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Pi and e, and the Most Beautiful Theorem in Mathematics

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Introduction

Following John Barrow's lecture on 0 (the nothingness number) and Raymond Flood's lecture on i (the imaginary number), I'm now going to look at two other mathematical constants, π (the circle number) and e (the exponential number), before combining them all into what many have claimed is 'the most beautiful theorem in mathematics'.

One person featuring throughout this story is the Swiss mathematician Leonhard Euler, who spent most of his life in the Imperial courts of St Petersburg and Berlin. The most prolific mathematician of all time, Euler published over 800 books and papers in over 70 volumes. Ranging across almost all branches of mathematics and physics at the time, these amounted to about one-third of all the maths and physics publications of the 18th century.

The 'Circle Number' π

First we meet the number π which, as we learned at school, is a little bit more than 3.14 and a little bit less than $^{22}/_{7}$. It arises in two ways – first as the ratio of the circumference C of a circle to its diameter d – that is,

$$\pi = C/d, \quad \textit{so} \quad C = \pi d, \ \textit{or} \ 2\pi r \quad \textit{where } r \textit{ is the radius.}$$

This ratio is the same for circles of any size – from a pizza to the moon. But it's also the ratio of the area A of a circle to the square of its radius r,

$$\pi = A/r^2, \ \textit{so} \ A = \pi r^2, \ \textit{or} \ \pi d^2/4.$$

This ratio is also the same for all circles, as Euclid proved in the 3rd century BC.

We can never write down π exactly – its decimal expansion goes on for ever. But amazingly, it has been memorised to over 100,000 decimal places, and calculated to over 20 trillion – but even that's only a beginning, and there's still a long way to go!

But why does the same number π appear in both the formula for the circumference and the formula for the area?

One way to answer this is to take two circles of radius r, one shaded and the other unshaded, and to assume that each has circumference $2\pi r$. We next divide each circle into a number of sectors and rearrange these sectors into a shape that looks a bit like a parallelogram – and, as the number of sectors increases without limit, this parallelogram increasingly resembles a rectangle with sides of lengths $2\pi r$ and r, and with area $2\pi r \times r = 2\pi r^2$. So the combined areas of the two original circles must also be $2\pi r^2$ – and each one has area πr^2, as expected.

We can also reverse this argument to show that if the area is πr^2, then the circumference must be $2\pi r$.

Such ideas are quite old: on the right this approach appears in a Japanese treatise from the year 1698.

Early Values

When did people start to measure circles? Several early civilizations obtained estimates for the circumference or area of a circle, and although they had no conception of π as a number, their results yield approximations to its value.

Let us begin with the Mesopotamians, who wrote their mathematical calculations on clay tablets, using a number system based on 60. One of these tablets, dating from around 1800 BC, gives the ratio of the perimeter of a regular hexagon to the circumference of the circle surrounding it as the sexagesimal number 0;57,36. If the radius of the circle is r, then each side of the hexagon also has length r, and so this ratio of $6r\,/\,2\pi r$ (or $3/\pi$) is ${}^{57}/_{60} + {}^{36}/_{3600}$. After some calculation, this gives a value for π of $3^1/_8$, or 3.125 in our decimal notation – a lower estimate that's within one per cent of its true value.

Around the same time, an Egyptian papyrus included the following problem:

Problem 50: Example of a round field of diameter 9 khet. What is its area?

The answer is given in steps.

Take away ${}^1/_9$ *of the diameter*, *which is* 1. *The remainder is* 8.

Multiply 8 *times* 8; *it makes* 64. *Therefore it contains* 64 setat *of land*.

From this calculation it seems that they found their value for the area of a circle of diameter d by reducing d by one-ninth and squaring the result. This method was probably discovered by experience: other explanations have been proposed, but none seems to be supported by historical evidence.

In terms of the radius, this area is ${}^{256}/_{81}\, r^2$, which corresponds to a value for π of about 3.160, an upper estimate that's also within one per cent of the true value.

Using Polygons

An important new method for estimating π was introduced by the Greeks, and would be used for almost 2000 years: it involves approximating a circle with polygons. But although it's often attributed to Archimedes, the method can be traced back a further couple of centuries, to the Greek sophists Antiphon and Bryson. Their aim was to obtain better and better bounds for π by repeatedly doubling the number of sides of a regular polygon within or surrounding the circle until the polygons eventually 'became' the circle.

Antiphon first drew a square inside the circle of radius r and found its area, giving a rather poor lower bound for π of 2. He then doubled the number of sides, giving an octagon and finding the better bound of $2\sqrt{2}$, or 2.828. Bryson's approach was the same, except that he also considered polygons surrounding the circle: this yields upper bounds of 4 for the square and about 3.32 for the octagon.

Archimedes became interested in circular measurement around 250 BC, proving that *a circle of radius r has area* πr^2, and that *a sphere has surface area* $4\pi r^2$ *and volume* ${}^4/_3\pi r^3$.

