GRESHAM COLLEGE

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# Escher and Coxeter - A Mathematical Conversation

Professor Sarah Hart

## 1 Introduction

In 1954 the artist Maurits Escher met the mathematician Donald Coxeter at the International Congress of Mathematicians in Amsterdam. This meeting sparked a lifelong correspondence which would influence the work of both men. In the talk, we will see examples of Escher's work in the plane and on the sphere and discuss why the possibilities in these geometries are finite. We'll look at how a diagram in an article by Coxeter opened up a new world of possibilities for Escher. Finally, we'll give an indication about what it was in Escher's work that Coxeter found mathematically fascinating.

## 2 Escher before Coxeter

Figure 1: *Hand with Reflecting Sphere*, M. C. Escher (1935)

Figure 1 is a self-portrait by Dutch artist Maurits Cornelis Escher. It is a lithograph made in 1935, when he was 37. Escher is well known for his intricate and beautiful designs that play with the ideas of geometry and perspective.

Escher was born on $17^{th}$ June 1898 in Leeuwarden, Holland, the youngest of five brothers. The family moved to Arnhem when he was five, and that is where he was brought up and educated. His father was a civil engineer, and all his older brothers became scientists.

In 1919 he was admitted to the School for Architecture and Decorative Arts in Haarlem; this was where he produced his first woodcuts. He had intended to study architecture but soon switched to graphic arts. He joked that it was only by a hair's breadth that he escaped becoming a useful member of society. At any rate he learnt and refined here some of the technical skills that he would use in his work – the making of lino cuts and woodcuts, as well as etching. Many of these processes require a huge level of skill. For example the woodcut produced must be the mirror image of the final intended picture; the printing process itself is also very delicate and precise.

Escher travelled around Italy and Spain in the summer of 1922, and woodcuts featuring Italian landscapes formed part of his first exhibition, in Holland in 1924. He began experimenting with lithography in 1929, still producing mostly landscapes. However in 1936 his work started to take a different direction. He began experimenting with more abstract designs (he said he had replaced landscapes by 'mindscapes'), as well as work playing with the ideas of perspective and geometry. Escher's work became more and more well-known. His repeating designs based on tilings of the plane, often featuring animals, were very popular, along with his 'false perspective' work, such as the impossible staircases of his 1953 lithograph *Relativity*.

A major influence on Escher's artistic development was the Alhambra palace in Granada, Spain, which he first visited in the summer of 1922. It is a place of pilgrimage for lovers of symmetry and pattern. The first building there was a small fortress in 889AD, but the buildings we know today were constructed in the mid-11th century by the Moorish king Mohammed ben Al-Ahmar. One feature of Moorish art is highly decorative geometric designs which are very symmetrical and often comprise repeating patterns of tiles which are tesselations that could repeat continually. An example of such a design is shown in Figure 2. Escher was struck with the Moorish style of decoration, and visited the Alhambra again in 1936. He was fascinated by the order and symmetry of the tiling patterns he saw there. Work incorporating regular tilings of the plane became a major focus, and he wanted to know all the possible forms his designs could take.

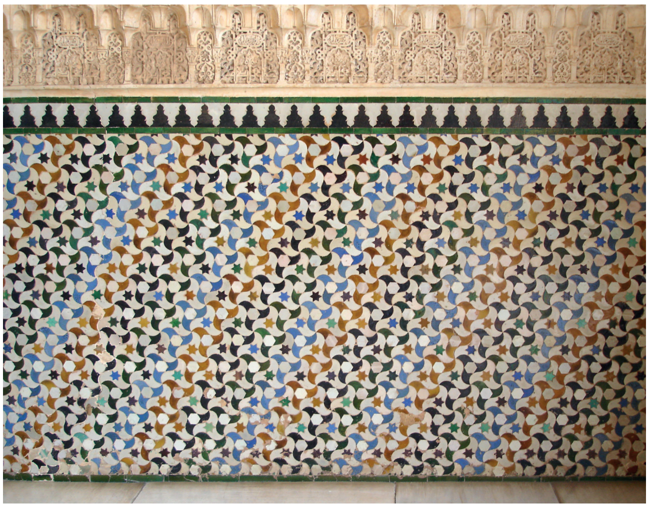

Figure 2: A tiled wall at the Alhambra Palace

## 3 Tilings of the plane

Escher had made a few sketches of tilings in the mid-1920s, following his first visit to the Alhambra. But it was after the 1936 visit that his obsession with plane-filling designs really took hold, and he began to produce designs like *Angels and Devils* (Figure 3).

