Escher and Coxeter

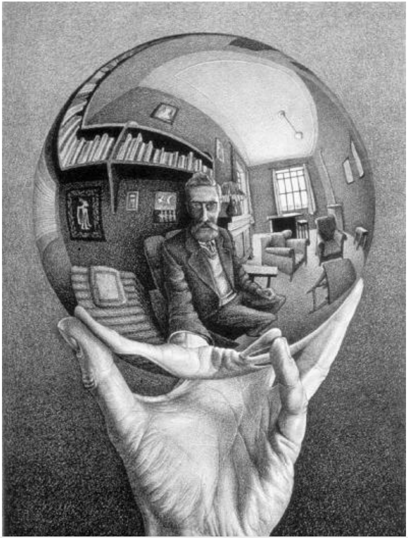
A Mathematical Conversation

Sarah Hart

Birkbeck, University of London

5 June 2017

M. C. Escher (1898 – 1972)



Hand with Reflecting Sphere
(Lithograph, 1935)

Education

- Early education in Arnhem.
- Admitted 1919 to School for Architecture and Decorative Arts in Haarlem.

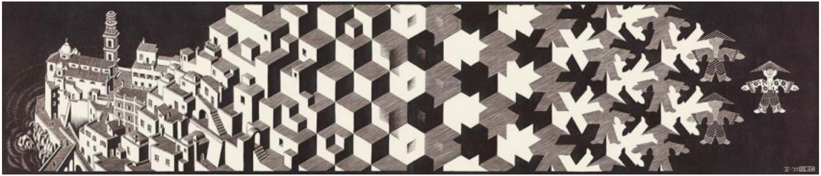
White Cat (Woodcut, 1919)

Travels in Italy and Spain

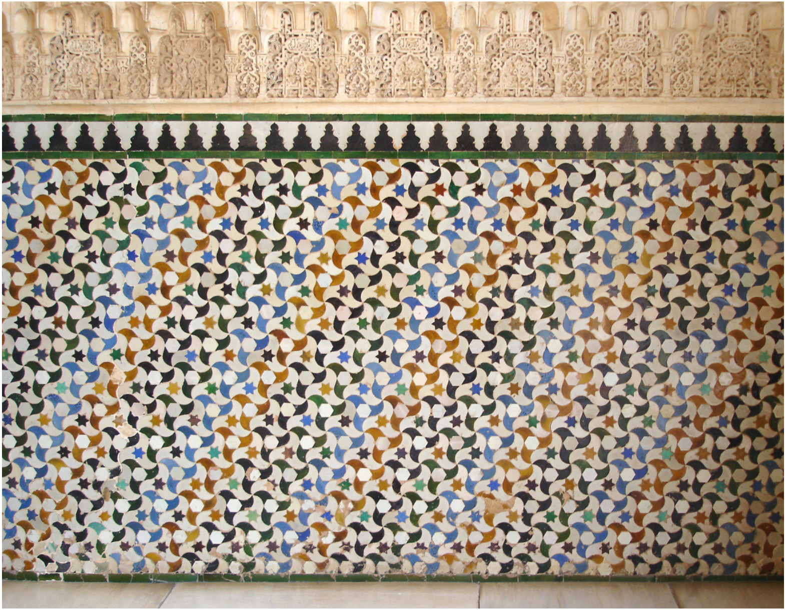
San Gimignano
(Woodcut, 1923)

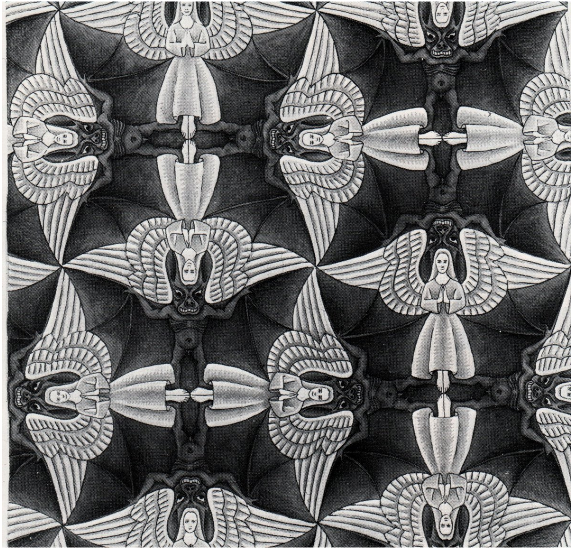
Atranti, Coast of Amalfi
(Lithograph, 1931)

