Euler's pioneering equation

'The most beautiful theorem in mathematics'

Robin Wilson

‘The most beautiful theorem in mathematics’

(*Mathematical Intelligencer*)

‘The greatest equation ever’

(almost – runner up to Maxwell’s equations)

(*Physics World*)

Richard Feynman (aged 14): ‘the most remarkable formula in math’

Sir Michael Atiyah: ‘the mathematical equivalent of Hamlet’s *To be or not to be*: very succinct, but at the same time very deep’

Keith Devlin: ‘Like a Shakespearian sonnet that captures the very essence of love, or a painting that brings out the beauty of the human form that is far more than just skin deep, Euler’s equation reaches down into the very depths of existence’

Featured in *The Simpsons* (twice) and in a criminal court case.

Euler's equation

– 'the most beautiful theorem in mathematics'

Five important constants:

1 – the counting number

0 – the nothingness number

π – the circle number

e – the exponential number

i – the imaginary number

$$\boxed{e^{i\pi} + 1 = 0 \quad (\text{or } e^{i\pi} = -1)}$$

Euler's equation

– 'the most beautiful theorem in mathematics'

Five important constants:

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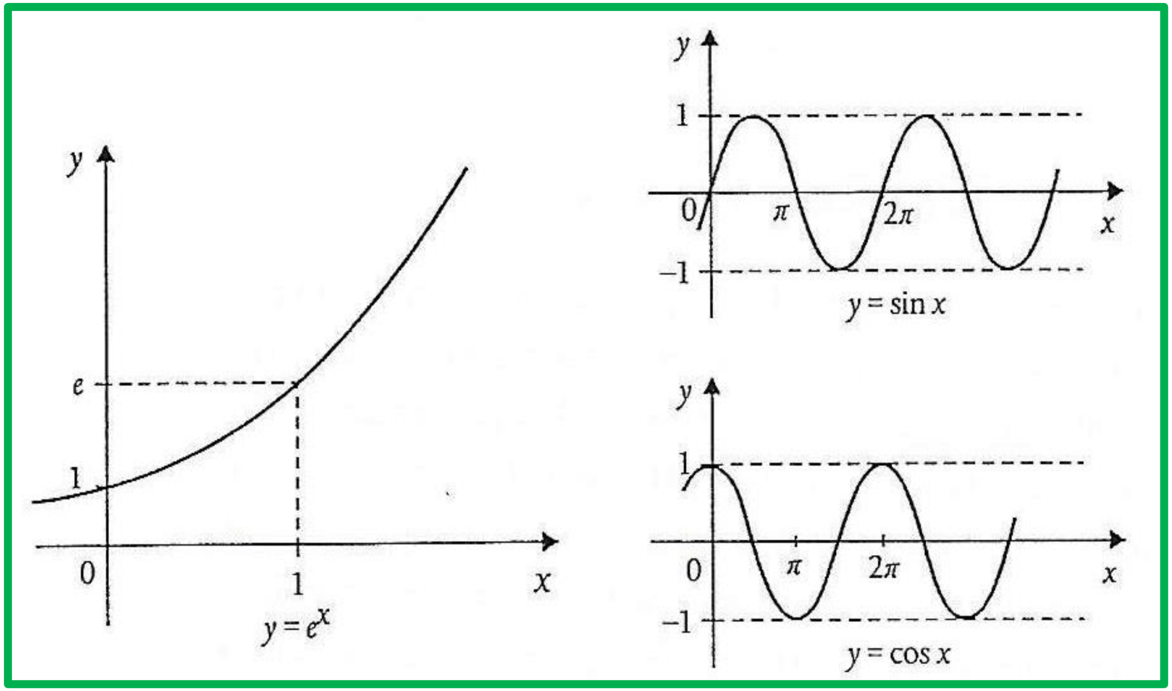
$$e^{i\pi} + 1 = 0$$

$$(\text{or } e^{i\pi} = -1)$$

Leonhard Euler had a farm,
e, i, e, i, 0,
And on that farm he had 1 π-g,
e, i, e, i, 0.

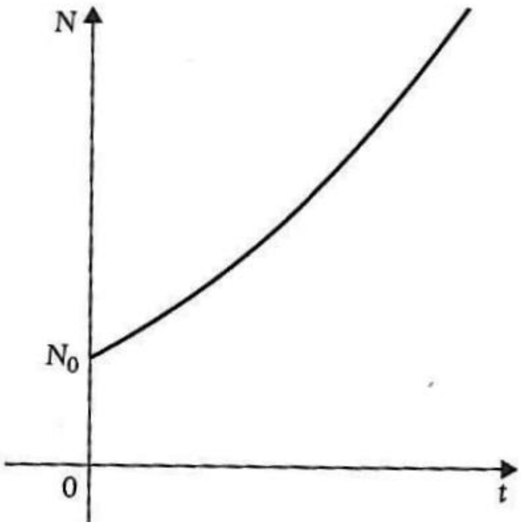
Euler's identity

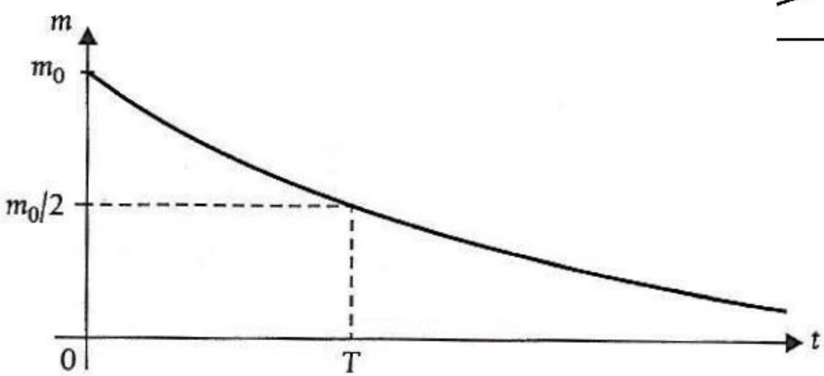
There's no real reason why there should be any connection between e^x and $\cos x$ and $\sin x$.

