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**The Mathematical Skyline**

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**Abstract.** The study of perspective in art is related to a branch of mathematics called projective geometry, which addresses the relationship between landscapes viewed from different locations. One of the key features of the subject is the concept of “point at infinity”, which represents the meeting point of parallel lines. The lecture will tackle the question of where to stand in order to photograph the restored four chimneys of Battersea power station equally spaced along the skyline. The answer will take us to the heart of projective geometry, to theorems about conics, and to generate images that are visible on a daily basis.

**1. Introduction**

This lecture is both lighthearted and serious. Lighthearted, because it was meant to be about the four chimneys of Battersea Power Station, but immediately the talk was conceived one of chimneys was dismantled and rebuilt, and others are probably being dismantled as I speak (slide 1). Hopefully, all four will again be visible long before 2020. This is a serious topic because the Battersea development raises important questions for urban planning. In particular, the need to renovate the chimneys in the first place has been challenged. But today's talk is a mathematical one, and it is serious to me because the mathematics is absorbing.

My aim is to explain the geometry associated to four points in the plane. These points will be those marked by the hypothetical chimneys of Battersea power station like pins on a map, and will be viewed by an observer or photographer standing at a fifth point. Of all the images of the power station available, I have chosen Matthew Train’s atmospheric photograph, taken from Ebury Bridge near Victoria Coach Station in 2013 (slide 2). I shall explain how the four points give rise to conics that can be superimposed on a map of London, and of direct relevance to the appearance of the chimneys.

The main object then to feature in the lecture is a *conic*, or *conic section*, which is a curve obtained by slicing a circular double cone. We shall in fact construct a hyperbola and its asymptotes, which form a pair of straight lines which “frame” the hyperbola and provide a good approximation to the curve far from its centre. A computer finds it difficult to plot a hyperbola without its asymptotes, since for the tracing it must jump from one branch of the curve to the other. Algebraically, conics are curves defined by equations of degree 2. A pair of intersecting straight lines is an example of a so-called degenerate or reducible conic, and is formed when slicing a cone through its vertex.

Today's lecture has its origin in a question posed by David Singmaster in the Open University's magazine M500 (slide 3). I myself was lucky enough to hear it from his own lips during an undergraduate retreat at Cumberland Lodge in Windsor Great Park on 22 February 2014. I remember that day three years ago very clearly as I had driven through floods to reach the Lodge early on the Saturday morning. What I want to tell you is the result of original joint work with colleagues John Silvester and John Armstrong. This is a typical model of mathematical collaboration: it starts over a meal, in this case breakfast.

Let me get to the point and describe David's “Problem 258.1”. As I am sure you know, Battersea Power Station is a massive rectangular brick building designed by Sir Giles Gilbert Scott, and eventually had four tall chimneys at its corners. The north-west chimney was erected in 1931, and the south-east one was only completed in 1955, so having less than four chimneys is nothing new. The completed station was an amalgamation of two halves: Battersea A consisted of the western block with two chimneys, and Battersea B the eastern block with the other two. Let us imagine we are back in time at least three years to 2014, when all four chimneys reached 100 meters.

David writes: “One can see the chimneys on the skyline as one drives into London from the west and their relative position shifts as one drives along the north side of the Thames. It appears to me that there will be some point where the chimneys will appear regularly spaced along the skyline. Where does one have to be to see this effect? I want to be able to go to a correct viewpoint and take a photo”. The word “photo” is crucial for this talk. Readers of M500 have interpreted this question in a different way with angles, and provided accurate answers accordingly. But today I shall mainly explain where one needs to stand to be able to take the sort of photo that David speaks about, so that one can place a print of the photo on the table and check that the three distances between the chimneys are equal.

In mathematics, half the battle is to understand the problem that has been set. It is my aim to show that David's is *well posed*, in the sense that it has an affirmative answer and an elegant solution. The latter involves some relatively advanced mathematics (slide 4), including:

* results concerning permutations (re-orderings) of four chimneys or of numbers ;
* the so-called *cross-ratio*, a projective invariant;
* theorems on conics, curves (including hyperbolae) defined by equations of degree 2, which I am ashamed to say that we do not currently teach in my department;
* the role played by asymptotes of hyperbolae;
* examples of cubic curves, defined by equations of degree 3, time permitting.