A new direction



Metamorphosis I (Woodcut, 1937)





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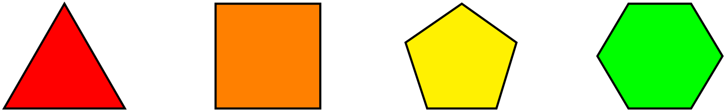

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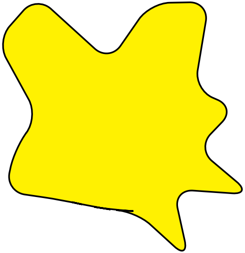

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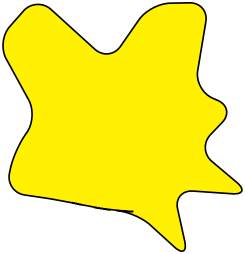

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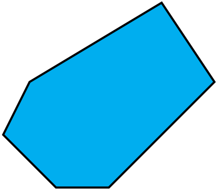

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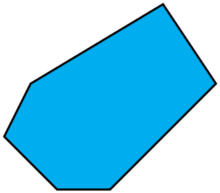

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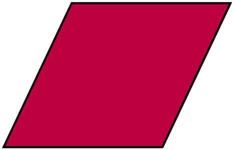

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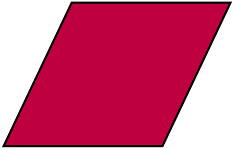

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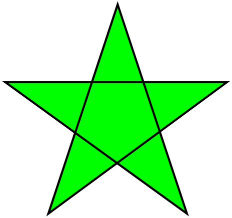

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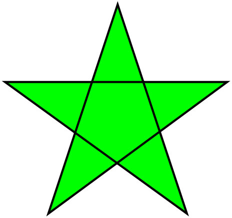

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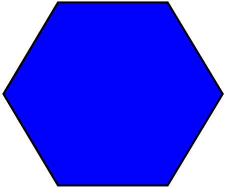

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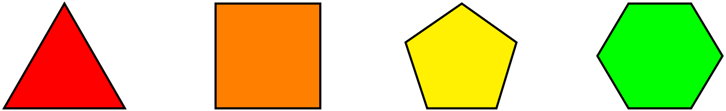
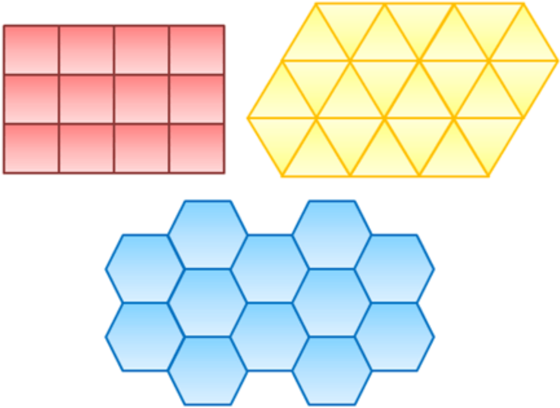
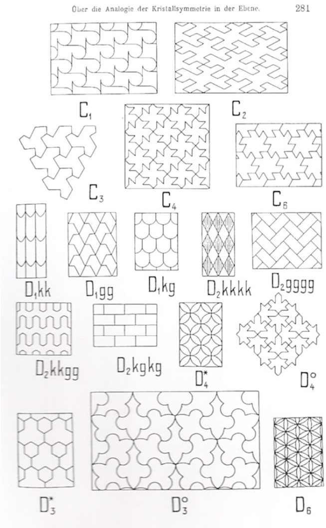




- A $\{k, n\}$-tiling has k tiles (n-gons) at each point.
- The regular tilings of the plane are $\{4, 4\}$, $\{6, 3\}$ and $\{3, 6\}$.