Unlike Antiphon and Bryson, who'd used areas, Archimedes worked with perimeters. He first approximated the circumference of a circle by the perimeters of regular hexagons drawn inside and outside the circle, and carried out the appropriate calculations to give lower and upper bounds for π of 3 and $2\sqrt{3}$ – so in our decimal notation, *π lies between* 3 *and* 3.464.

He then doubled the number of sides of the polygons, replacing the hexagons by dodecagons and getting the better estimates of 3.105 and 3.215. Three more doublings to polygons with 24, 48, and 96 sides then gave ever closer values, with his bounds for polygons with 96 sides presented as $3\,{}^{10}/_{71}$ and $3\,{}^{1}/_{7}$ – or, in decimal notation, between 3.14084 and 3.14286, correct to two decimal places. As Archimedes expressed it: *The circumference of any circle exceeds three times its diameter by a part which is less than ${}^{1}/_{7}$ but more than ${}^{10}/_{71}$ of the diameter.*

What was happening elsewhere? In China, an early value for π was given by Zhang Heng, inventor of the seismograph for measuring earthquakes. This was $\sqrt{10}$, which is about 3.162 – a useful approximation for the time.

Around the year 263, in his commentary on the Chinese classic *Nine Chapters on the Mathematical Art*, Liu Hui used inscribed regular polygons to approximate π. Starting with hexagons and dodecagons, he developed simple methods for relating the successive areas and perimeters when one doubles the number of sides, and for polygons with 192 sides he obtained lower and upper bounds of about 3.141 and 3.143. Four more doublings led to polygons with 3072 sides and to his approximation of 3.14159.

Even more impressively, around the year 500 Zu Chongzhi and his son doubled the number of sides three more times, extending their calculations to polygons with over 24,000 sides and obtaining estimates that give π to six decimal places. They also replaced Archimedes' fractional approximation of ${}^{22}/_{7}$ by the more accurate ${}^{355}/_{113}$, which also gives π to six decimal places. As we'll see shortly, this latter approximation wasn't rediscovered in Europe for another thousand years.

After this, everyone got in on the game.

In Italy, in a geometry book of 1220, Leonardo of Pisa (known to us as Fibonacci) cited earlier calculations and used polygons with 96 sides to give $\pi = 3.141818$.

Then, in 1424, the Persian astronomer al-Kashi, who was working in Ulugh Beg's observatory in Samarkand, used polygons with over 800 million sides to find π to a remarkable 9 sexagesimal (or 16 decimal) places. This remained the best value for almost 200 years.

Meanwhile, European mathematicians from several countries were using similar methods. In 1579 the French lawyer and mathematician François Viète used polygons with over 393,000 sides to find π to 9 decimal places, while six years earlier, the German mathematician Valentin Otho had proposed the fraction ${}^{355}/_{113}$: as we saw earlier, this value was already known to Zu Chongzhi 1000 years previously, and gives π to six decimal places. In 1585 the Dutch cartographer Adriaan Anthonisz obtained the same value accidentally: having found the lower and upper bounds of ${}^{333}/_{106}$ and ${}^{377}/_{120}$ he then averaged their numerators and denominators to give the result.

Also in the Netherlands, Adriaan van Roomen used polygons with 2^{30} sides (that's over a billion) sides to find π to 15 decimal places. But best of all was Ludolph van Ceulen who used polygons with over 500 billion sides to find π to 20 decimal places. Not content with this, he then used polygons with 2^{62} sides to find π to 35 decimal places. He asked for this latter value to appear on his tombstone in Leiden, and for many years π was known in Germany as the *Ludolphian number.*

Infinite Products

Up to this time, most estimates for π had been bounds on its value. New approaches were taken by François Viète and John Wallis, who obtained exact expressions involving products of infinitely many terms.

In 1579 Viète showed that we can find $2/\pi$ by multiplying the cosines of $\pi/4$, $\pi/8$, $\pi/16$, and so on for ever: here, the angles are given in radian measure, where π corresponds to 180 degrees, so that the first term is the cosine of 45 degrees. Noting that this is $^1/_2\sqrt{2}$, he was then able to rewrite his result in terms of expressions involving successive square roots:

$$2/\pi = \cos \pi/4 \times \cos \pi/8 \times \cos \pi/16 \times \cos \pi/32 \times \ldots$$

$$= \tfrac{1}{2}\sqrt{2} \times \tfrac{1}{2}\sqrt{(2+\sqrt{2})} \times \tfrac{1}{2}\sqrt{(2+\sqrt{(2+\sqrt{2})})} \times \tfrac{1}{2}\sqrt{(2+\sqrt{(2+\sqrt{(2+\sqrt{2})})})} \times \ldots$$

Later, in 1656, another infinite product was presented by John Wallis, the Savilian Professor of Geometry at the University of Oxford. It expresses the number $4/\pi$ as $^3/_2 \times {}^3/_4 \times {}^5/_4 \times {}^5/_6$, and so on – the pattern should be clear.

$$\frac{4}{\pi} = \frac{3 \times 3 \times 5 \times 5 \times 7 \times 7 \times 9 \times 9 \times \ldots}{2 \times 4 \times 4 \times 6 \times 6 \times 8 \times 8 \times 10 \times \ldots}$$

Unfortunately, although such products have theoretical significance, they converge very slowly to π and have no practical value.