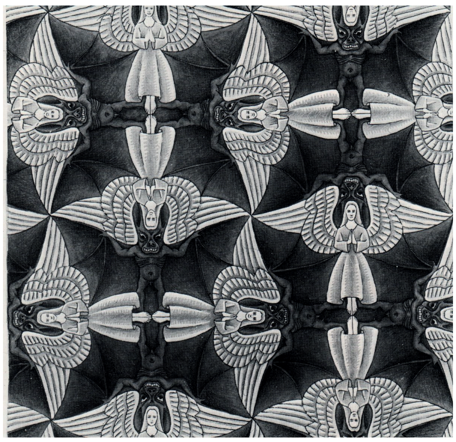

Figure 3: *Angels and Devils* (1941)

For *Angels and Devils* Escher worked in ink rather than woodcut. It has two 'tiles' that meet exactly and form a repeating design that could be extended indefinitely.

Such drawings hint at infinity, in that the patterns could in principle be extended and repeated forever, but no finite diagram can actually show the whole tiling. Escher found this frustrating, and wanted a better solution to represent infinity.

Another issue is that one is somewhat limited in the options for tilings. To see why, notice that underlying the *Angels and Devils* design is a 'squareness' - we could superimpose a repeating grid of squares - a so-called 'regular tiling'. To define this concept we need the idea of a regular polygon, that is, a convex shape made from straight edges where all the edges and internal angles are equal. If it has $n$ sides we call it an $n$-gon. A regular 3-gon is an equilateral triangle, a regular 4-gon is a square, and so on.

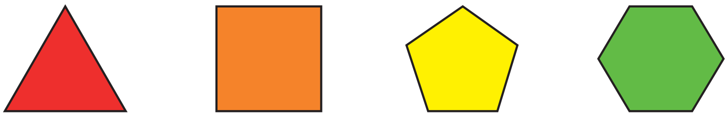

Figure 4: Some regular polygons

We define a regular tiling to be a tiling that uses just one shape of tile, which must be a regular polygon, with the same number of tiles meeting at each point. The angles around a point add up to $360^{\circ}$, so if we are going to have any hope of a regular tiling, we have to work with polygons whose interior angle divides 360 exactly. A little thought shows that the only possibilities are equilateral triangles, squares, and hexagons. (One can show that the interior angle of a regular $n$-gon is $180(\frac{n-2}{n})^{\circ}$.)

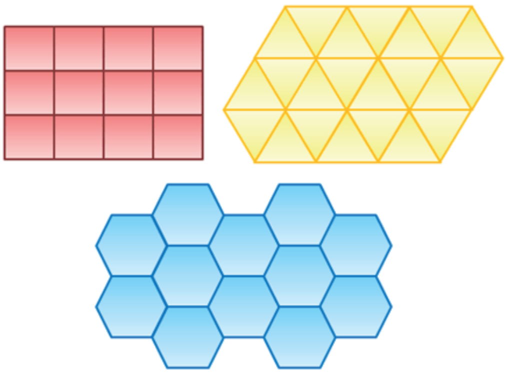

Figure 5: Regular Tilings of the Plane

We see (Figure 5) that there are just three regular tilings of the plane, and every tiling of the plane has one of these three regular tilings underlying it. A regular tiling with $n$-gons, where $k$ of them meet at each point, is called a $\{k, n\}$-tiling. Thus the possible regular tilings of the plane are $\{6, 3\}$ (six equilateral triangles), $\{4, 4\}$ (four squares) and $\{3, 6\}$ (three hexagons).

Of course tiles don't have to be regular polygons, and we don't have to just have one kind of tile. However, always underlying any repeating tiling (or wallpaper pattern or tesselation) is one of the three regular tilings. For example underlying *Angels and Devils* is the $\{4, 4\}$ tiling. Incidentally, you may have heard of the 'seventeen wallpaper patterns'. This refers to a categorisation of the possible sets of symmetries that repeating designs covering the plane can have. The particular collection of symmetries will depend not only on the underlying regular tiling but also on the designs used for the tiles – whether they have mirror symmetry, rotation symmetry and so on. We won't discuss this in detail here.