What about the 17 wallpaper patterns?



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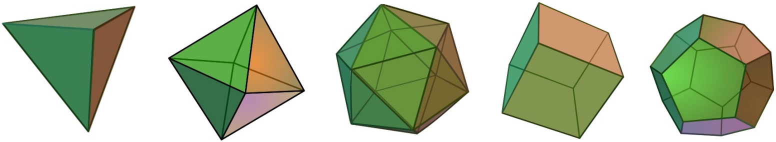

Angels and Devils on a Sphere (1942)

Donald Coxeter (1907 – 2003)

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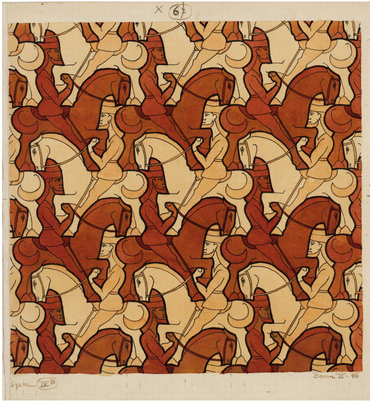
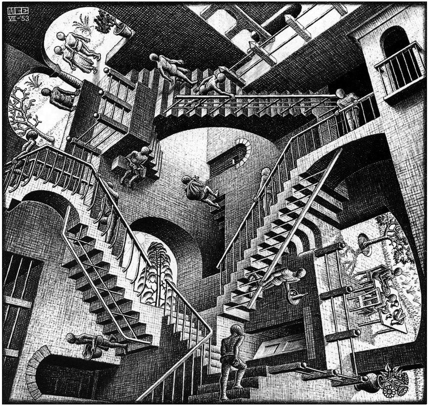
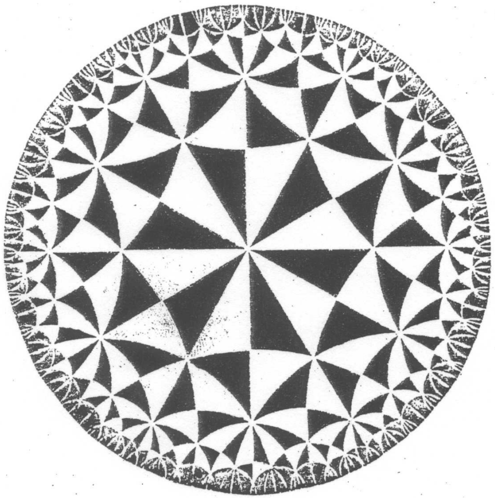
‘No one asks artists why they do what they do. I’m like any artist, it’s just that the obsession that fills my mind is shapes and patterns.’