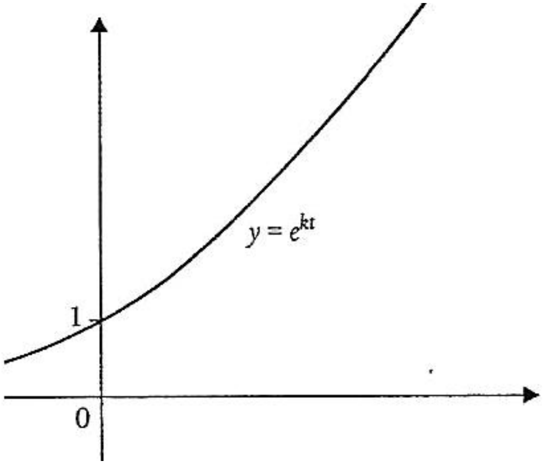


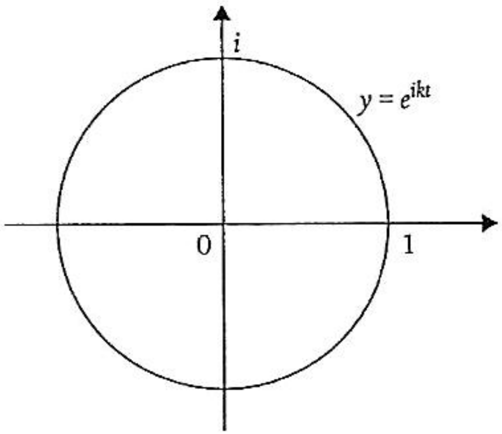
$$e^{ix} = \cos x + i \sin x$$

Some applications









$y = e^{kt}$ and $y = e^{ikt}$

Exponential growth / decay:
population growth
and the decay of radium

electric current,
quantum mechanics,
radio waves
signal analysis
image processing

1 : the counting number

Our decimal place-value system uses only
1, 2, 3, 4, 5, 6, 7, 8, 9, and 0.

It's a place-value system: 5157 means

$$(5 \times 1000) + (1 \times 100) + (5 \times 10) + 7$$

or $(5 \times 10^3) + (1 \times 10^2) + (5 \times 10^1) + (7 \times 10^0)$

We can then carry out calculations in columns
– units, tens, hundreds, thousands, etc.

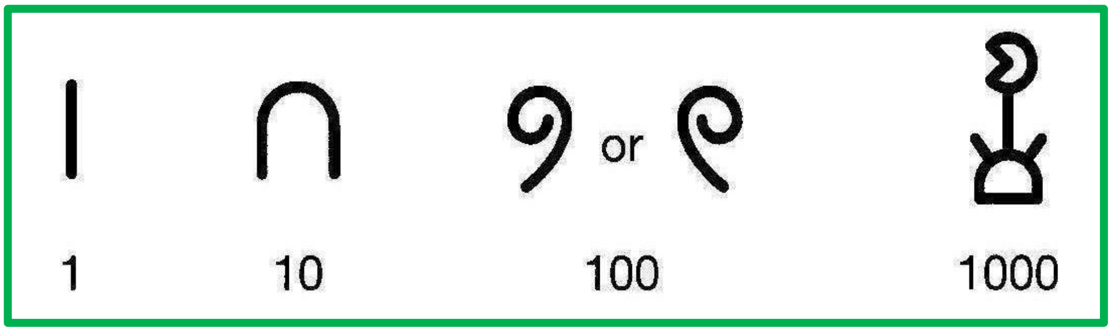
Similarly the binary system uses only 0 and 1:

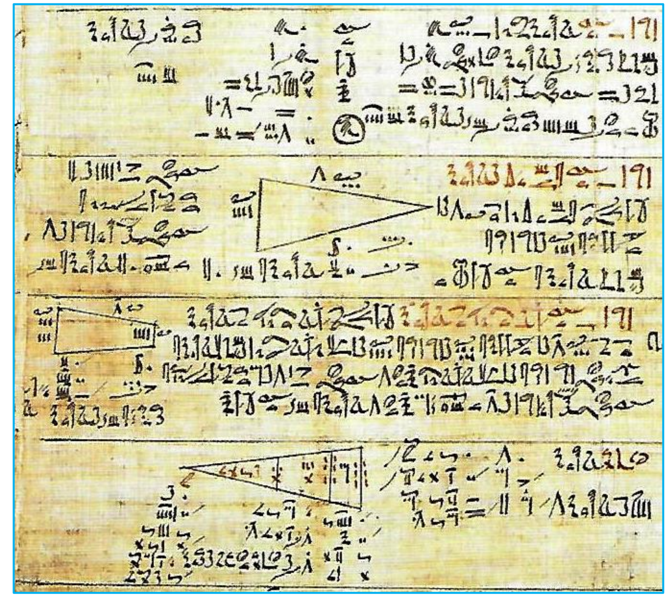
for example, 1101 means

$(1 \times 2^3) + (1 \times 2^2) + (0 \times 2^1) + (1 \times 2^0)$ [= 13]

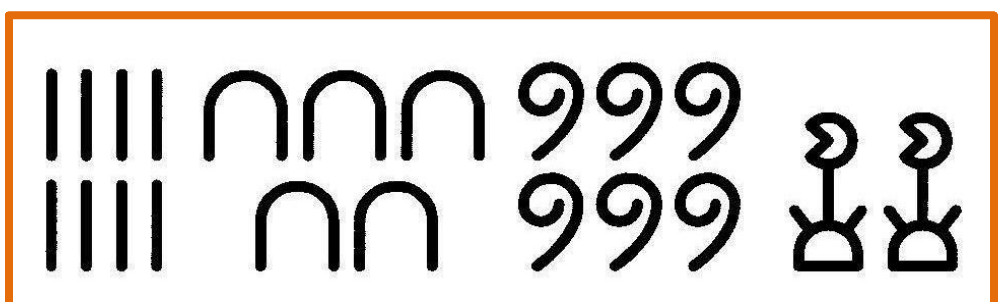
Egyptian counting

Decimal system, written on papyrus with different symbols for 1, 10, 100, etc.