Using the $\tan^{-1}$ (or Arctan) Function

A new and highly productive method for estimating π, which came to be used extensively throughout the 18th and 19th centuries, involved the *inverse tangent function*, usually written as $\tan^{-1} x$ or $\arctan x$: if $y = \tan x$, then $x = \tan^{-1} y$;

for example, $\tan \pi/4 = 1$, so $\tan^{-1} 1 = \pi/4$, and $\tan \pi/6 = 1/\sqrt{3}$, so $\tan^{-1} 1/\sqrt{3} = \pi/6$.

We can combine different values of $\tan^{-1}$ – for example, if we add $\tan^{-1}\, {}^1/_2$ and $\tan^{-1}\, {}^1/_3$ we get $\pi/4$: this can then be proved by simple geometry.

Many mathematical functions can be written as infinite series. For example, we can write $\tan^{-1} x$ as an infinite series with only odd powers of x, and with odd numbers as denominators:

$$\tan^{-1} x = x - {}^1/_3\, x^3 + {}^1/_5\, x^5 - {}^1/_7\, x^7 + \ldots;$$

this result was already known to Madhava in 15th-century India, but is usually named after the Scotsman James Gregory, who rediscovered it 300 years later.

If we now let $x = 1$, we get a series expression for $\pi/4$, a result also due to Madhava, but usually credited to Leibniz:

$$\pi/4 = 1 - {}^1/_3 + {}^1/_5 - {}^1/_7 + \ldots$$

This is one of the most remarkable results in the whole of mathematics: by simply adding and subtracting reciprocals of whole numbers we get a result involving the circle number π.

Unfortunately, the Leibniz series converges exceedingly slowly, and we cannot use it to find π in practice; for example, the first 300 terms of the series give π to only two decimal places, while the first half-a-million terms give us only five correct.

But we *can* still use Gregory's series to estimate π if we substitute values other than 1. Remembering that $\tan^{-1}\, {}^1/_2$ and $\tan^{-1}\, {}^1/_3$ add up to $\pi/4$, we can substitute $x = {}^1/_2$ and $x = {}^1/_3$ into the series for $\tan^{-1} x$, giving the two series shown here.

$$\pi/4 = \tan^{-1}({}^1/_2) + \tan^{-1}({}^1/_3)$$
$$= \{{}^1/_2 - {}^1/_3({}^1/_2)^3 + {}^1/_5({}^1/_2)^5 - {}^1/_7({}^1/_2)^7 + \ldots\} - \{{}^1/_3 - {}^1/_3({}^1/_3)^3 + {}^1/_5({}^1/_3)^5 - {}^1/_7({}^1/_3)^7 + \ldots\},$$

And because of the increasing powers of 2 and 3 in the denominators, these series converge much faster, yielding good estimates for π. Indeed, in 1861 a certain W. Lehmann of Potsdam used these same series to find π to 261 decimal places.

The search was now on to find new $\tan^{-1}$ identities where the series converge even faster. In 1706 John Machin used the addition formula several times over to prove that

$$\pi = 16 \tan^{-1} ({}^{1}/_{5}) \;-\; 4 \tan^{-1} ({}^{1}/_{239}) \,,$$

and then wrote out these two $\tan^{-1} x$ series:

$$\begin{aligned} \pi \;=\; & 16\ \{{}^{1}/_{5} - {}^{1}/_{3}\,({}^{1}/_{5})^{3} + {}^{1}/_{5}\,({}^{1}/_{5})^{5} - {}^{1}/_{7}\,({}^{1}/_{7})^{7} + \ldots\} \\ & - 4\ \{({}^{1}/_{239}) - {}^{1}/_{3}\,({}^{1}/_{239})^{3} + ({}^{1}/_{239})^{5} - {}^{1}/_{7}\,({}^{1}/_{239})^{7} + \ldots\} \end{aligned}$$

Both of these series converge rapidly because of the powers of 5 and 239 in the denominators – for example, we get the value 3.14 from just first three terms of each series. Also, 5 is an easy number to divide by, and Machin was thereby able to calculate π by hand to 100 decimal places, a great improvement on anything that had gone before.

Incidentally, John Machin later became Gresham Professor of Astronomy for almost 40 years.

1706 was a good year for π. As well as Machin's result, a Welsh maths teacher called William Jones wrote *A New Introduction to the Mathematicks*, in which he introduced the symbol π for measuring circles. In one extract is Machin's series, and just below it is the first ever appearance of the symbol π. And another extract from the same book includes Machin's value in full – *True to above a* 100 *places; as Computed by the Accurate and Ready Pen of the Truly Ingenious Mr. John Machin.*

But it was Euler who popularised the use of the letter π – first in a work of 1737, and then in many later writings – so that it soon came to be used universally.