International Congress of Mathematicians, 1954

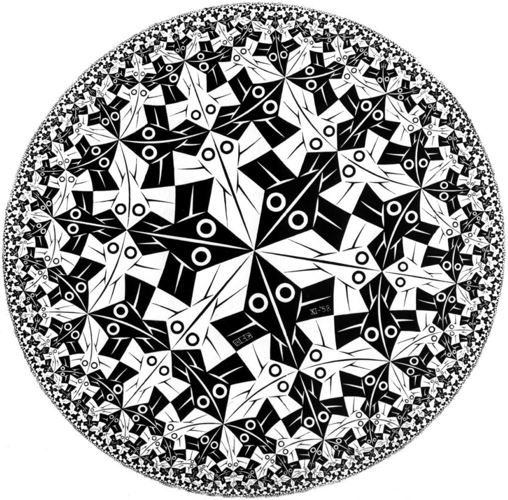




Coxeter's Diagram



Escher's New Tiling



Three Geometries

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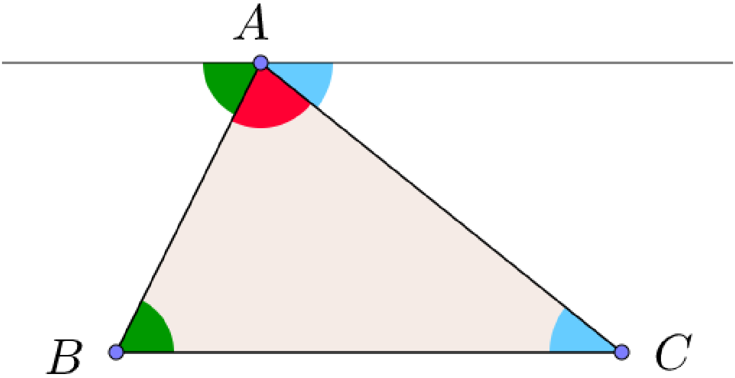
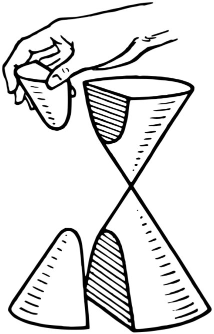
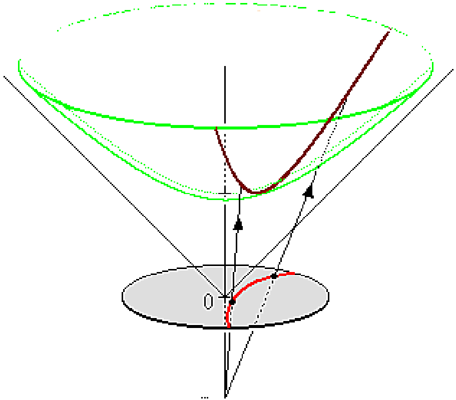

A
B
C

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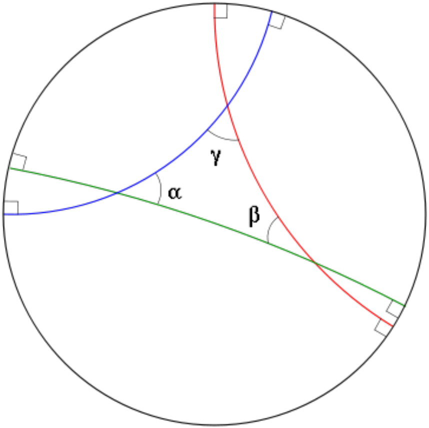
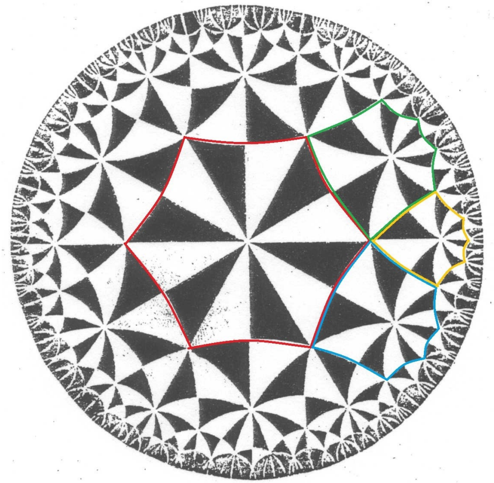
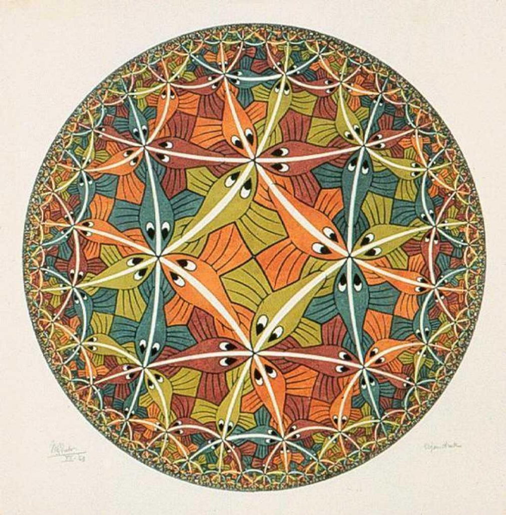
The Poincaré Disc