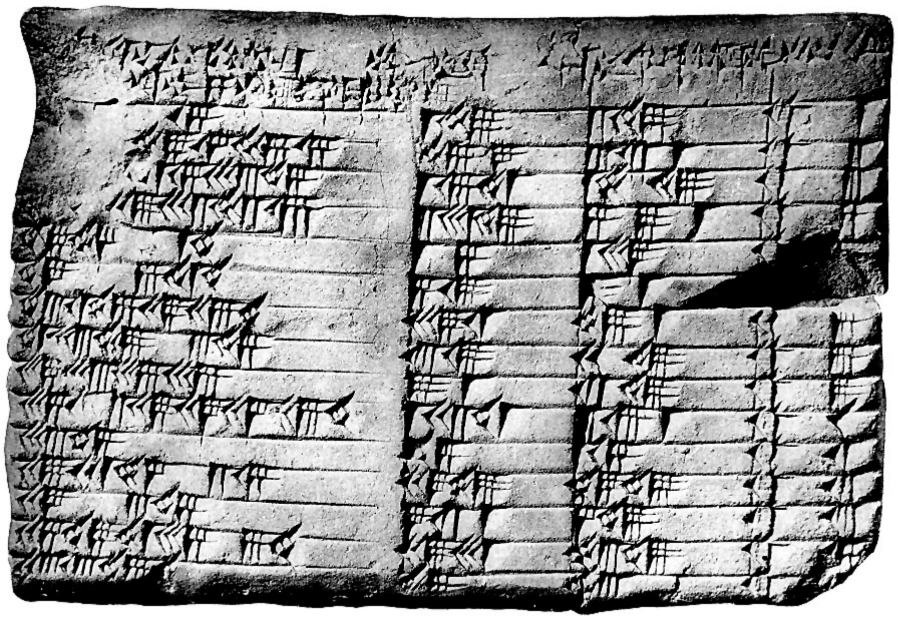




1 = rod; 10 = heel bone;
100 = coiled rope; 1000 = lotus flower

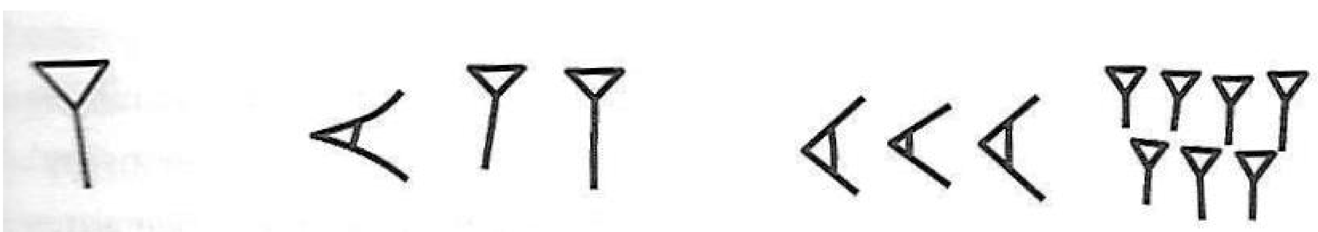


Mesopotamian counting



Cuneiform writing:
place-value system
based on 60:
symbols for 1 and 10

For example:



or 1, 12, 37, means

$$(1 \times 60^2) + (12 \times 60^1) + (37 \times 60^0) = 3600 + 720 + 37 = 4357$$

The classical world

Roman numerals: decimal system using the symbols

I = 1, V = 5, X = 10, L = 50, C = 100, D = 500, M = 1000:

so **2018** is **MMXVIII.**

Greek number system:

decimal system with different symbols (Greek letters) for

1, 2, . . . , 9,

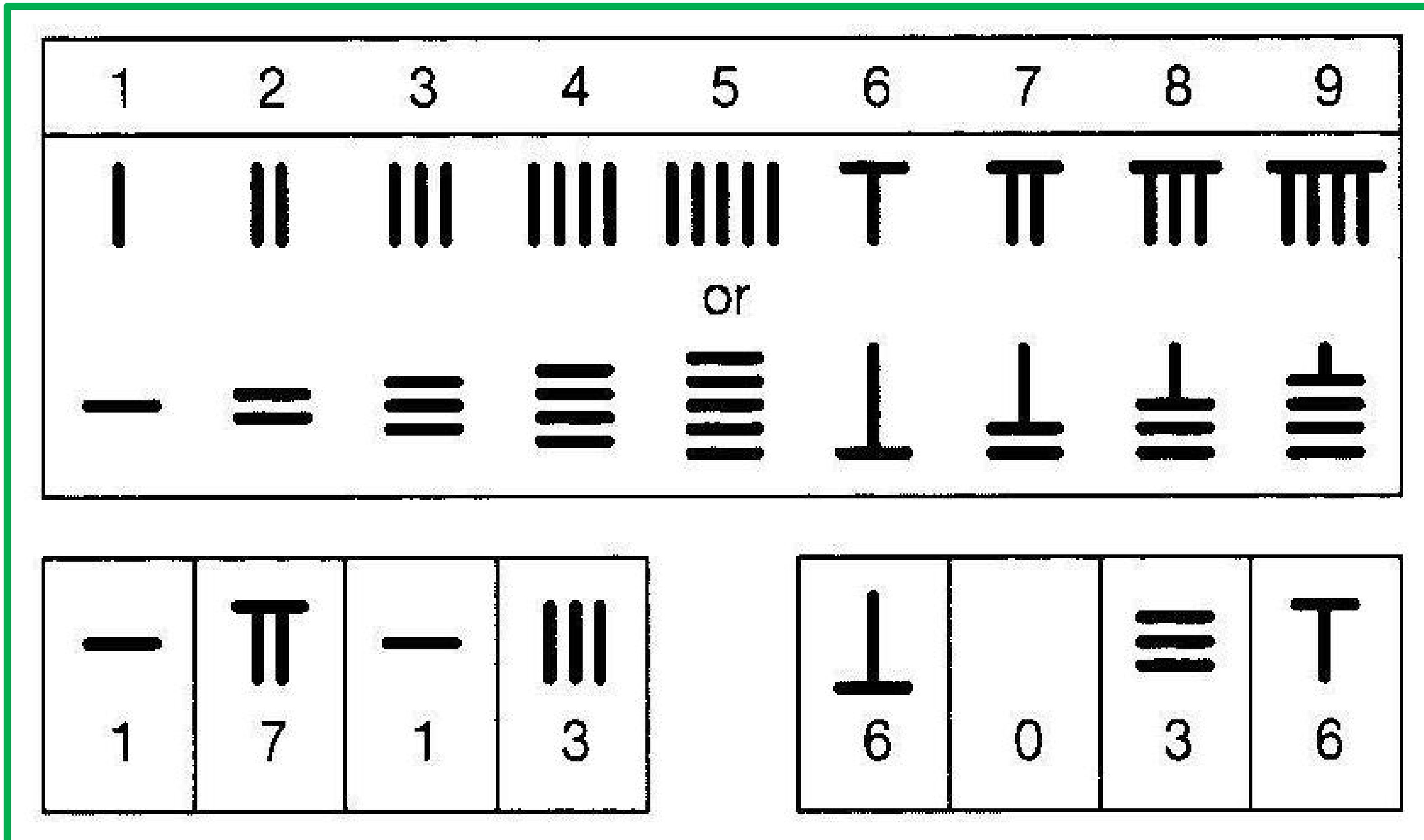
10, 20, . . . , 90,

100, 200, . . . , 900.