One of Euler's many results involving π was the following $\tan^{-1}$ identity, which enabled him to calculate 20 decimal places of π in one hour:

$$\pi \;=\; 20 \tan^{-1} {}^{1}/_{7} \;+\; 8 \tan^{-1} {}^{3}/_{79} \,.$$

It was used again in 1794 by the Slovenian Jurij Vega to calculate π to 136 decimal places, and for many years this was the most accurate value known. But there were persistent references in the literature to an earlier and more accurate value, seen by the Hungarian Baron von Zach while visiting Oxford's Bodleian Library in the 1780s. This reference was eventually located in 2014 by BSHM member Benjamin Wardhaugh, and confirmed that in 1721 a resident of Philadelphia used 314 terms of the series for $\tan^{-1} 1/\sqrt{3}$, calculated with great accuracy, to obtain π correctly to 152 decimal places:

$$\pi = 6 \tan^{-1} ({}^{1}/_{\sqrt{3}}) \;=\; \sqrt{12}\ \{1 - {}^{1}/_{3}\,({}^{1}/_{3}) + {}^{1}/_{5}\,({}^{1}/_{3})^{2} - {}^{1}/_{7}\,({}^{1}/_{3})^{3} + \ldots\}.$$

This was indeed the world's most accurate value of π for over 100 years, even though it was largely unknown at the time.

But most notorious of all was the value obtained by William Shanks, who in 1873 used Machin's formula to calculate π to an impressive 707 decimal places. These were later inscribed in a ceiling frieze in the π-room of the Palace of Discovery in Paris, where they can still be seen. Unfortunately for him, and for the Palais, it was later found that only the first 527 of these decimal places are correct.

Some Weird Results

The 20th century saw a number of discoveries about π – many of them completely bizarre.

In 1914 the Indian mathematician Ramanujan found several remarkable exact formulas for $1/\pi$, including an infinite series in which strange numbers, such as 1103 and 26,390, seem to appear from nowhere. Many years later, in 1989, David and Gregory Chudnovsky of New York produced a similar, but even more complicated, result with even larger numbers. Such series converge extremely rapidly and form the basis of some of today's fastest algorithms for calculating π.

A very different type of result was discovered in 1995 by David Bailey, Peter Borwein and Simon Plouffe, and caused a great deal of surprise. It's a much simpler series, and its importance is that, if we work in a base-16 number system rather than in base 10, we can calculate each digit of π one at a time without having to re-calculate all the preceding digits first.

Enter the Computer

By this time, computers had entered the scene, and it was now possible to calculate π to a much greater accuracy.

The first advance was in 1949 when Machin's series were put to good use on the American ENIAC machine to calculate π to 2037 places in 70 hours. Machin's result was again used in 1955 to find π to 3089 decimal places in just 13 minutes, on the Naval Ordnance Research Calculator.

Meanwhile, progress was being made in England: in 1957 a different $\tan^{-1}$ series was used on the Ferranti PEGASUS computer to calculate over 10,000 decimal places in 33 hours, though not all were correct. Then IBM entered the scene, and the number of decimal places rose rapidly while the calculation time plummeted.

1973 saw one million decimal places reached in 23.3 hours on a CDC 7600 machine. The scene then moved to Japan, where the number places increased to 10 million in 1983, 100 million in 1987, and over 500 million in 1989. Using very sophisticated $\tan^{-1}$ formulas, and carrying out their calculations in base 16, the Japanese were able to calculate the individual digits of π one at a time (as we saw earlier), before translating their results back into base 10.

Meanwhile, in New York, the Chudnovsky brothers were developing algorithms for their home-built supercomputers to push the numbers even higher, and in 1989 they were the first to exceed one billion places. There was then a frantic race with the Japanese group, with a trillion places being achieved in 2002 and 10 trillion in 2011. Since then, the number of calculated places has increased to over 20 trillion.

Circling the Earth

We'll end our discussion of π with a simple puzzle that appeared in 1702, in a book on Euclid's *Elements* by the Cambridge mathematician William Whiston. If you haven't seen it before, you may find its answer surprising.

The circumference of the Earth is about 25,000 miles. Assuming the Earth to be a perfect sphere, suppose we tie a piece of string of this great length tightly around it. We then extend this string by just 2π (that's just over 6.3) feet, and prop it up equally all around. How high above the ground is the string? Most people think that the resulting gap must be extremely small – perhaps a tiny fraction of an inch – but the correct answer is *one foot*!

In fact, we get the same answer whether we tie the string around the Earth, a tennis ball, or any other sphere. For, if the sphere has radius r feet, then the original string has length $2\pi r$. When we extend it by 2π feet, the new circumference is $2\pi r + 2\pi$, which is $2\pi \times (r + 1)$. So the new radius is $r + 1$: one foot more than before.

The 'Exponential Number' e

How fast do things grow? We often use the phrase 'exponential growth' to indicate something that grows very fast, but how quickly is this?