Escher started exploring tilings in earnest after the 1936 visit to the Alhambra. He studied the sketches he had made of the Alhambra tilings, and the different possibilities for symmetries that they contained. From this he was able to construct several new drawings with interlocking motifs. In 1937 he showed his brother Berndt these drawings. Berndt, who was a Professor of Geology, recognised straightaway that these patterns were like the ones crystallographers were studying to categorise different crystal structures (crystals after all are defined by the underlying repeating molecular structure). Berndt sent his brother several articles about crystal structures, including ones by Pólya, who had independently discovered the 17 tilings in 1924 (he was unaware that these had been classified more than 30 years earlier by Fedorov, in 1891), and Haag, who gave a clear definition of a regular tiling.

So, on the plane we cannot fully depict an infinite tiling, and we are limited to three underlying regular tilings. After a while Escher had explored all of these – though one can change the motifs on the tiles, the number of ways they can be fitted together is small. He was curious as to whether other geometries could lead to other tilings. As we happen to reside on the surface of a sphere, the geometry of the sphere was a natural next step.

## 4 Tilings of the Sphere

If we want a regular tiling of a sphere our tiles will be regular polygons. To start off we can just think of ways to fit together regular planar polygons in three dimensions to make a solid. We can then put this on a sphere by 'inflating'. In other words, imagine that the polygonal faces are made of stretchy rubber, and then inflate the polyhedron so that we get a sphere with a regular tiling on it.

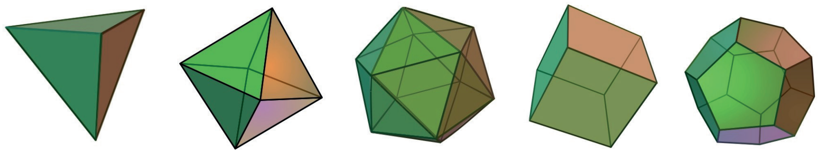

Figure 6: The five Platonic solids

It turns out that there are just five ways to do this, and the resulting shapes are called the Platonic solids. These are shown in Figure 6.

How do we know there are only five possibilities? First we must ask which regular polygons could be faces of such a solid. At any vertex we must have more than two faces meeting, otherwise we would not create a 3-dimensional object: equilateral triangles have internal angle 60, so we can fit three, four of five of them round a point in three dimensions. Six equilateral triangles would give total angle at the point of 360° so we would get a flat surface. With squares we can fit three only, and the same holds for regular pentagons. Hexagons and higher are not possible; three hexagons already give 360°. Four or more hexagons or three or more higher polygons give more than 360°, which would cause the faces to overlap.

We end up with just five possibilities: three, four or five triangles; three squares; or three pentagons. Each of these produces one of the five Platonic solids shown in Figure 6. They are, in order, the tetrahedron, octahedron, icosahedron, cube and dodecahedron.

These platonic solids give rise to the five regular tilings of the sphere, which gave Escher a few more tilings to play with.

Figure 7: *Angels and Devils*: wood carving (1942)

Here (Figure 7) is a version from 1942 of the *Angels and Devils* tiling, but this time carved on a sphere. In some sense such tilings carry on for ever as we get back to where we first started but we can only ever use finitely many tiles as the surface area of a sphere is finite. So actually in some senses this is worse and we still have a problem.

## 5 Coxeter enters the picture

Donald Coxeter was born on the 9th of February 1907, in London. His full name was Harold Scott Macdonald Coxeter – in fact originally the plan had been 'Harold Macdonald Scott' but they realised just in time that he'd then be HMS Coxeter, which sounds more like a ship than a baby. He is widely regarded as having been the greatest geometer of the 20th century with a career that spanned almost nine decades.

Coxeter was interested in mathematics, and particularly geometry, from early childhood. The picture shown over the page was taken when he was a young boy. At school he was spending so much time on his geometry that his other subjects were starting to suffer – so much so that a teacher said he was only allowed to think in four dimensions on Sundays! Coxeter studied at Cambridge, where he became Senior Wrangler (the name give to the person scoring the top marks on the mathematics final examinations). He took up a post at the University of Toronto in 1936, and lived in Canada for the rest of his life. More detail is available in the excellent biography by Siobhan Roberts [1].

