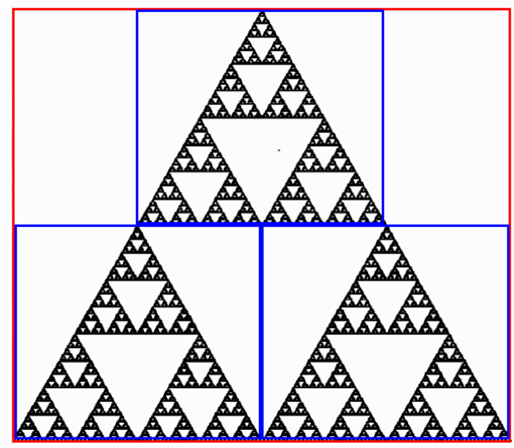

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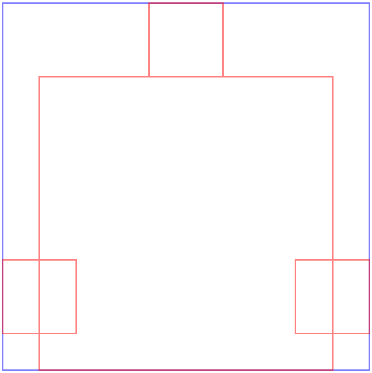

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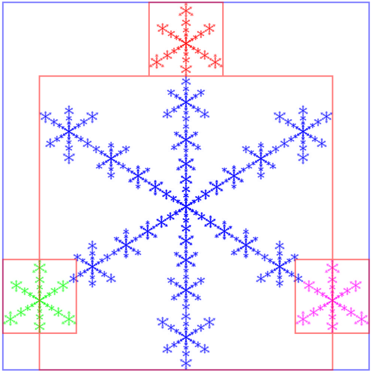

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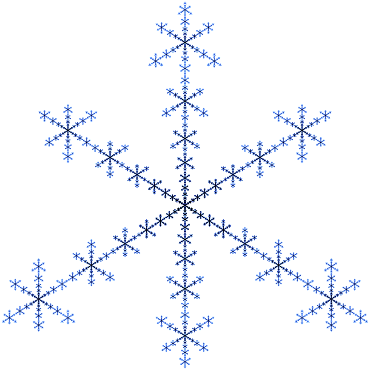

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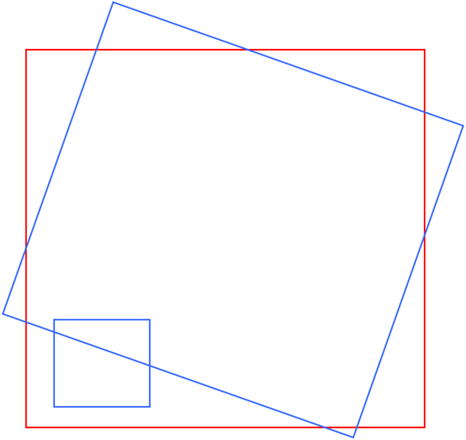

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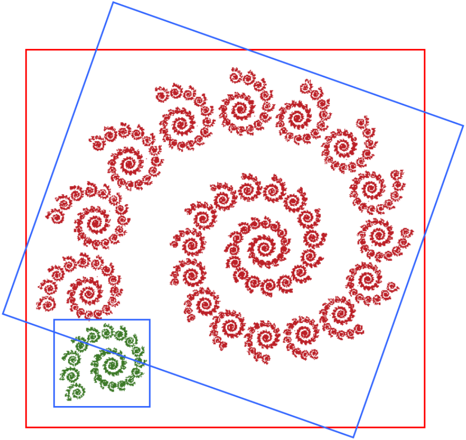

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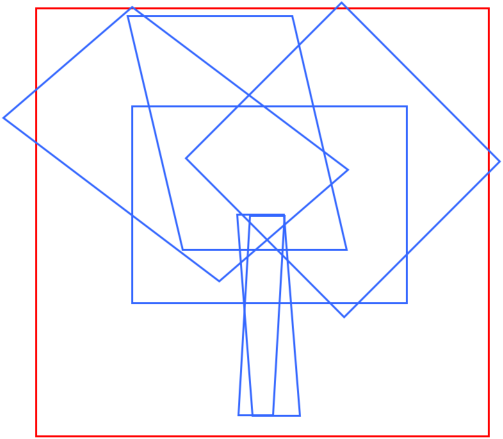

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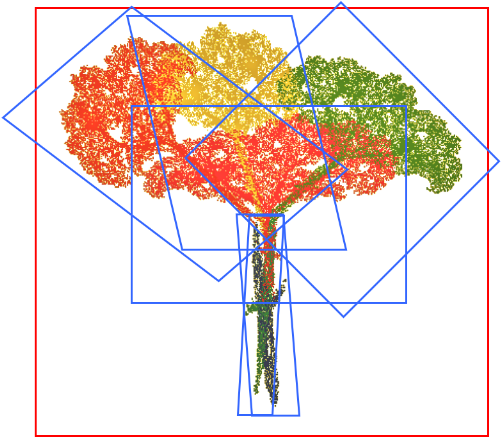
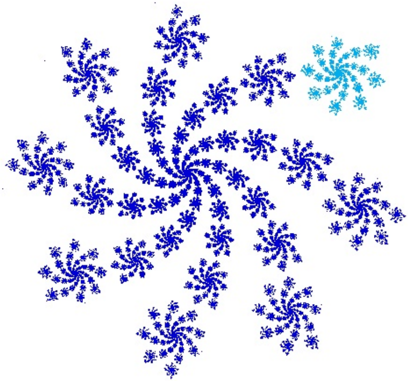


Templates

Conclusion

• Some of the best known ‘fractal’ curves were introduced around the start of the 20th century, but purely as ‘one-off’, ‘esoteric’, constructions to illustrate mathematical properties. There was no intrinsic interest in fractal objects for many years.

• Since the 1980s, particularly with the advent of fast computing, fractal curves have been studied as a vast class in their own right, with fractal geometry now a major area of mathematics and with fractal phenomena observed and studied across science.



Thank you!