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Just Imagine: The Tale of i

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I am delighted to be contributing to this joint meeting between Gresham College and the BSHM. I am going to talk about one of the most remarkable constants in mathematics the square root of -1, now usually denoted by the letter *i*.

Clearly when you square any number you can never get a negative result. So what could possibly be meant by the square root of -1? It is no wonder that it was called imaginary and it took centuries for it to be accepted. I will discuss why the square root of -1, arose in mathematics, how it was represented and how it was all eventually demystified by the 19th century Irish mathematician William Rowan Hamilton. As well as his work on imaginary numbers Hamilton also discovered or created quaternions, a non-commutative algebraic system and this work helped free algebra from the constraints of arithmetic.

$$\boldsymbol{i = \sqrt{-1}}$$

What could the square root of minus 1 mean since every real number, positive or negative, gives a positive result when squared? So the square root of minus 1 cannot be a real number.

Euler quote
Euler, in the eighteenth century, who worked with this type of number a great deal, also said:

Of such numbers we may truly assert that they
are neither nothing, nor greater than nothing,
nor less than nothing, which necessarily
constitutes them imaginary or impossible.

Even in the early 19th century there was still a great deal of unhappiness about so-called 'imaginary' numbers that don't seem to exist.

De Morgan quote
For example,

Augustus De Morgan, Professor of Mathematics at University College, London, declared that:

We have shown the symbol $\sqrt{-1}$ *to be void of meaning, or rather self-contradictory and absurd.*

But however strange imaginary numbers seemed they were of great use in solving quadratic and other equations and in other areas of mathematics.

(2 + 3$\sqrt{-1}$) or (4 + 5$\sqrt{-1}$)

But if we allow the imaginary number $\sqrt{-1}$ we can form many more numbers such as (2 + 3$\sqrt{-1}$) or (4 + 5$\sqrt{-1}$). These are called complex numbers.

Suppose that we try to calculate with them

Addition
We find that addition is easy:

$$(2 + 3\sqrt{-1}) + (4 + 5\sqrt{-1}) = 6 + 8\sqrt{-1},$$

Multiplication
And so is multiplication (replacing $\sqrt{-1} \times \sqrt{-1}$ whenever it appears by -1).

$$(2 + 3\sqrt{-1}) \times (4 + 5\sqrt{-1})$$
$$= (2 \times 4) + (3\sqrt{-1} \times 4) + (2 \times 5\sqrt{-1}) + (15 \times \sqrt{-1} \times \sqrt{-1})$$
$$= (8 - 15) + (12 + 10)\sqrt{-1} = -7 + 22\sqrt{-1}.$$

We can carry out all the standard operations of arithmetic on these new objects.

Complex Numbers
As I've said we call the object $a + b\sqrt{-1}$ a *complex number*:

the number *a* is its *real part*, and the number *b* is its *imaginary part*.

Nowadays, we usually follow Euler who used the letter *i,* the first letter of 'imaginary' to mean $\sqrt{-1}$ so that $i^2 = -1$
So a complex number is usually written $a + bi$ where $i^2 = -1$

If $b = 0$ we get the real number *a* and if $a = 0$ we get the imaginary number *bi*.

Possibly the first time many people meet complex numbers is when they come to solving quadratic equations. Examples of quadratic equation are:

Real Distinct Roots
$x^2 – 6x + 8 = 0$ which can be factorized as $(x – 2)(x – 4) = 0$ so there are two real solutions $x = 2$ and $x = 4$.

Equal Real Roots
$x^2 – 6x + 9 = 0$ which can be factorized as $(x – 3)(x – 3) = 0$ so there is a repeated real root $x = 3$.

Complex Roots
$x^2 – 6x + 10 = 0$. To factorise this quadratic we need to bring in $i = \sqrt{-1}$
$x^2 – 6x + 10 = (x – 3 – i)(x – 3 + i) = 0$ so there are two complex solutions $x = 3 + i$ and $x = 3 – i$.