Figure 8: A young Donald Coxeter

Coxeter became a prominent mathematician, well-known for his work on geometry and symmetry, for example studying and classifying symmetries of higher dimensional figures. He published several several influential and important papers and books; many mathematicians came to geometry through his work. The creative act of doing new mathematics was to him akin to making art, and in this spirt he had the following response for people who ask what is the point of pure mathematics: '*No one asks artists why they do what they do. I'm like any artist, it's just that the obsession that fills my mind is shapes and patterns.*'

The seeds for Coxeter's interaction with Escher were sown in 1954, when the International Congress of Mathematicians was held in Amsterdam. This is the largest mathematical conference in the world, held once every four years, at which the famous Fields medal is awarded. In 1954, to coincide with the Congress, a major exhibition of Escher's work was held at the Stedelijk Museum in Amsterdam. It was here that Coxeter met Escher for the first time, and he bought a couple of prints from the exhibition. (Another mathematician who visited the exhibition was Roger Penrose, who came up with his 'Penrose triangles' after seeing an impossible staircase in Escher's *Relativity* print.)

A couple of years after the 1954 Congress, Coxeter wrote to Escher to request permission to use some of his regular tiling pictures for his upcoming Presidential Address to the Royal Society of Canada. Escher agreed, and Coxeter in due course (probably early in 1958) sent him a copy of the finished transcript, after it appeared as an article in the Transactions of the Royal Society of Canada [2]. By this point in time Escher had really reached the apotheosis of what he wanted to do with regular tilings of the plane - 1958 had also seen the publication of his book *Regelmatige vlakverdeling (The Regular Division of the Plane)*, featuring a collection of his tiling designs. When Escher looked at Coxeter's paper, he was fascinated by one of the other diagrams it included – an illustration of a tiling in what's called hyperbolic geometry. Escher instantly realised that this diagram opened up an entirely new possibility for his designs. He got to work straightaway and by the end of 1958 had produced a woodcut called *Circle Limit I* (Figure 10).

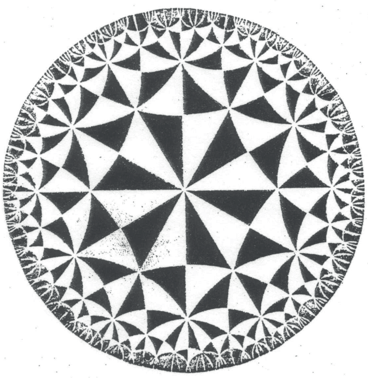

Figure 9: The hyperbolic tiling from Coxeter's paper

Figure 10: *Circle Limit I*, M.C. Escher (1958)

But what actually was this figure in Coxeter's paper? It is a tiling of the so-called Poincaré disc, which is one representation of hyperbolic geometry. Escher realised that this was the perfect way to represent infinity because in hyperbolic geometry all the triangles shown are actually the same size. Infinity is the edge of the circle and the pattern, by appearing to shrink, can in fact continue infinitely in a bounded shape. So this was a good solution to the conundrum of representing infinity in a finite picture. To explore these ideas we need a geometrical interlude.

# 6 Three Geometries

The geometry we all learned at school is *Euclidean geometry*. We all know, for example, that the shortest path between any two points is a straight line, that the angles in a triangle add up to $180^\circ$, and so on. This is the geometry of the plane.

But there are other geometries – for example we live on a sphere (ish). On a sphere the angles in a triangle don't add up to $180^\circ$. As one instance of this, a triangle whose vertices are the north pole and any two points on the equator will have two right angles in addition to the angle at the north pole. The example in Figure 11 has an angle sum of $270^\circ$. How can this be? The first thing to notice is that the 'lines' on a sphere cannot be 'straight lines' as they are drawn on a curved surface, so we have to decide what we are going to call 'lines'. In Euclidean geometry we could actually define the line between two points (strictly speaking a line segment) to be the path of shortest length between those points. If we wanted to do a practical experiment we could run a piece of string between two points and pull it taut – it would be a straight line. If we try this on a sphere we get arcs of 'great circles' (equators). If you've ever flown long-haul you'll know that airline routes do indeed follow these arcs, rather than travelling along what look like straight lines on a map. So, on a sphere, the 'lines' are arcs of great-circles. These paths of shortest distance are called geodesics, and it is these that we call the lines of the geometries we work in. A triangle is then a shape bounded by three geodesics.