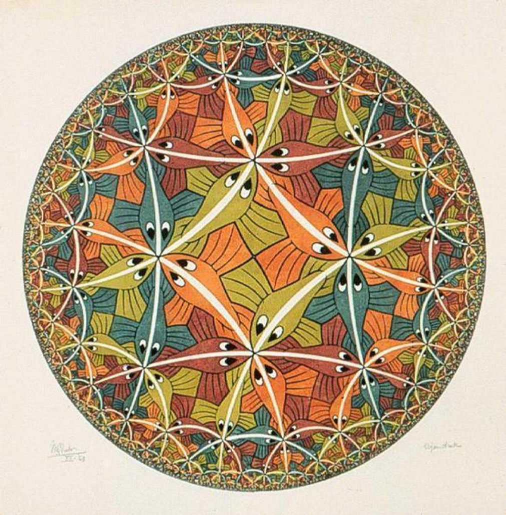
Hyperbolic Tilings



Circle Limit III (woodcut, 1959)

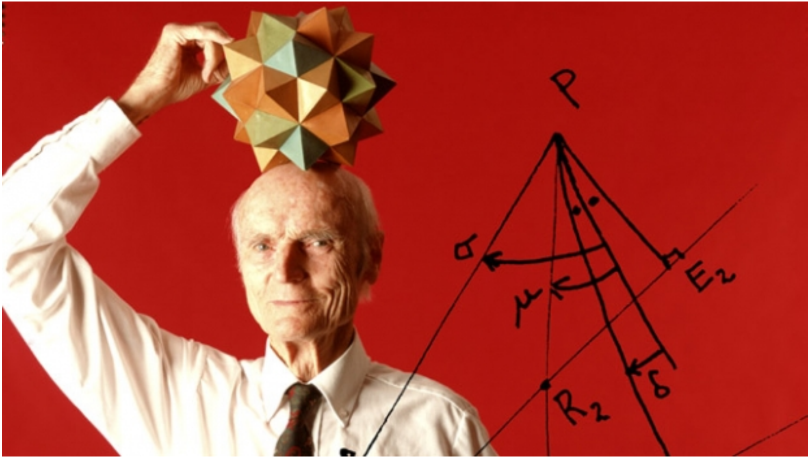


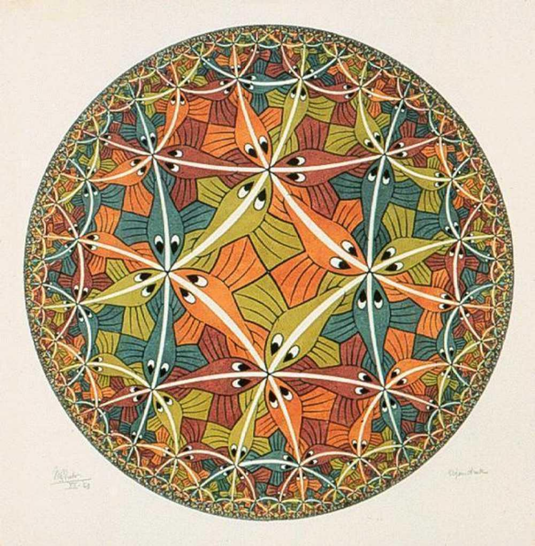
Circle Limit III (woodcut, 1959)