This part of the talk concerns the number $e = 2.71828\ldots$ – like π, its decimal expansion goes on for ever. The letter e was first used for this number around 1727 in an unpublished paper of Euler, and its first appearance in print was in 1736, in his *Mechanica* on the mathematics of motion.

Exponential Growth

To illustrate what we mean by 'exponential growth', let's start with a story about the invention of the game of chess.

The wealthy king of a certain country was so impressed by this new game that he offered the wise man who invented it any reward he wished – to which the wise man replied:

My prize is for you to give me 1 grain of wheat for the first square of the chessboard, 2 grains for the second square, 4 grains for the third square, and so on, doubling the number of grains on each successive square until the chessboard is filled.

The king was amazed to be asked for such a tiny reward (or so he believed), until his treasurers calculated the total number of grains of wheat. This is $1 + 2 + 2^2 + 2^3 + \ldots + 2^{63}$, which works out at $2^{64} - 1$ grains, enough wheat to form a pile the size of Mount Everest. Placed end to end they'd reach to the nearest star, *Alpha Centauri,* and back again!

Let's see how quickly various other sequences can grow.

A simple form of growth is *linear growth*, illustrated by the counting numbers $n = 1, 2, 3, 4, 5, \ldots$ Somewhat faster is *quadratic growth*, involving the perfect squares $n^2 = 1^2, 2^2, 3^2, 4^2, 5^2, \ldots$, and even more rapid is *cubic growth*, involving the cubes $n^3 = 1^3, 2^3, 3^3, 4^3, 5^3, \ldots$. These are all examples of *polynomial growth*, since they involve powers of n.

Alternatively, we could look at powers of 2, or of any other number. As we saw in the chessboard story, the sequence 2^n of powers of 2 starts off fairly slowly – 1, 2, 4, 8, 16, 32 – but soon gathers pace because each successive term is twice the previous one. The sequence 3^n of powers of 3 takes off even more quickly: 1, 3, 9, 27, 81, 243. These are examples of *exponential growth*, where n appears as the exponent.

To compare these types of growth, we can calculate the running times of some polynomials and exponentials when n is 10, 30, and 50, on a computer performing a million operations per second. For polynomial growth, such as n^5, such a computer takes about 5 minutes when $n = 50$. But exponential growth, such as 2^n, is much faster, as we've seen: when $n = 50$, the computer would take over 35 years, and would be vastly greater than this for 3^n.

So, in the long run, exponential growth tends to exceed polynomial growth, often by a huge margin. Algorithms that run in polynomial time are generally thought to be 'efficient', whereas those that run in exponential time normally take much longer to implement as the input size increases, and are considered as 'inefficient'.

An Interest-ing Problem

Returning to e, what exactly is this number, and how did it arise?

In 1683 the Swiss mathematician Jakob Bernoulli was calculating compound interest. Given a sum of money to invest at a given rate of interest, how does it grow? The answer depends on how often we calculate the interest.

How much is earned if we calculate it yearly? Twice a year? Every month? Every day? n times a year? continuously?

To answer this, suppose that we invest £1 at the unlikely annual rate of 100 per cent. After one year the amount has increased to £2. But what happens if we calculate the interest twice a year? After six months the amount has increased by 50 per cent to £1.50, and after the second six months this has increased by a further 50 per cent to one-and-a half times £1.50, which is £2.25.

Similarly, if the amount is calculated four times a year, then every three months the amount increases by 25 per cent, so that by the end of the year it has become £1 multiplied by 1.25 four times – that is £$(1 + {}^{1}/_{4})^{4}$, which is about £2.44.

And if the interest is calculated n times a year, then after each period of time the amount is multiplied by $1 + {}^{1}/_{n}$, so that at the end of the year it has become $(1 + {}^{1}/_{n})^{n}$.

One can draw up a table to show how these amounts increase as we calculate the interest with increasing frequency, and we see that, as n increases indefinitely, these numbers tend to a limiting value that corresponds to when the interest is calculated continuously. This limiting value of about 2.81828 is the exponential number that Euler called e.

Leonhard Euler (1707-83)

The greatest advances in understanding exponentials were made in the early 18th century. After Bernoulli, the main figure in this story was Euler, who investigated the properties of e and of the exponential function e^x. In 1748 one of the most important mathematics books ever written, his *Introduction to the Analysis of Infinites*, brought together many of his results from earlier works.

Some Properties of *e*

Here are some of his main findings.

We've just seen that e is the limit of the numbers $(1 + 1/n)^n$ as n increases indefinitely, and similarly we can show that e^x is the limit of $(1 + x/n)^n$ for any number x.

But, as Isaac Newton had already discovered, the number e is also the sum of the infinite series shown here, where the denominators are the factorials: 1, 1×2, $1 \times 2 \times 3$, and so on. More generally, there's a similar series for e^x which converges for all values of x. These series converge very quickly because the factorials increase so rapidly; for example, the first ten terms of the series for e already give e to five decimal places.

We can also consider the graph of $y = e^x$. One of its most important features is that, at each point x, the slope of the graph is also e^x – that is, the slope at any point is the y-value – so the curve becomes ever steeper as x increases.