Figure 11: A $270^\circ$ spherical triangle (Image credit: rollingalpha.com)

In Euclidean geometry the proof that the angles in a triangle add up to $180^\circ$ is very quick – given a triangle $ABC$, draw the line parallel to $BC$ that passes through $A$. Now alternate angles are equal, so we can see that the green angles are equal and the blue angles are equal. The angles around $A$ (red, green, blue) sum to the angle on a straight line – namely $180^\circ$. But these angles are precisely the same as the ones in the triangle. Therefore the angles in the triangle also sum to $180^\circ$. (This is Proposition 32 of Euclid Book 1.)

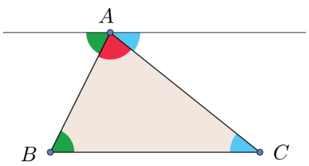


Figure 12: Angles in a plane triangle

Vital to this proof is the existence of exactly one line parallel to $BC$ passing through $A$. This is essentially the famous 'parallel postulate' of Euclidean geometry – and it does not hold in spherical geometry. There, any two geodesics always meet. So there are no parallel 'lines' at all. (Remember lines of latitude are not great circles, except the equator, so are not geodesics.) It turns out that every triangle on a sphere has angles adding up to more than $180^\circ$. The amount by which the sum exceeds $180^\circ$ turns out to be proportional to the area of the triangle, and you can see a nice explanation of this on the NRICH website [3].

On a sphere there are no lines parallel to a given line, passing through a given point. On the plane there is exactly one such line. Is there a kind of geometry where there are many such lines? On a sphere angles in a triangle add up to more than $180^\circ$. On the plane they add up to exactly $180^\circ$. Is there a geometry where the angles add up to less than $180^\circ$? The answer to both questions is yes – and this is where we encounter *hyperbolic geometry*.

We can see hyperbolic geometry on the surface of a shape called a *hyperboloid*. A hyperbola is one of the kinds of conic section, made by slicing a cone (if you slice in other ways you get ellipses and parabolas).

If we rotate this hyperbola about its vertical axis of symmetry, we get what's called a 'hyperboloid of revolution'. This is a curved surface, and its geodesics are hyperbolas (in particular they are the intersections with the hyperboloid and planes passing through the origin). We tend just to keep the top half of the surface. Now this does give a geometry where angles in a triangle are less than $180^\circ$, but it is quite hard to visualise.

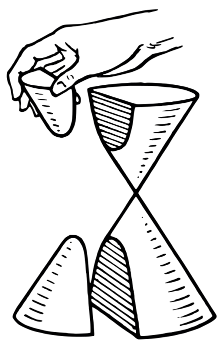

Figure 13: Hyperbola (Image by Melikamp / Wikipedia)

The Poincaré disc is a projection of this hyperboloid onto a flat circular disc, illustrated in Figure 14. We won't go into the exact details of the algebra involved but the idea is similar to the methods used to produce maps of the

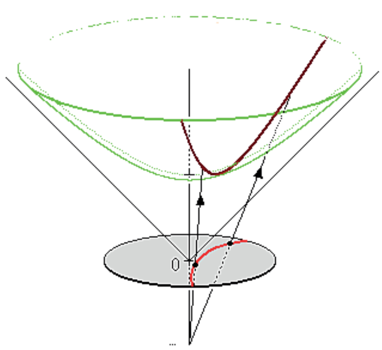



Figure 14: Projecting a hyperboloid onto a disc

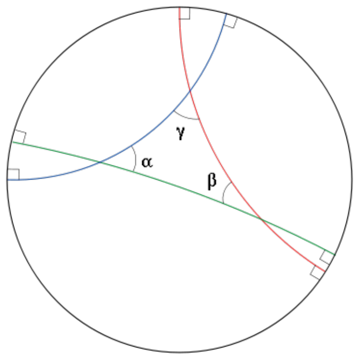



Figure 15: A hyperbolic triangle

world on flat surfaces, when after all the world is a sphere. What happens when we do this is that the geodesics, those hyperbolas on the curved surface, become arcs of circles that meet the edge of the disc at right angles. (In the limiting case these are just diameters of the disc.) Figure 15 shows three such lines forming a hyperbolic triangle on the Poincaré disc – its angles add up to less than 180 degrees!