So 888 is written as $\omega\pi\eta$.

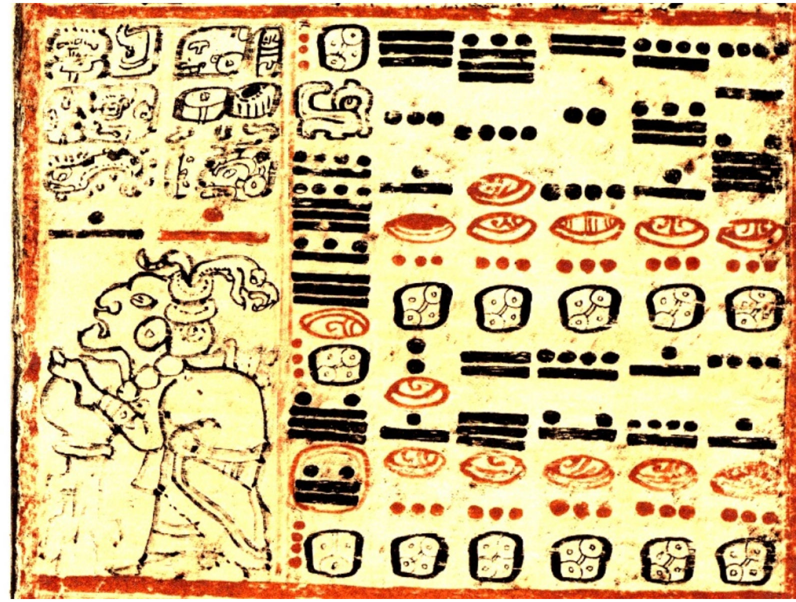
1	2	3	4	5	6	7	8	9
α	β	γ	δ	ε	ς	ζ	η	θ
10	20	30	40	50	60	70	80	90
ι	κ	λ	μ	ν	ξ	o	π	ϙ
100	200	300	400	500	600	700	800	900
ρ	σ	τ	υ	ϕ	χ	ψ	ω	ϡ

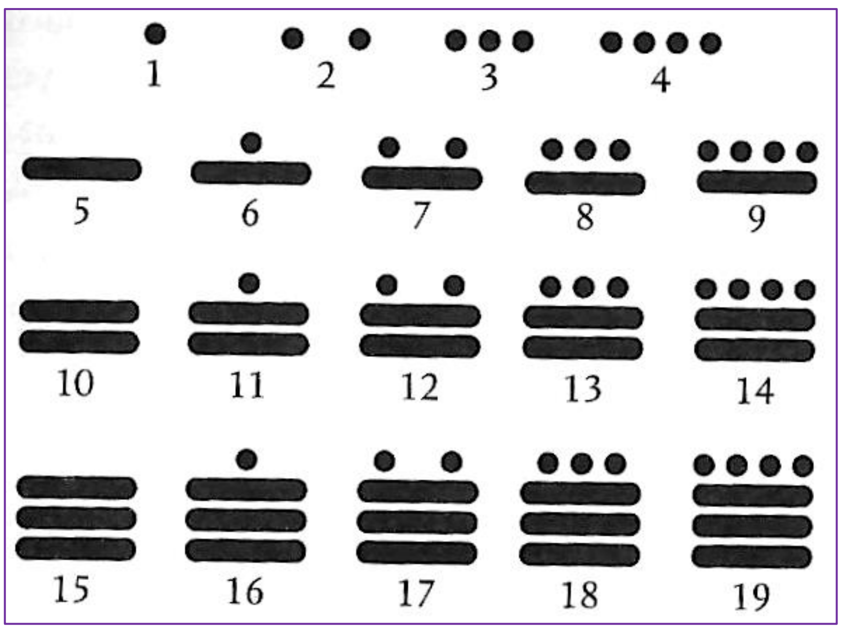
Chinese decimal counting boards

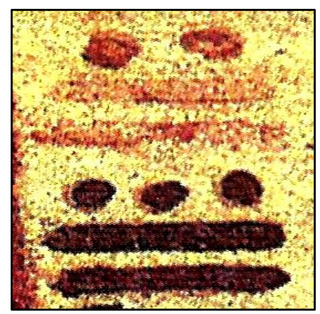


Mayan counting

Place-value system based mainly on 20, using symbols for 1 and 5.