The conics arise from the problem concerning lengths defined by four chimneys, the main focus of this talk. The cubics arise from a study of angles subtended by three chimneys, and could be the subject of a separate lecture.

**2. The Battersea site**

Let me fill in some recent history. After Battersea B ceased producing electricity in the 1980's, the power station had been described as a dead upturned table. Various plans came to nothing. Real Estate Opportunities (REO) purchased the power station in 2006 for £400 million, and a design by Rafael Viñoly (of “walkie-talkie” and 432 Park Avenue fame) introduced what he described as a “fluid geometry for the new residential buildings”. This Battersea masterplan included a tower described by Boris Johnson as an inverted toilet roll holder, and the whole project floundered. REO was unable to secure funding in the wake of the financial crisis, and in February 2012 the site was put on the open market. There had been interest from Chelsea Football Club, but it was a Malaysian consortium, Setia and Sime Darby, that placed the winning bid to secure the sale.

Architects Frank Gehry and Norman Foster and Partners began work in 2013 on Viñoly's revised masterplan, in which the power station is surrounded by curved geometrical forms (slide 5). Our own master analysis will achieve the same result in a virtual sense.

The old rectangular boiler house will host offices including (we are told) Apple's London campus. The east-west Electric Boulevard is designed to resemble “a thoroughfare like New Bond Street rather than a shopping mall”, and will lead to apartment complexes, such as the Gehry’s flower building (slide 6). I want to draw your attention instead to the swimming pool planned on the roof of the Art’Otel just south of the boiler house, for the infinity design of this pool is a good symbol for the powerful mathematics underlying the historic site. Infinity is our cue to study parallel lines, because parallel lines meet at infinity. In projective geometry, there are lots of points at infinity: the so-called *real projective plane* is formed by adding one point at infinity for each set of parallel lines with a given direction.

The equal-spacing problem involves photography and perspective, lengths not angles, and four chimneys not three. Given the publicity shots from the pool, one might be forgiven for believing that the developers are superstitious of four chimneys, since often one chimney is shown hiding behind another. Maybe they were anticipating the reconstruction: in the 2016 film *London Has Fallen* starring Gerard Butler and Morgan Freeman, the only believable shot is that of Battersea with one chimney missing. Three chimneys give rise to a different problem concerning angles that I shall mention later, but let's go back to four.

**3. An approximate solution**

One can see the chimneys in different orders in various regions external to the power station. We shall pay special attention to viewing from the north-west, from under Chelsea Bridge, or from Ebury Bridge as in our early photo. I have chosen to label the chimneys so that one sees them in the order 1234 from left to right from Ebury Bridge (so chimney 1 is NE and chimney 4 is SW). Looking diametrically opposite, from the south-east, one sees them in reverse order 4321. There is a total of 4! = 24 ways of re-ordering four objects, and the diagram (slide 7) shows that only 12 of these 24 permutations are possible.

As Ken Greatrix remarked in response to David' published question, it is easy to construct sets of four equally-spaced parallel lines, one passing through each chimney. One just needs to ensure that the two middle lines pass through the midpoints of opposite walls (slide 8). The boiler house rectangle is approximately 160 meters by 50 meters, so the lines shown have slope 8/5 because 80 meters vertically corresponds to 50 meters horizontally. Any fifth line (a “transversal”) that cuts all four parallel lines will do so with three equal distances, irrespective of its angle of inclination; this is a consequence of the theory of similar triangles and Euclid's axioms.

If we also use the midpoints of the shorter sides, we can produce a total of four sets of parallel lines (slide 9). These four sets represent 8 different orders in which to view the chimneys, and they realize 8 of the 24 possible permutations that we saw previously.