Graphs of the Three Quadratics
The difference between the solutions can most easily be seen graphically.
And indeed the formula for solving a quadratic equation shows us that all we need are real or complex numbers to solve any quadratic.
If $ax^2 + bx + c = 0$ then $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Polynomial of Degree 6
But what happens for higher degree (polynomial) equations

$$x^6 - 12x^5 + 60x^4 - 160x^3 + 239x^2 - 188x + 60 = 0\ ?$$

Can this be solved with only real and complex numbers or do we need to introduce yet another type of number?

There are several possibilities;

We only need real and complex numbers
We need to introduce new 'hyper-complex numbers'
Some equations have solutions which are not numbers
Some equations have no solutions.

Let us try to solve the simpler cubic equation $x^3 = i$ or $x^3 - i = 0$ and try to see if complex numbers are sufficient. Well, because the equation is so simple, we can see that a solution is

$$x = -i \text{ because } (-i)(-i)(-i) = i$$

Then the equation factorises as

$$x^3 - I = (x + i)(x^2 - ix - 1) = 0$$

and we can use the quadratic formula on the second factor to get three solutions

$$-i \text{ and } \tfrac{1}{2}(\surd 3 + i) \text{ and } \tfrac{1}{2}(-\surd 3 + i)$$

So for this example complex numbers are all we need.

In fact complex numbers are enough to solve any polynomial equation.
For our earlier example

$$x^6 - 12x^5 + 60x^4 - 160x^3 + 239x^2 - 188x + 60 = 0$$

we have

$$\begin{aligned} & x^6 - 12x^5 + 60x^4 - 160x^3 + 239x^2 - 188x + 60 \\ &= (x^2 - 4x + 3)\,(x^2 - 4x + 4)\,(x^2 - 4x + 5) \\ &= (x - 1)\,(x - 3)\,(x - 2)^2\,(x^2 - 4x + 5) \\ &= (x - 1)\,(x - 3)\,(x - 2)^2\,(x - 2 - i)\,(x - 2 + i), \end{aligned}$$

So there are six solutions of

$$x^6 - 12x^5 + 60x^4 - 160x^3 + 239x^2 - 188x + 60 = 0$$

They are $x = 1, 3, 2$ (repeated twice), $2 + i$, and $2 - i$.

This result is a particular case of what is known as the Fundamental theorem of Algebra. There are various equivalent ways of stating it.

Fundamental Theorem of Algebra
Every polynomial equation of degree n has exactly n real or complex solutions (as long as we count them appropriately).

Representing Complex Numbers Geometrically

Towards the end of the eighteenth complex numbers were given a geometrical form by the Norwegian-Danish surveyor Caspar Wessel. In this representation, called the *complex plane*, two axes are drawn at right angles (the *real axis* and the *imaginary axis*) and the complex number $a + b\sqrt{-1}$ is represented by the point at distance *a* in the direction of the real axis and height *b* in the direction of the imaginary axis.

The same idea was subsequently discovered by Argand and Gauss and is now often known as the Argand diagram.

The Argand diagram is useful for understanding another representation of complex numbers using polar coordinates.

Polar Form of a Complex Number

From the diagram we can see that for the complex number $a + bi$

$a = r\cos\theta$ and $b = r\sin\theta$

where $r = \sqrt{(a^2 + b^2)}$ is the distance from $a + bi$ to the origin and θ is the angle between the line segment from the origin to $a + bi$ and the positive *x*-axis.

We use the notation $[r, \theta]$ for these polar coordinates of the complex number $a + bi$

r is called the *modulus* of the complex number

Multiplying Two Numbers in Polar Form

Multiplying two numbers in polar form

$$[r, \theta] \times [s, \phi] = [rs, \theta + \phi]$$

To multiply two numbers in polar form just multiply their moduli and add their angles.

$$r(\cos\theta + i\sin\theta) \times s(\cos\varphi + i\sin\varphi)$$
$$= rs\,\{(\cos\theta\cos\varphi - \sin\theta\sin\varphi) + i(\sin\theta\cos\varphi + \cos\theta\sin\varphi)\}$$
$$= rs\,\{\cos(\theta + \varphi) + i\sin(\theta + \varphi)\}.$$

Using the addition formulas for sine and cosine

Multiplying by *i*

Multiplying a complex number by $i = [1, \pi/2]$ rotates it anticlockwise through a right angle.