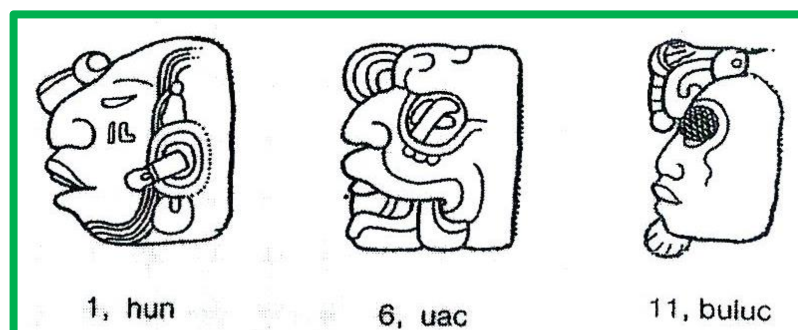




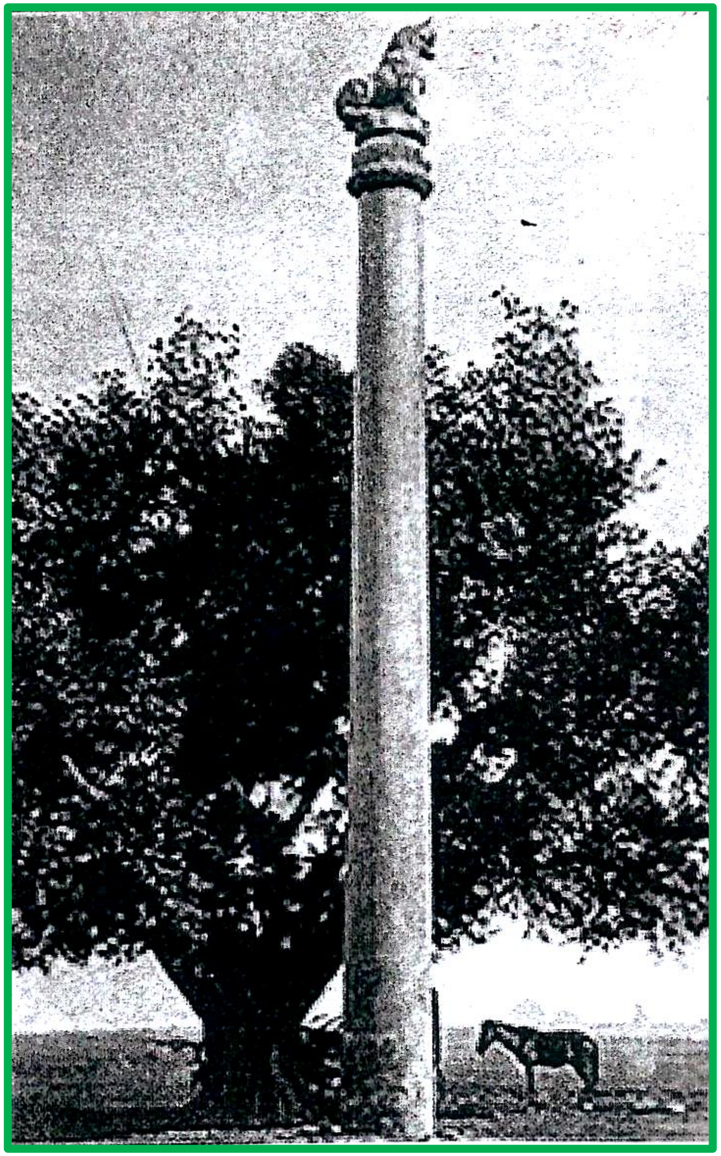


means

$$(12 \times 20) + 13 = 273$$



Indian counting



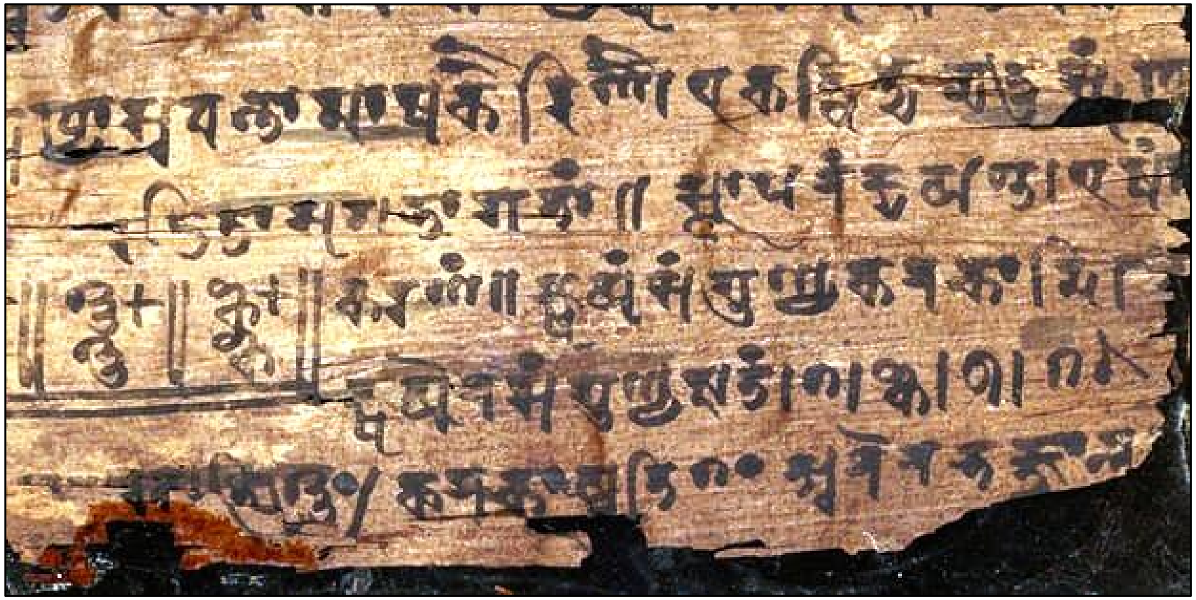
King Asoka (c. 250 BC),
first Buddhist monarch:
numbers inscribed on
pillars around the kingdom

They used a place-value
system based on 10,
with only
1, 2, 3, . . . , 9
– and (later) also 0

0 – the ‘nothingness’ number

← Gwalior, c. 800

↓ Peshawar, c. 300-400



↑

Brahmagupta (c.AD 600)

Calculating with zero and negative numbers.

$4 \times 0 = 9 \times 0$, so $4 = 9$??