On a smaller scale, each set of four lines coalesces into a single thickened line (slide 9). This shows how our sets of parallel lines give the solution to the Battersea question if one is sufficiently far away. The “bandwidth” of each set, and so the uncertainty of the precise location, remains constant (at about 100 or 170 meters) but the further one is away from Battersea, the less significance this distance has. For viewing from afar (even from Ebury Bridge), the power station can be regarded as a distant star from which rays of light arrive in parallel. Mathematically, we have therefore found the points at infinity that solve the problem.

This solution is inaccurate closer to the power station, for example if one walks east along Grosvenor Road, il “Lungo-Tamigi”. Yes, there will be four positions on the pavement from which the chimneys appear equally spaced (slide 10), and each time one passes from one position to the next two chimneys are transposed, but our parallel lines of each set are too far apart to accurately mark the “correct” location. The situation is even worse on the south bank: think of someone who wants to buy a flat with a view in which the chimneys are equally spaced with access only to a show-flat (this is the geometer's analogue of a number theorist asking the phone company to have a number that is prime).

**4. The cross-ratio**

For the accurate solution, we need to resort to a fundamental concept in projective geometry, called the *cross-ratio*. This is a number that can be associated to any set of four lines that meet in one point. Once we fix the position of an observer , we have such a set (slide 11). Draw a line from to each of the four chimneys

and consider the “pencil” P of four lines

To compute the cross-ratio of P, we draw a fifth line that cuts the other four. The transversal can lie anywhere, provided it intersects all four lines of the pencil. If it is close to it could represent the film of our camera, the screen of an iPad, or a picture canvas, but it could also pass through one or two of the chimneys (this is convenient for doing the calculations).

Denote by x1, x2, x3, x4, the four points of intersection of with the four lines of P, or rather their distances measured along from some fixed origin on . Our cross-ratio is then the number

.

Notice that this does not depend on the choice of origin (because only involves the distances *between* points) Nor does it depend whether we use feet, meters or any other units of measurement (because the scaling factors cancel out, top and bottom). It is therefore permissible to set , , and scale accordingly. But much more is true, provided we fix the first four lines:

*The quantity does not depend on the position of the fifth line .*

This is a theorem of projective geometry. One proof relies on Menelaus’ Theorem concerning the lengths defined by a transversal line cutting through a triangle. A sophisticated way of expressing the theorem is to say that is a *projective invariant* of the pencil P.

A dual way of looking at all this is as follows. The formula for the cross-ratio allows us to define it for any four collinear points (rather than four lines). If and are two transversals of P, then projection from defines a correspondence or *mapping*

because each point of corresponds to a point of by drawing a line between them that passes through (slide 12). Such a mapping f is called a *projective transformation* or *perspectivity* with vanishing point . The theorem above tells us that the cross-ratio of four points on equals the cross-ratio of their images.

Here we are talking about transformations of 1-dimensional objects (lines), but we can visualize projective transformation of 2-dimensional objects (planes) by taking photographs of the same scene from different viewpoints (slide 13). Lines are transformed to lines by a projective transformation, and the cross ratio of four points is preserved. One the other hand, circles can become ellipses. Indeed, any two conic curves are equivalent by a suitable projective transformation.

To sum up, the cross-ratio is a number associated to four points that are *collinear*, or to four lines that all pass through a common point . Now suppose that is (in David Singmaster's language) a “correct” point on the map of London, meaning one from which a photographer can position a camera so as to capture the chimneys equally spaced along the horizon. That means, with an appropriate ordering of the chimneys, that there exists a transversal (the camera film or screen) for which

In this case, the arithmetic is very easy (slide 14): the cross-ratio equals

Our original problem has therefore led us to ask:

*What is the locus of points O in the plane for which ?*

**5. Battersea’s conics**

The answer is well known to geometers and, as I shall explain, the locus must be a conic. But one can simply do the calculations, and find the equation of the curve directly. I did this by choosing Cartesian coordinates parallel to the sides of the rectangle with origin at the centre of the rectangle. For convenience, I then chose to pass through two chimneys (slide 15), and did a lot of scribbling. The correct equation is

which can also be written

where and This is a hyperbola H. Its asymptotes are found by ignoring the constant on the right and factorizing the left-hand, giving

or Its eccentricity equals about 1.18, and the two foci lie less than 2 meters outside the power station.