John Napier's Logarithms (1614)

The number e is also intimately linked with logarithms, so let's look briefly at these.

Since the Middle Ages, ways had been sought for turning lengthy calculations involving multiplications and divisions into simpler ones involving additions or subtractions, and in the 16th century, the German Michael Stifel and others developed a new method for doing so. This was to turn *geometric* progressions whose successive terms have a common ratio into *arithmetic* ones whose successive terms have a common *difference.* This process was called *prosthaphairesis*, from the Greek words for 'addition' and 'subtraction'.

In 1614 the Scotsman John Napier, Eighth Laird of Merchiston (near Edinburgh), produced his *Description of the Wonderful Canon of Logarithms*, shown here. This important work contained extensive tables of the logarithms of sines and tangents of all the angles from 0 to 90 degrees in steps of 1 minute of arc; Napier's emphasis on these functions arose from his interest in spherical geometry, so that his 'excellent briefe rules' (as he called them) could be used by navigators and astronomers.

Napier's logarithms originated from this idea of prosthaphairesis. He considered two points moving along straight lines – an upper one (PQ) of finite length and a lower one (L_0L) of infinite length – as follows:
the upper point moves from P towards Q in such a way that its speed at each point is proportional to the distance that it still has to travel;
the lower point, representing its 'Naperian logarithm', starts from L_0 and travels at constant speed towards L for ever.

So, in successive periods of time, the distances still to be travelled by the first point form a geometric progression, and the distances already travelled by the second point form an arithmetic progression.

Napier took 10^{-7} as his successive time intervals and then multiplied his results by 10^7 in order to avoid the use of decimal fractions which were still largely unfamiliar at the time. It followed from his construction that the log of 10 million is 0, and that as n decreases its logarithm increases (unlike those we use today). It also followed that

$$\log (a \times b) = \log a + \log b - \log 1,$$

so that for each calculation he had to subtract the cumbersome term log 1 = 161,180,956.

Henry Briggs's Logarithms (1617)

In 1615 Henry Briggs, the first professor of geometry at Gresham College in London, heard about Napier's logs and was wildly excited by them. He included them in his Gresham lectures, enthusing that Napier had
set my Head and hands a Work with his new and remarkable logarithms . . . I never saw a Book which pleased me better or made me more wonder.

But Napier's logs were cumbersome to use, and Briggs wanted to redefine them so as to avoid having to subtract log 1 in every calculation:

I myself, when expounding this doctrine to my auditors in Gresham College, remarked that it would be much more convenient that 0 *should be kept for the logarithm of the whole sine* [namely, 1].

Briggs twice visited Edinburgh to stay with Napier, and it's recorded that when they first met they spent the first quarter-hour looking at each other in admiration without speaking a word. The outcome of their meetings was that Briggs started to construct 'logarithms to base 10', where log 1 = 0, log 10 = 1, log 100 = 2, and so on.

Other values he found by interpolation. In order to find these accurately, he calculated the square root of 10, then the square root of that, and so on, eventually taking square roots fifty-four times, all to thirty decimal places! Since log 1 = 0, as he'd demanded, Briggs's logarithms satisfied the simpler fundamental rule: $\log (a \times b) = \log a + \log b$.

In 1617 Briggs produced a small pamphlet containing his calculations. Seven years later, after he'd left London to become the first Savilian Professor of Geometry at Oxford University, he followed this with his *Arithmetica Logarithmica*, an extensive collection of logs to base 10 of the integers from 1 to 20,000 and from 90,000 to 100,000, all calculated by hand to fourteen decimal places. The gap in these tables, between 20,000 and 90,000, was later filled by a Dutch mathematician, Adriaan Vlacq.

Exp and log are Inverse to Each Other

The fundamental connection between the functions e^x and log x (where e is the base of the logarithms) is that they're 'inverses' of each other: in symbols, $\log e^x = x$ and $e^{\log y} = y$. It follows that if we take x, calculate e^x, and take the log (to base e) of the result, we get back to x – and if we take x, calculate log (to base e) of x, and take the exponential of the result, we get back to x.

This inverse relationship had been noticed by John Wallis back in 1685, and was developed by Euler in his *Introductio* with more available machinery. Using it, we can show that the multiplicative law for exponentials and the basic law of logarithms are essentially the same result.

Notice also that, since $y = e^x$ and $y = \log x$ are inverses of each other, their graphs can be obtained from each other by reflection in the line $y = x$.

Derangements

Let's now change pace by looking at two applications of the exponential function – to derangements and to population growth.

Around 1710, De Moivre, de Montmort and others posed the following problem:
Given any n letters, in how many ways can we rearrange them so that no letter is in its original position?
A more popular form of this problem is:
If we randomly place a number of messages into addressed envelopes, what is the probability that no message ends up in its correct envelope?