## 7 Escher's hyperbolic tilings

If we look again at Coxeter's diagram, we can see that it is a tiling of the hyperbolic disc using just one shape of tile; a hyperbolic triangle whose angles are $30°$, $45°$ and $90°$. The shapes in the tiling look different sizes because we are trying to draw them on a flat disc, a bit like countries on a map may look different sizes because we are actually representing a sphere. This is not a regular tiling, because the tiles are not regular polygons. However, there is a regular tiling underlying it. It is the $\{4, 6\}$ regular tiling with 4 regular hexagons meeting at each point.

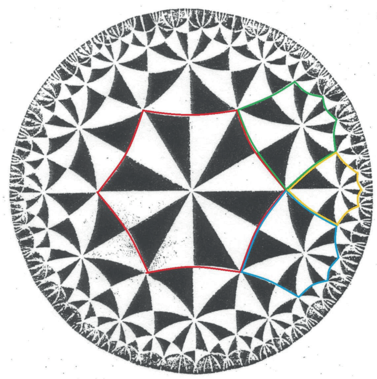

Figure 16: Underlying regular tiling of the Coxeter triangular tiling

Escher wrote to Coxeter in December 1958, sending him a copy of *Circle limit I*, and asking for advice: '*If you could give me a simple explanation how to construct the following circles, whose centres approach gradually from the outside till they reach the limit, I should be immensely pleased and very thankful to you! Are there other systems besides this one to reach a circle limit?*' Coxeter wrote back with some suggestions, and Escher kept working with these new ideas, producing further 'Circle Limit' woodcuts. He considered *Circle Limit III* (Figure 17) to be his best of these: it has a sense of flow as the fish are following each other along curves, and look more like fish than the slightly angular ones of *Circle Limit I*. It is also considerably more ambitious technically, featuring four colours as well as black and white.

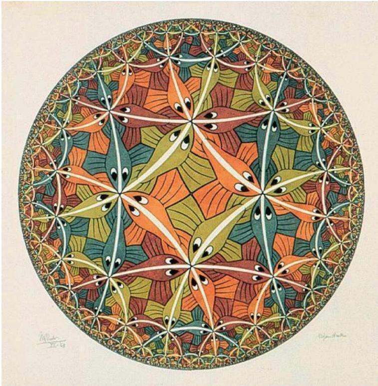

Figure 17: *Circle Limit III* (1959)

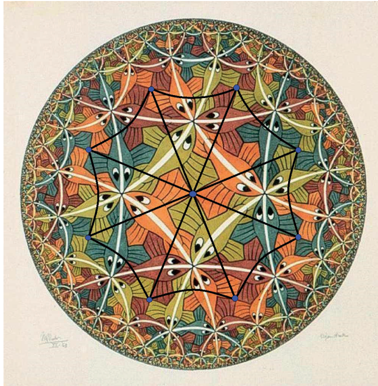

Figure 18: The underlying regular tiling

If we look at the places where four fish fins meet, some of which are highlighted in Figure 18, we can see that eight of these points surround the centre, forming an octagon. The octagon is divided into eight triangles which are actually equilateral in the hyperbolic world (they do not look it with our Euclidean eye but remember again that this is a projection). So underlying Escher's fish is a $\{8, 3\}$ regular tiling of the hyperbolic disc, where eight equilateral triangles meet at each vertex.

## 8 Influence on Coxeter's work

There are many instances of artists using mathematics – one obvious example being the rules of perspective which use the mathematics of projective geometry, though it's fair to say this does not necessarily imply the artists in question actually understand the mathematics behind the rules. What is much rarer is mathematicians producing new work from art. What makes the interaction between Escher and Coxeter special is that there was a genuine exchange of ideas. Escher learnt from Coxeter but Coxeter learnt from Escher too.