When the hyperbola is superimposed on a satellite view, one notices that it passes through the four chimneys (slide 16). It consists of two separate branches, one extending infinitely to the south, and the other to the north. One can see exactly where it crosses Grosvenor Road. The asymptotes of H are parallel to our outermost set of parallel lines with slope 8/5. A few other conics passing through the chimneys are also relevant.

Let's return to analyse the link between points and conics (slide 17). Just as two points determine a straight line and three determine a circle, so five points (no three of which are collinear) determine a unique conic curve. Take the four chimneys and any fifth point that represent a viewpoint that is “correct” in the sense that the chimneys are equally spaced in the order 1234. Such a point exists because we could take it to be the point at infinity with slope 8/5 beyond Ebury Bridge. Then it turns out that *our hyperbola H is the unique conic through these five points!*

Why so? It is a consequence of a theorem due to the French mathematician Michel Chasles:

*Given four fixed points*  *on a conic, the cross-ratio of the lines is a constant independent of the position of a fifth point on the conic.*

If the conic is a circle, then this theorem is a corollary of the theorem about inscribed angles subtended by points on a circle. The general case follows because the cross-ratio is unchanged by a projective transformation, which can be used to convert the circle into a hyperbola.

Our hyperbola H can in fact be defined as the locus of points whose cross-ratio relative to four of the five points is constant (slide 18). It follows that H contains some of the points where a photographer should stand to capture Battersea's chimneys equally spaced along the skyline.

A word about Michel Chasles. He was the first honorary foreign member of the London Mathematical Society. His name is a household one in France, because the rule of vector addition

is known to French schoolchildren a “la relation de Chasles”. We might call it the parallelogram law. There are other theorems attributed to his name, including one describing the rigid motion of a body in terms of rotation and translation. The cross-ratio theorem appears in Chasles' monograph on conics (slide 19) published in 1865, incidentally the date the London Mathematical Society was founded. It was no doubt known to him much earlier. Indeed, it appears on page 3 (slide 20), with the words “rapport anharmonique” or anharmonic ratio, another name for cross-ratio.

**6. More results**

So far, I have shown that *if* one can take a photo with the chimneys equally spaced in a certain order, then one must be standing somewhere on the hyperbola H. I have not shown the converse, or at least that one can take such a photo from any point of H outside the power station. To convince you that this is true, I shall explain how to position the camera to ensure that the chimneys are equally spaced.

The asymptotes of the hyperbola help to answer this question. They divide the power station rectangle into quarters (slide 21). Note that the asymptotes do not pass through the chimneys; indeed, H is the only conic that passes through the chimneys and the two points at infinity. It turns out that taking a photo from (say) the south-east, near one asymptote, one should align the camera film or screen parallel to the other asymptote. In fact, its point at infinity acts as what John Silvester calls a “pivot” for the problem.

I shall skip the details. But the reason this works is that if we fix three chimneys and a point at infinity on H, then the constant cross-ratio that allows these four points to define H equals rather than . Four points with such a cross-ratio are said to form a *harmonic range* (slide 22).

Up to now, I have glossed over the importance of the *order* of the four chimneys (or lines, or points) in defining the cross-ratio. A re-ordering may or may not affect the cross-ratio. For example, the four permutations

corresponding to the four legs of our hyperbola H have the same cross-ratio. Each is obtained from the first by a double transposition (like and ), which generates the so-called Klein group of order 4. But swapping just two chimneys changes , and in theory we can form six different values

generated by substituting by or , and repeating. Each value corresponding to a conic passing through the chimneys (slide 23).