Such rearrangements are now known as *derangements*. For example, when n is 4 there are 4! (or 24) permutations of the four letters a, b, c, d, but only the nine are derangements, with no letter in its usual position:

badc, bcda, bdac, cadb, cdab, cdba, dabc, dcab, dcba

To investigate this question, we'll let D_n be the number of derangements of n letters (so that $D_4 = 9$). The following table gives the values of D_n, for all n up to 8.

n		1	2	3	4	5	6	7	8
D_n	0	1	2	9	44	265	1854	14,833	

Around 1779 Euler became interested in the derangement problem and used a counting argument to show that D_n has the following value, a result that De Moivre had obtained some years earlier.

$$D_n = n!\ \{1 - 1/1! + 1/2! - 1/3! + \ldots \pm 1/n!\}.$$

Unfortunately, this value can be time-consuming to evaluate for all but very small values of n – but there's a quicker way. Because the expression in brackets is just the beginning of the series for e^{-1}, D_n is very close to $n!/e$. In fact, for every n, D_n is the integer closest to $n!/e$ – for example, when n is 8, $n!/e$ is about 14,832.9… while the value of D_8 is 14,833.

Exponential Growth

We'll end our present discussion of exponentials by returning to exponential growth. In 1798 Thomas Malthus wrote his *Essay on Population*, where he contrasted the steady linear growth of food supplies with the exponential growth in population. He concluded that, however one may cope in the short term, the exponential growth would win in the long term, and that there'd be severe food shortages – a conclusion that was borne out in practice.

How fast does a population grow? If $N(t)$ is the size of a population at time t, and if the population grows at a fixed rate k proportional to its size, then we have the differential equation $dN/dt = kN$. This can be rewritten as $dN/N = k\,dt$, which can be integrated to give $\log N = kt + \text{constant}$, or (in terms of exponentials) N is a multiple of e^{kt}, where the multiple turns out to be the initial population N_0. So $N(t) = N_0e^{kt}$ – an example of exponential growth. In the same way we can model exponential decay as, for example, in the decay of radium, or in the cooling of a cup of tea.

Euler's Equation

We come at last to the equation which regularly tops the polls among mathematicians as 'the most beautiful theorem in mathematics' – namely, $e^{i\pi} + 1 = 0$ (or $e^{i\pi} = -1$). It's remarkable for combining five separate and important constants – each with deep mathematical significance, and each with its own story. These are:

1 – the basis of our counting system
0 – the number expressing 'nothingness'
π – the basis of circle measurement
e – the number linked to exponential growth
i – the 'imaginary' square root of −1.

It also involves the fundamental mathematical operations of addition, multiplication and taking powers, and the notion of equality. As one participant in a poll in *Physics World* was moved to remark:
What could be more mystical than an imaginary number interacting with real numbers to produce nothing?

Indeed, at the age of only 14, the future Nobel prize-winning physicist Richard Feynman called it 'the most remarkable formula in math', while the Fields-medal winner Sir Michael Atiyah has described it as 'the mathematical equivalent of Hamlet's '*To be or not to be*': very succinct, but at the same time very deep'. It has even featured twice in *The Simpsons*, and was crucial in a criminal court case – but those are for another day.

A Near-miss: Johann Bernoulli

Although we've called this result 'Euler's equation', it was nearly discovered a few years earlier by Johann Bernoulli.

As we've seen, the logarithm function $\log x$ is defined for all positive values of x – but can it be defined when x is negative? This question caused disagreement between Leibniz who believed the logarithm of a negative number to be 'impossible', and Bernoulli who used the basic property of logarithms to prove that log (–1) is 0, as shown here:

$$2 \times \log(-1) = \log(-1) + \log(-1) = \log(-1 \times -1) = \log(1) = 0.$$

So $\log(-1) = 0$ – with a similar proof that $\log(-x) = \log x$ for all x.

In 1702 Bernoulli was investigating the area A of a sector of a circle of radius a – the shaded area bounded by the circle, the x-axis, and the line from the origin to the point (x, y) – and found it to be

$$(a^2/4i) \times \log\{(x + iy) / (x - iy)\}.$$

Leaving aside what is meant by the logarithm of a complex number, Euler later observed that when $x = 0$ this formula simplifies to $(a^2/4i) \times \log(-1)$. Because such a sector clearly has a non-zero area, he deduced that the logarithm of –1 cannot be 0, contradicting Bernoulli's result above. Moreover, since this sector is a quarter-circle with area $\pi a^2/4$, this area must be equal to $(a^2/4i) \times \log(-1)$, and so $\log(-1) = i\pi$.

Although Euler wrote down this last result explicitly, he doesn't seem to have taken exponentials to deduce Euler's equation in the form $e^{i\pi} = -1$. Indeed, Euler often credited Bernoulli with discovering this value for log (−1), but Bernoulli didn't include it in his 1702 paper or in any later work, continuing to insist that log (–1) = 0.