Multiplying by $i \times i$

Multiplying a complex number by *i* **twice** rotates it anticlockwise through two right angles corresponding to $i \times i = -1$.

De Moivre's Theorem

For any number *n*,

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

Complex nth Roots of Unity

This enables us to find all the solutions of the equation $z^n = 1$ which are called the complex nth roots of unity. For example suppose n = 4 then by the Fundamental Theorem of Algebra we know there are four and only four roots.

Four Solutions

Apply de Moivre's theorem with n =4 to each of the complex numbers

$$\cos 2\pi/4 + i \sin 2\pi/4$$
$$\cos 4\pi/4 + i \sin 4\pi/4$$
$$\cos 6\pi/4 + i \sin 6\pi/4$$
$$\cos 8\pi/4 + i \sin 8\pi/4$$

to get

Raising to the Power of Four

$$\{\cos 2\pi/4 + i \sin 2\pi/4\}^4 = \cos 2\pi + i \sin 2\pi = 1$$
$$\{\cos 4\pi/4 + i \sin 4\pi/4\}^4 = \cos 4\pi + i \sin 4\pi = 1$$
$$\{\cos 6\pi/4 + i \sin 6\pi/4\}^4 = \cos 6\pi + i \sin 6\pi = 1$$
$$\{\cos 8\pi/4 + i \sin 8\pi/4\}^4 = \cos 8\pi + i \sin 8\pi = 1$$

This tells us that the original 4 numbers all solutions of $z^4 = 1$. They are

The Four Solutions

$$\cos 2\pi/4 + i \sin 2\pi/4 = i$$
$$\cos 4\pi/4 + i \sin 4\pi/4 = -1$$
$$\cos 6\pi/4 + i \sin 6\pi/4 = -i$$
$$\cos 8\pi/4 + i \sin 8\pi/4 = 1$$

They are the four different numbers *i, -1, -i and 1*. Hence by the Fundamental Theorem of Algebra they are all the solutions.

Let us plot them on the Argand diagram.

The Case n = 4

The four solutions lies on the circle of radius 1 centred at the origin and lie at the vertices of a square.

Plot The Solutions of $z^n = 1$

If we plot the solutions of $z^n = 1$, we find that they always lie on the unit circle at the corners of a regular polygon with n sides. On the left we see the case of n = 3, a triangle; in the middle n= 4, a square; on the right n = 6 and we have a regular hexagon.

I have been showing you some uses of complex numbers. Robin, in his book, discusses their origin in the solution of equations and in particular tells the exciting story of how in 16th century Italy formulae were discovered for the cubic and for the quartic, i.e. polynomial equations of degree three and four.

But I have not answered the question as to what is the square root of minus one? It was William Rowan Hamilton who in the 1830s eliminated the confusion and suspicion surrounding them.

William Rowan Hamilton

Hamilton was born in Dublin in 1805. He achieved international recognition in his own lifetime as shown by his election (shortly before his death) the first in the list of the Foreign Associates of the newly formed American National Academy of Sciences. These Associates were in the view of the Academy the most important scientists working outside the United States of America. Although Hamilton had performed world class research in geometrical optics, as well as in dynamical systems, it was his work on algebra which won him recognition from the National Academy.

TCD and Dunsink Observatory

He took first place in the entrance examinations for Trinity College Dublin and quickly scaled the academic ladder, becoming Professor of Astronomy and Astronomer Royal of Ireland at the age of 22 before he even graduated.

So, what did Hamilton do to resolve the problem of the square root of minus 1? In hindsight it seems very obvious and is now generally accepted!