Only four conics are needed to photograph the chimneys with equal spacing from each of our 12 external regions, two hyperbolas and two ellipses (slide 24). Only the hyperbolas relate to our discussion with straight lines, because the lines are parallel to their asymptotes. To see the chimneys equally spaced in other orders, such as , one must be positioned on an ellipse and relatively close to the boiler house, on the same side of the river. The line of the camera film should be pivoted from a point on an axis of the ellipse. What’s more, our problem can in theory be generalized to points inside the power station by allowing views both ahead and behind, but that is beyond the scope of this analysis.

An older theorem on conics is relevant to our discussion about where to place and how to position the camera. This was formulated by Blaise Pascal in 1639 when he was 16:

*The three points of intersection of the opposite sides of a hexagon inscribed in a conic are collinear.*

An inscribed “hexagon’’ really just means six points lying on the conic; one is allowed to order the vertices as one pleases and draw in the edges accordingly. One can also use the theorem to construct a conic through five points (slide 25). Here we have chosen to be a fifth fixed point on an ellipse; by choosing an arbitrary point on , we can construct the ``Pascal line'' and sixth vertex of the hexagon.

If we take the conic to be degenerate (two intersecting lines) then Pascal’s Theorem reduces to the more ancient Theorem of Pappus. Despite its relevance, the cross-ratio appears to be a more modern invention. Lazare Carnot (whose son studied heat engines) introduced the “rapport anharmonique” in an 1803 treatise in advance of Poncelet’s work on projective geometry, and the English version is due to Willian Kingdom Clifford, who studied at King’s College London as a teenager just before Maxwell write down his celebrated equations of electromagnetism there.

**7. Different constructions**

So far, we have been discussing geometry in the plane. Let me conclude the discussion of conics by reminding you that conics arise in the 3-dimensional setting of a double circular cone. Many textbooks show in great detail the various conic sections one can obtain by slicing a cone, but (to my knowledge) none fix a conic and consider all the circular cones that have that conic as a section. If we consider our hyperbola H, one can show that the locus of the vertices of these cones traces out a giant ellipse in a vertical plane. The ellipse passes through the foci of H that lie about 2 metres outside the power station rectangle. It is best imagined as an arch like the one over Wembley stadium (designed, incidentally, by Foster+Partners, slide 26).

I promised to say something about cubic curves. Such a curve arises if one fixes attention on three chimneys, and seeks points for which the two angles subtended by adjacent pairs of chimneys are equal. I have plotted the locus of points for which the angle subtended by chimneys 1 and 4 and the angle subtended between chimneys 4 and 3 are equal (slide 27). This is a cubic curve that passes through the three chimneys, with a node at the common chimney (number 4). The cubic is defined similarly. The fact that these are no external points of intersection between the two cubics means that there are no points where the three angles , , are equal.

A rather different configuration appears when we choose different pairs of chimneys for which to define equal angles (slide 28). It is remarkable that one can discover a rather familiar looking figure in this way. It is also clear that there exist points where some triples of angles (such as , ,) between adjacent chimneys are equal.

Our final slides are tributes to musicians in association with Battersea. The power station was immortalized by Pink Floyd's cover to their 1977 album *Animals*, with its inflatable pink pig (slide 29). One can purchase a mobile phone cover featuring this image but it only has the pig and *two* chimneys.

The final image is that of Sir Elton John at a party to launch the Battersea development (slide 30). The photo was taken by Dave Benett and published in the London Evening Standard in May 2014. Elton John is wearing “correct” glasses, and an analysis of the heights of the chimneys shows that it represents a 1234 or 4321 view from our hyperbola, consistent with Chasles’ Theorem.

**8. Acknowledgments**

Selected references:

* *Géométrie de Position,* par L.N.M. Carnot, J.M.B. Duprat, 1803.
* *Traité des Sections Coniques,* par M. Chasles, Gauthier-Villers, 1865.
* *Projective Geometry*, by H.S.M. Coxeter, Springer, 2013.
* *Traité des Propriétés Projectives des Figures*, by J.V. Poncelet, Bachelier, 1822.
* *Geometry Ancient and Modern*, by John R. Silvester, Oxford University Press, 2001.

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I apologize in advance for any inaccuracies in the lecture.

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