Euler's Identity

As we'll see, Euler's equation is a special case of a general result that Euler published in 1748 in his *Introductio*. This celebrated result relates the exponential function e^x and the trigonometric functions cos x and sin x. But why should the exponential function which goes 'shooting off to infinity' as x becomes large, have anything to do with these trigonometric functions which forever oscillate between the values 1 and −1? Indeed, there's no *real* reason why there should be any such relationship, but there are *complex* reasons! Introducing complex numbers leads to such connections, and realising this was one of Euler's greatest achievements.

To see the connection, recall that these functions can all be expanded as series, valid for all values of x.

$$e^x = 1 + x/1! + x^2/2! + x^3/3! + x^4/4! + x^5/5! + x^6/6! + x^7/7! + \dots;$$
$$\cos x = 1 - x^2/2! + x^4/4! - x^6/6! + \dots; \qquad \sin x = x - x^3/3! + x^5/5! - x^7/7! + \dots .$$

What happens if we now allow ourselves to introduce the complex number i, the square root of −1, as Euler did in 1737? Let's take the series for e^x, and replace x by ix. We get:

$$e^{ix} = 1 + ix/1! + (ix)^2/2! + (ix)^3/3! + (ix)^4/4! + (ix)^5/5! \text{ and so on.}$$

But since i^2 is −1, it follows that $i^3 = -i$, $i^4 = 1$, etc., and we can collect terms to give

$$(1 - x^2/2! + x^4/4! - \dots) + i\,(x - x^3/3! + x^5/5! - \dots)$$

– the series for cos x plus i times the series for sin x: that is:

$$e^{ix} = \cos x + i \sin x.$$

This is *Euler's identity*, one of the most remarkable equations in the whole of mathematics, beautifully connecting these seemingly unrelated functions. Euler gave more than one proof of his identity. As Euler himself commented:

> *From these equations we can understand how complex exponentials can be expressed by real sines and cosines.*

Another near-miss: Roger Cotes

At this stage let's see another near-miss, by the English mathematician Roger Cotes, the first Plumian Professor of Astronomy in the University of Cambridge. Born in 1682 and dying at the age of 33, he introduced radian measure for angles, and worked closely with Isaac Newton on the second edition of the *Principia Mathematica*.

Around 1712 Cotes was investigating the surface area of an ellipsoid. The details are somewhat complicated, but he managed to find two different expressions for the area involving logarithms and trigonometry – and both involving an angle φ. He first proved that the surface area is a certain multiple of log (cos φ + i sin φ), and then proved it to be the same multiple of $i\varphi$. Equating these he deduced the identity

$$\log(\cos\varphi + i\sin\varphi) = i\varphi,$$

which gives a connection between logs and trig functions. If he'd then taken exponentials, he'd have discovered Euler's identity in the form $e^{i\varphi} = \cos\varphi + i\sin\varphi$ – but he didn't. Another near miss!

Consequences of Euler's Identity

Euler's identity has many simple, yet profound, consequences. Here are three of them.

The most important one follows when we put $x = \pi$ (the radian form of 180°) to give

$$e^{i\pi} = \cos \pi + i \sin \pi = -1 + 0i = -1,$$

so $e^{i\pi} + 1 = 0$ (Euler's equation). Although Euler must surely have made this deduction, it doesn't appear explicitly in any of his writings.

Next, we note that Euler's identity gives us a one-line proof of De Moivre's theorem: for any number n,
$(\cos x + i \sin x)^n = (e^{ix})^n = e^{i(nx)} = \cos nx + i \sin nx$.

Since Euler had used De Moivre's theorem to obtain his identity, the two results are, in some sense, equivalent.

Note also that if in Euler's identity $e^{ix} = \cos x + i \sin x$ we replace x by $-x$, we get

$e^{-ix} = \cos(-x) + i \sin(-x)$, which is $\cos x - i \sin x$. Adding and subtracting these two equations now gives

$$\cos x = (e^{ix} + e^{-ix})/2 \text{ and } \sin x = (e^{ix} - e^{-ix})/2i.$$

These remarkable results show how, by allowing complex numbers, we can rewrite the standard trig functions in terms of the exponential function.

Who Discovered 'Euler's Equation'?

To end with, what should we call the equation $e^{i\pi} + 1 = 0$?

We've seen how it can easily be deduced from results of Johann Bernoulli and Roger Cotes, but that neither of them seems to have done so. Even Euler seems not to have written it down explicitly – and certainly it doesn't appear in any of his publications – though he surely realised that it follows immediately from his identity, $e^{ix} = \cos x + i \sin x$.

In fact, we don't know who first stated the equation explicitly, though it certainly appears in a French maths journal of 1813–14.

But almost everybody nowadays attributes the result to Leonhard Euler, so we're surely justified in naming it 'Euler's equation', to honour the achievements of this truly great mathematical *pioneer*, a word that describes him so well, and which appropriately includes among its letters our five constants *pi*, *i*, *o*, *one*, and *e*.

Euler's Pioneering Equation

May I conclude by inviting you to the Gresham College launch of this forthcoming book, to be published by Oxford University Press on 25 January 2018. It'll take place at Barnard's Inn Hall in High Holborn on 15 February at 6 pm.