Hamilton on Complex Numbers
The geometrical representation of complex numbers by the so-called Argand diagram raised the following question in Hamilton's mind.

Is there any other algebraic representation of complex numbers that reveals all valid operations on them?

Definition
He proposed that the complex number should be *defined* as a pair (a, b) of real numbers.

Addition
We combine such pairs (a, b) and (c, d) by using the following rules:

Addition:

$$(a, b) + (c, d) = (a + c, b + d);$$

Multiplication
Multiplication:

$$(a, b) \times (c, d) = (ac - bd, ad + bc);$$

Identification
The pair $(a, 0)$ then corresponds to the real number a. The pair $(0, 1)$ corresponds to the imaginary number $i = \sqrt{-1}$, and we have the equation

$$(0, 1) \times (0, 1) = (-1, 0),$$

which corresponds to the equation $i \times i = -1$.

Hamilton avoided talking about imaginary quantities by dealing with pairs or couples of real numbers and giving formal rules for adding and multiplying these couples. These rules then gave an exact parallel to the rules for adding and multiplying complex numbers but without introducing the mysterious square root of minus 1.

The logical development was to extend his ideas to three dimensions and to consider number triples or *triadic fancies* as Hamilton called them.

Triadic Fancies
He now looked at number triples such as (a, b, c) and wanted to find rules for their addition and multiplication. By analogy with complex numbers he wrote them as

$$a + bi + cj \text{ where } i^2 = j^2 = -1.$$

Adding triples was easy, for example:

$$(1 + 2i + 3j) + (4 + 5i + 6j) = (1 + 4) + (2i + 5i) + (3j + 6j)$$
$$= 5 + 7i + 9j$$

Multiplying Triples

But what about multiplying triples? What is, or rather how could we define:

$$(1 + 2i + 3j) \times (4 + 5i + 6j)?$$

If we multiply them out in the analogous way to complex numbers and use $i^2 = j^2 = -1$ we obtain
$(4 - 10 - 18) + (5 + 8)i + (6 + 12)j + (12 + 15)ij$

What is this term *ij*?

Hamilton thought about his problem for many years exploring different possibilities for the term *ij*.

For example we cannot set $ij = 0$ because then $0 = (ij)^2 = i^2 j^2 = (-1)(-1) = 1$

In fact it became something of a family joke as we can read in this letter from Hamilton to one of his sons, Archibald, written many years later, shortly before his death.

Hamilton and Son
Every morning on my coming down to breakfast, your brother Wlliam Edwin and yourself used to ask me. 'Well, Papa, can you multiply triplets?' Whereto I was always obliged to reply, with a sad shake of the head 'No, I can only add and subtract them

All his attempts failed because Hamilton wanted to construct a way to multiply triples which had similar properties to those enjoyed by the real and complex numbers. He spent over a decade on the problem but the strange thing is that if he suspected it could not be done then that is easily proved.

No Multiplication Law for Triples

As we saw the problem is:

what value to give to *ij*.

Hamilton wanted it to be another triple so suppose we say
let $ij = a + bi + cj$ for some real numbers *a, b and c* that we want to determine.
$ij = a + bi + cj$

Multiply both sides by i

$$i^2 j = ai + bi^2 + cij$$

Now use $i^2 = -1$, $ij = a + bi + cj$ and collect like terms to get
$0 = (ac - b) + (a + bc)i + (1 + c^2)j$

This means each coefficient on the right is zero, in particular $1 + c^2$ is zero which is impossible for any real number c!

And then suddenly one day he had the flash of inspiration that resolved the problem.