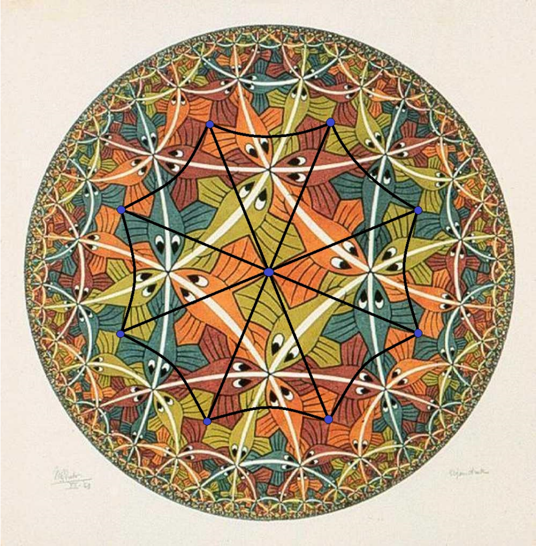


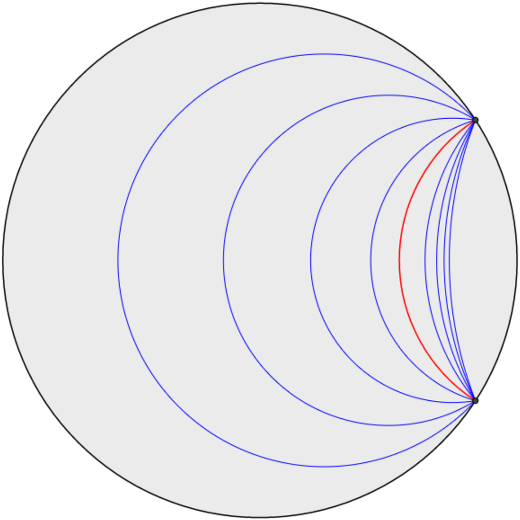
'*I've been killing myself, [...] for four days with clenched teeth, to make another nine good prints of that highly painstaking circle-boundary-in-colour. Each print requires twenty impressions: five blocks, each block printing four times.*' (Escher to his son Arthur, March 1960)

Influence on Coxeter

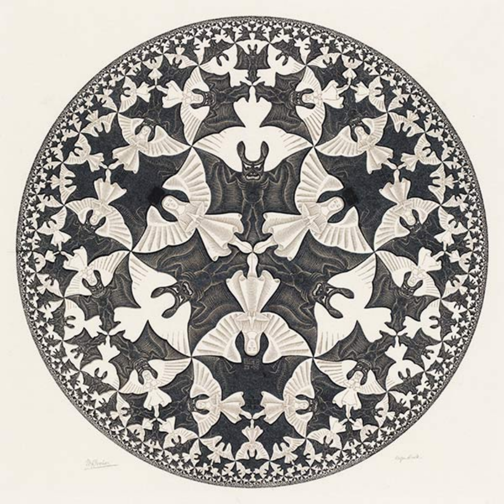








Circle Limit IV (woodcut, 1960)



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- We must have $n \geq 3$ as the smallest number of sides a polygon can have is 3.

Theorem

For any k and n with $k \geq 3$ and $n \geq 3$, there exists a regular tiling in exactly one of plane, spherical and hyperbolic geometry.

- $\frac{1}{k} + \frac{1}{n} > \frac{1}{2} \rightarrow$ *regular spherical tiling.*
- $\frac{1}{k} + \frac{1}{n} = \frac{1}{2} \rightarrow$ *regular plane tiling.*
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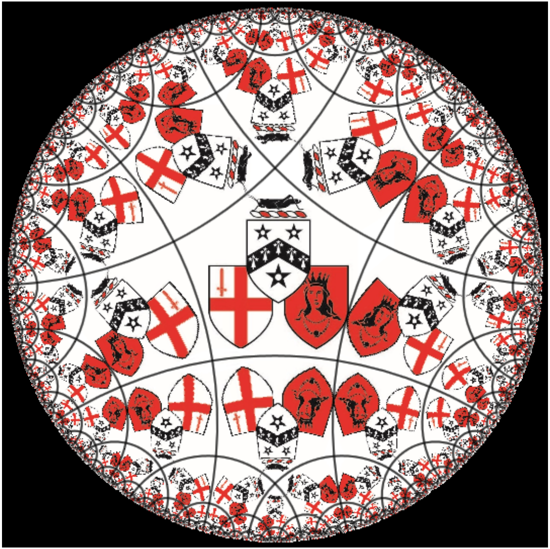
Example

For a tiling with equilateral triangles $(n = 3)$ we have $\frac{1}{2} - \frac{1}{n} = \frac{1}{6}$. So if k is 3, 4 or 5 we get a spherical tiling; if $k = 6$ it's a plane tiling; if k is 7 or higher it's a hyperbolic tiling.



Thank you!