Figure 19: *Circle limit IV (Heaven and Hell)*, 1960

An example of this is a paper Coxeter wrote where he described the mathematics in *Circle Limit III* – he looked at the white lines, the 'spines' of the fish, and having initially thought they were slightly inaccurate hyperbolic lines (ie circle arcs) he realised they weren't, and that in fact Escher had found equidistant curves (that is, curves at a constant distance from a given hyperbolic line - analogous to parallel lines in Euclidean geometry) and produced them incredibly precisely. Coxeter said this had led him to a new understanding of the hyperbolic disc. These curves make 'triangles' whose angles are 60 degrees – impossible for genuine lines in hyperbolic space, where angles in a triangle are less than 180 degrees. There is more detail in [4] and [5].

Escher corresponded regularly with Coxeter about new ideas – in fact when Escher was creating a new picture based on this kind of geometry, he called it 'Coxetering', surely the only verbing of a mathematicians name in the word of art. Escher kept working with this new idea, and we see in Figure 19 his angels and devils in a third geometry, based on a $\{4, 6\}$ regular tiling of the hyperbolic disc.

The technical skill to produce these woodcuts is quite breathtaking. Escher wrote, in a letter to his son Arthur (20 March 1960): '*I've been killing myself, [...] for four days with clenched teeth, to make another nine good prints of that highly painstaking circle-boundary-in-colour. Each print requires twenty impressions: five blocks, each block printing four times.*'

Coxeter explained to Escher that there are infinitely many regular tilings of the hyperbolic disc. Remember that a $\{k, n\}$ tiling denotes $k$ regular $n$-gons meeting at each vertex. With the obvious restriction that $k \geq 3$ and $n \geq 3$, it turns out that there is a tiling for every possible choice of $k$ and $n$. There is a $\{k, n\}$-tiling of the plane precisely when $\frac{1}{n} + \frac{1}{k} = \frac{1}{2}$; of the sphere precisely when $\frac{1}{n} + \frac{1}{k} > \frac{1}{2}$; and of the hyperbolic disc precisely when $\frac{1}{n} + \frac{1}{k} < \frac{1}{2}$. As an example of this, suppose you want to make a regular tiling with equilateral triangles. With three, four or five meeting at each point, you need a spherical tiling; with six at each point you get a tiling of the plane; with more than six at each point you get a hyperbolic tiling.

Escher and Coxeter continued to correspond until Escher's death in 1972. Coxeter wrote several mathematical papers about Escher's work, and it was a source of regret to him that his article about *Circle Limit III* did not appear until 1979, seven years after Escher had died. Coxeter was still actively researching until his death on March 31st 2003 at the age of 97.

To finish, here is a $\{4, 5\}$ regular tiling of the hyperbolic disc using the Gresham College crest. You can create your own hyperbolic tilings based on an image of your choice at Malin Christersson's excellent website [6].

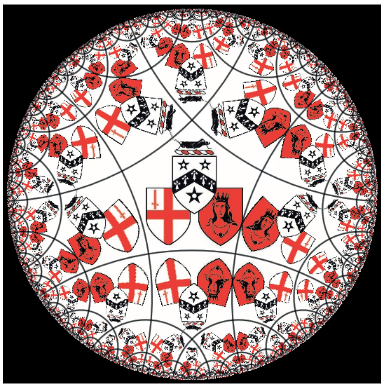

Figure 20: A hyperbolic tiling of the Gresham crest

## References


[1] Siobhan Roberts. *King of Infinite Space: Donald Coxeter, the man who saved geometry*, Profile Books, London (2007).

[2] H.S.M. Coxeter. Crystal symmetry and its generalizations, in *A Symposium on Symmetry*, Trans. Royal Soc, Canada 51, ser.3, sec. 3 (1957), 1–13.

[3] https://nrich.maths.org/1434

[4] Doris Schattschneider. Coxeter and the artists: Two-Way inspiration, in *The Coxeter Legacy: Reflections and Projections*. Providence/Toronto: American Mathematical Society/Fields Institute, 2006, 255–80.

[5] Doris Schattschneider. The Mathematical Side of M.C.Escher, Notices of the American Mathematical Society, Vol. 57, no. 6 (2010), 706–718.

[6] http://www.malinc.se/m/ImageTiling.php


## Note