Notebook and Sketch of Hamilton
It was on the 16th October 1843 and Hamilton was walking along the Royal Canal in Dublin to a meeting of the Royal Irish Academy when as he later wrote:

an undercurrent of thought was going on in my mind which gave me at last a result, whereof it is not too much to say that I felt at once the importance. An electric circuit seemed to close; and a spark splashed forth, the herald as I foresaw immediately of many long years to come of definitely directed thought and work by myself if spared, and at all events on the part of others if I should ever be allowed to live long enough distinctly to communicate the discovery. I pulled out on the spot a pocket book, which still exists, and made an entry there and then. Nor could I resist the impulse— unphilosophical as it might have been—to cut with a knife on a stone on Brougham Bridge, as we passed it, the fundamental formula with the symbols *i, j*, and *k* namely $i^2 = j^2 = k^2 = ijk = -1$
which contains the solution of the problem.

Hamilton was a compulsive scribbler. According to his son, if there was no paper available, he would write on his fingernails or if at breakfast scribble on his hard-boiled egg.

Well, what did he see in his flash of inspiration? Instead of triples he added another term k which also satisfied $k^2 = -1$ giving him quadruples or as he called them quaternions.

Quaternions
Instead of triples $a + bi + cj$ he was now working with quadruples
$a + bi + cj + dk$

and which he could multiply using the relations $i^2 = j^2 = k^2 = ijk = -1$.
These relations imply that ij is not equal to ji but is equal to $-ji$.

ij = k
We have $ijk = -1$

Multiply both sides by k

$ijkk = -k$
But $k^2 = -1$ so
$-ij = -k$ or $ij = k$

k = -ji
Start again with $ijk = -1$
But this time multiply by ji

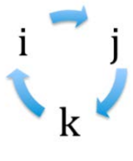
$jiijk = -ji$
But $i^2 = -1$ so
$-jjk = -ji$

But $j^2 = -1$so
$k = -ji$
So $ij = k = -ji$

Hamilton's great insight was to realise that he could sacrifice commutativity (which means that the order of multiplying is unimportant) and still have a consistent and meaningful algebra.

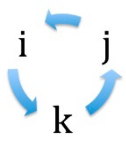
Multiplying Quaternions
To multiply
$(a + bi + cj + dk)(w + xi + yj + zk)$

We use the rules for multiplying quaternions that can be summarised as in the following diagrams:
Multiplying clockwise



Clockwise: $ij = k$, $jk = i$, $ki = j$

Multiplying anticlockwise



Anticlockwise: $ji = -k$, $ik = -j$, $kj = -i$

Quaternions satisfy all the usual arithmetical laws, apart from the commutative law. It was the first non-commutative algebra and was important in the development of abstract algebras. Hamilton was so pleased with his discovery that he wrote a poem about it.

Hamilton Poem

And how the One of Time,
of Space the Three,
Might in the Chain of Symbols girdled be

Although they did not become the powerful and widespread tool that Hamilton hoped they would be, quaternions were however important in the development of vector analysis.

Computer Graphics

Quaternions could be used to achieve the transformation of any directed line in three dimensions to any other directed line which is why they are of use in computer graphics. They overcome various issues which affect other methods of rotating points in three dimensional space.

We have met the reals, the complex numbers and the quaternions. Are there any other similar number systems? It can be proved that there is only one more if we are prepared to abandon another arithmetical law. They are called the *octonians* introduced independently by John Graves, a friend of Hamilton, and Arthur Cayley, an English mathematician.

The Octonians

Each octonion consists of eight terms of the form
$a + bi + cj + dk + el + fm + gn + ho$,

where a, b, c, d, e, f, g, h are real numbers and
$i^2 = j^2 = k^2 = l^2 = m^2 = n^2 = o^2 = -1$.

Once again addition is easy but multiplication is more complicated.

As with the quaternions the multiplication is not commutative (the order matters) but we now also lose the associative law so for example, if A, B and C are octonians:

$$(A \times B) \times C \text{ need not equal } A \times (B \times C)$$

With the octonians we come to the end of the tale. There are no more similar number systems, a result proved at the end of the nineteenth century by the German mathematician Adolf Hurwitz.

Reference

You can find out more about Hamilton at David Wilkins fine website at

http://www.maths.tcd.ie/pub/HistMath/People/Hamilton/

Thank you!