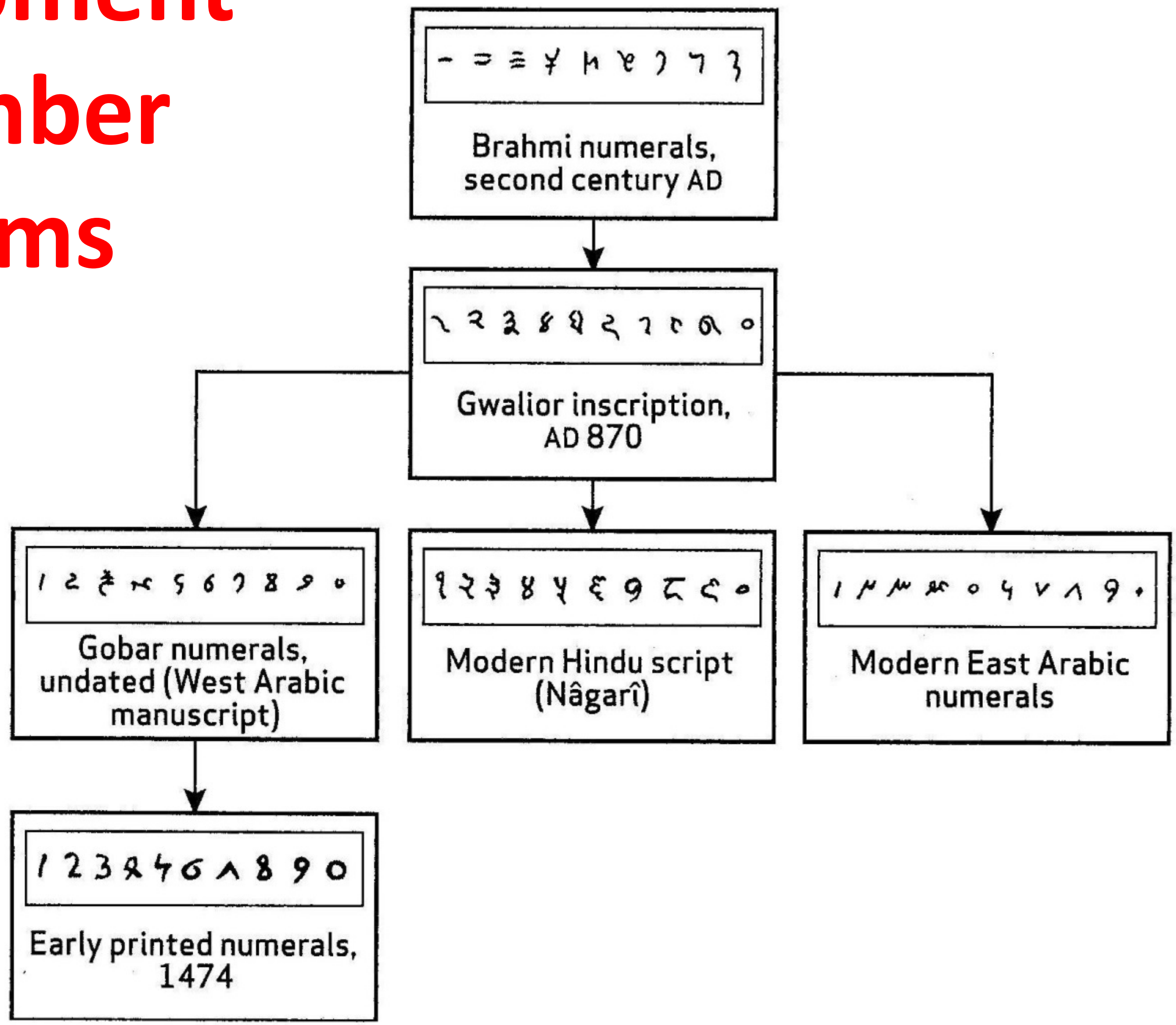
The sum of cipher and negative
is negative;
Of positive and nought, positive;
Of two ciphers, cipher.

Negative taken from cipher
becomes positive,
and positive from cipher is negative;
Cipher taken from cipher is nought.

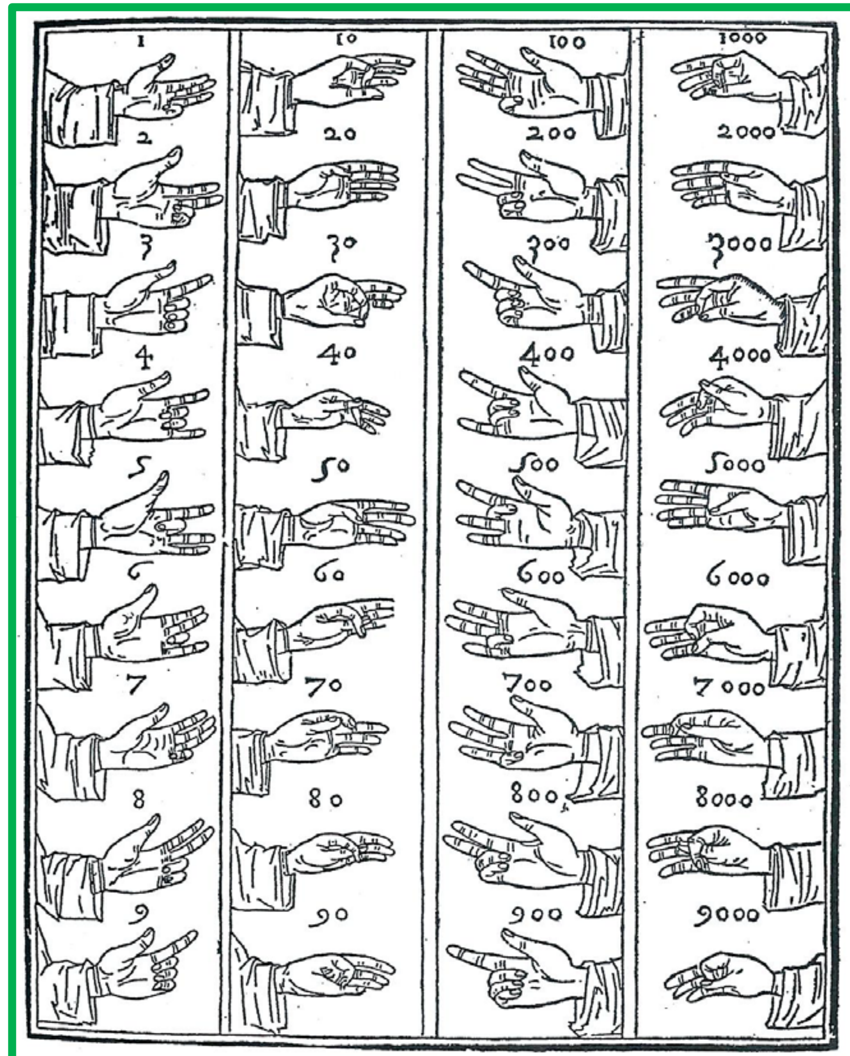
The product of cipher and positive,
or of cipher and negative, is nought;
Of two ciphers, it is cipher.

Cipher divided by cipher is nought.

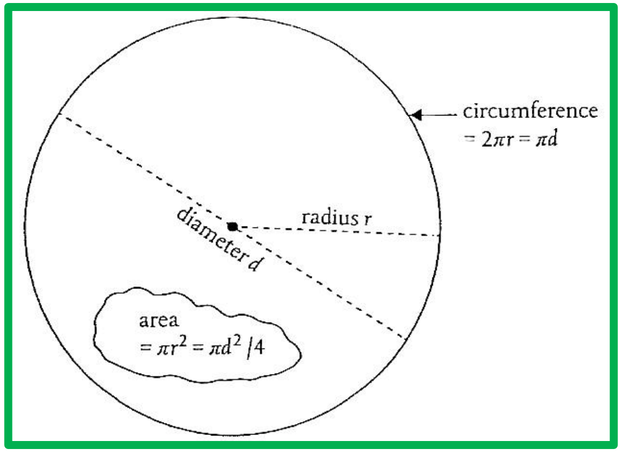
Development of number systems



The Hindu-Arabic numerals



π: the circle number



$\pi = 3.14159\ldots$ ($< {}^{22}/_{7}$) is

the ratio of the circumference C to the diameter d:

$\pi = C/d$, so $C = \pi d = 2\pi r$ (r = radius)

the ratio of the area A to the square of the radius r:

$\pi = A/r^2$, so $A = \pi r^2$

π to 500 decimal places

3.1415926535897932384626433832795028841971693993751058209749
44592307816406286208998628034825342117067982148086513282306
64709384460955058223172535940812848111745028410270193852110
55596446229489549303819644288109756659334461284756482337867
83165271201909145648566923460348610454326648213393607260249
14127372458700660631558817488152092096282925409171536436789
25903600113305305488204665213841469519415116094330572703657
59591953092186117381932611793105118548074462379962749567351
88575272489122793818301194912983367336244065664308602139494
63952247371907021798609437027705392171762931767523846748184
67669405132000568127145263560827785771342757789609173637178
72146844090122495343014654958537105079227968925892354201995
61121290219608640344181598136297747713099605187072113499999
98372978049951059731732816096318595024459455346908302642522
30825334468503526193118817101000313783875288658753320838142
06171776691473035982534904287554687311595628638823537875937
51957781857780532171226806613001927876611195909216420198 ...

The Vienna metro (Karlplatz)

π to 2000 decimal places

3.14159265358979323846264338327950288419716939937510582097494459230781640628620899862803 4
8253421170679821480865132823066470938446095505822317253594081284811174502841027019385211
0555964462294895493038196442881097566593344612847564823378678316527120190914564856692346
0348610454326648213393607260249141273724587006606315588174881520920962829254091715364367
8925903600113305305488204665213841469519415116094330572703657595919530921861173819326117
9310511854807446237996274956735188575272489122793818301194912983367336244065664308602139
4946395224737190702179860943702770539217176293176752384674818467669405132000568127145263
5608277857713427577896091736371787214684409012249534301465495853710507922796892589235420
1995611212902196086403441815981362977477130996051870721134999999837297804995105973173281
6096318595024459455346908302642522308253344685035261931188171010003137838752886587533208
3814206171776691473035982534904287554687311595628638823537875937519577818577805321712268
0661300192787661119590921642019893809525720106548586327886593615338182796823030195203530
1852968995773622599413891249721775283479131515574857242454150695950829533116861727855889
0750983817546374649393192550604009277016711390098488240128583616035637076601047101819429
5559619894676783744944825537977472684710404753464620804668425906949129331367702898915210
4752162056966024058038150193511253382430035587640247496473263914199272604269922796782354
7816360093417216412199245863150302861829745557067498385054945885869269956909272107975093
0295532116534498720275596023648066549911988183479775356636980742654252786255181841757467
2890977772793800081647060016145249192173217214772350141441973568548161361157352552133475
7418494684385233239073941433345477624168625189835694855620992192221842725502542568876717
9049460165346680498862723279178608578438382796797668145410095388378636095068006422512520
5117392984896084128488626945604241965285022210661186306744278622039194945047123713786960
95636437191728746776465757396241389086583264599581339047802759010

How I wish I could calculate pi! (3.141592)

May I have a large container of coffee

3. 1 4 1 5 9 2 6 …

How I need a drink, alcoholic of course, after all these lectures informing Gresham audiences . . .

3.14159265358979 …

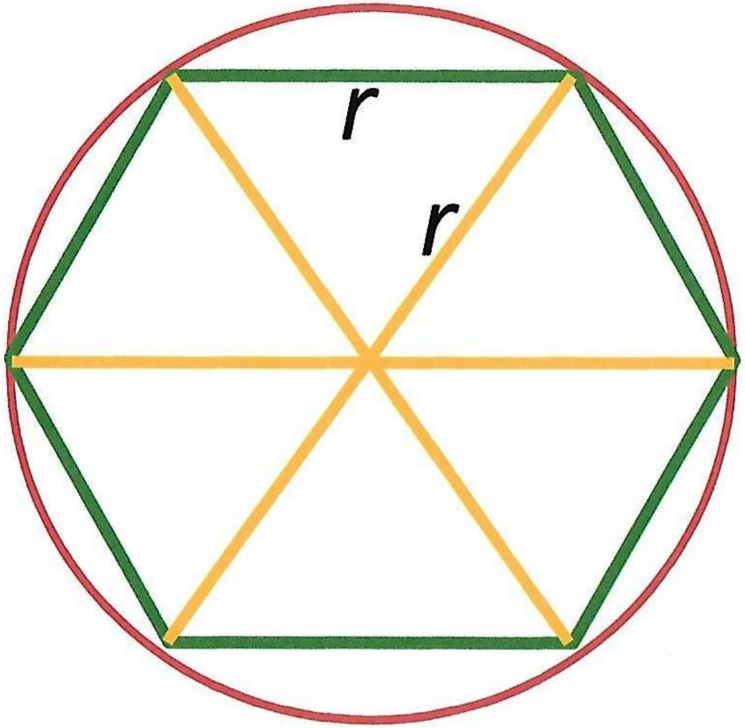
ἀει ὁ Θεος ὁ Μεγας γεωμετρει / το κυκλου μηκος ἰνα ὁριση διαμετρω / παρηγαγεν ἀριθμον ἀπεραντον / και ὀν φευ οὐδεποτε ὀλον / θνητοι θα εὑρωσι

[Great God ever geometrizes – to define the circle length by its diameter . . .]

3.14159265358979323846 26…

Mesopotamian value of π

(c.1800 BC)



Ratio of the perimeter of the hexagon to the circumference of the circle is 0; 57, 36

$$^{6r}/_{2\pi r} = {}^{3}/_{\pi} = {}^{57}/_{60} + {}^{36}/_{3600}$$

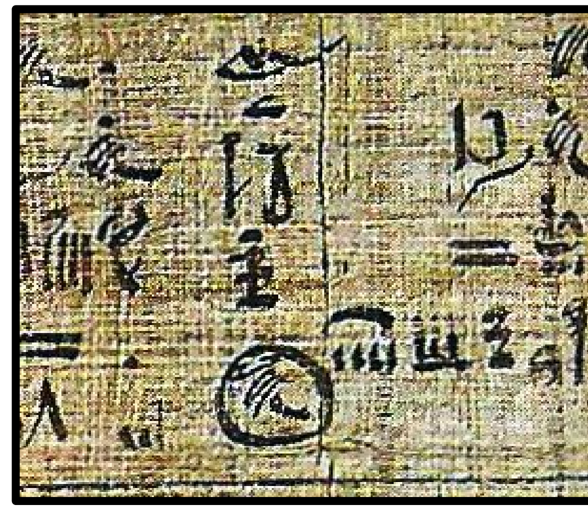
So $\pi = 3\,^{1}/_{8} = 3.125$

An Egyptian problem in geometry

Problem 50. **Example of a round field of diameter 9 khet. What is its area?**

Answer:

Take away $^1/_9$ of the diameter, namely 1; the remainder is 8. Multiply 8 times 8; it makes 64. So it contains 64 setat of land.



$$\text{Area} = (d - {}^1/_9 d)^2 = ({}^8/_9\, d)^2 = {}^{256}/_{81}\, r^2$$

which is about $3.160\, r^2$

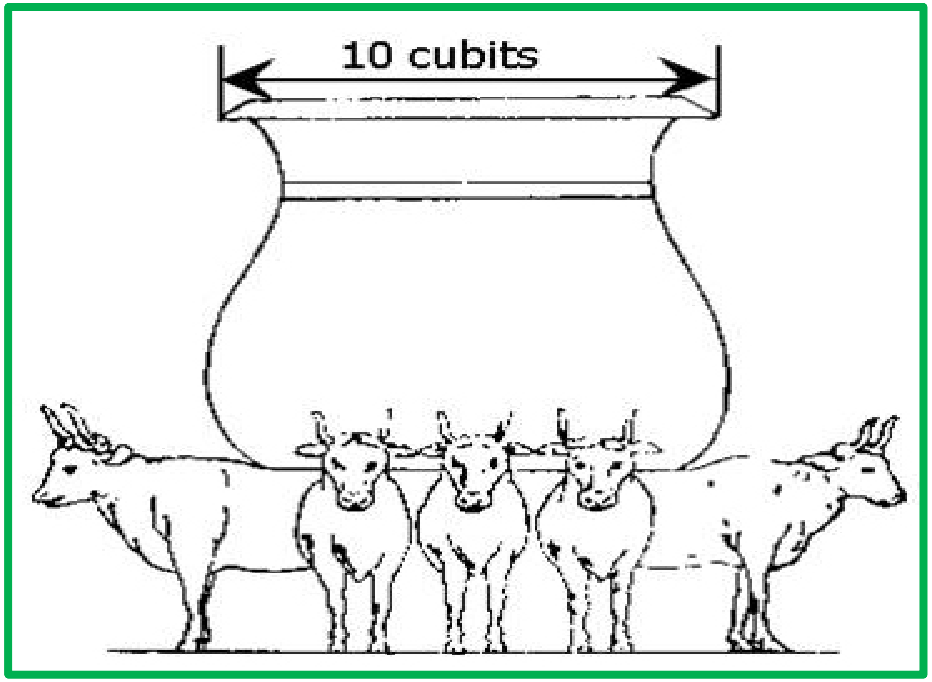
The Biblical value (c.550 BC)

I Kings VII, 23 and II Chronicles IV, 2

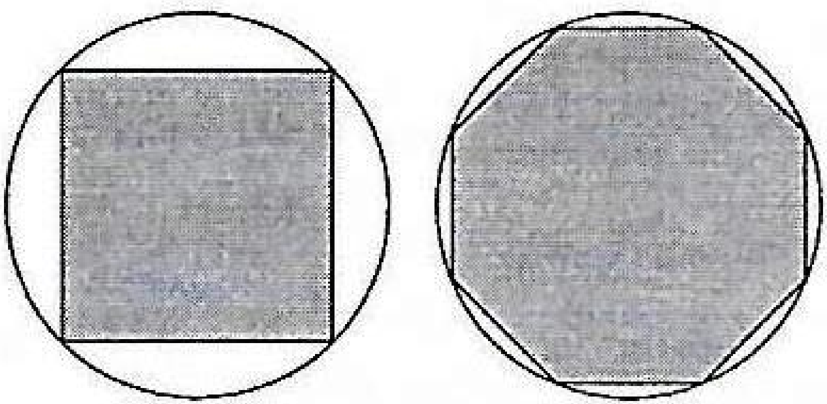
Also, he made a molten sea of ten cubits from brim to brim, round in compass, and five cubits the height thereof; and a line of thirty cubits did compass it round about.

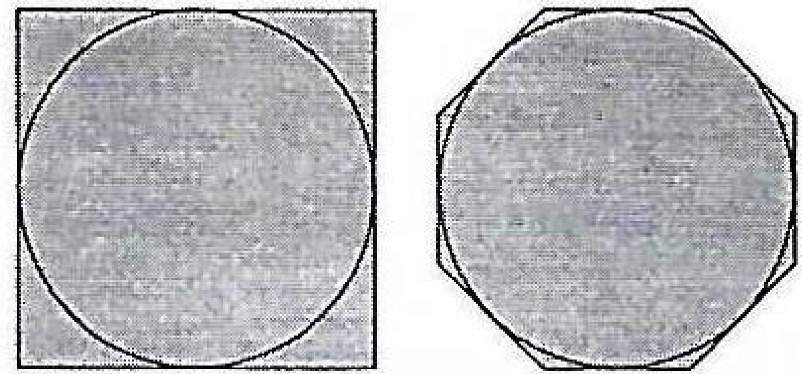
So:

$$\pi = {}^{30}/_{10} = 3$$

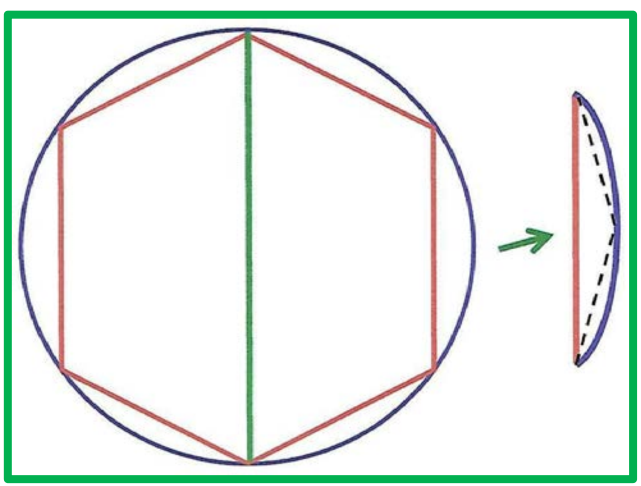


Using polygons





Antiphon: $\pi > 2$, $\pi > 2.828$ **Bryson:** $\pi < 4$, $\pi < 3.32$



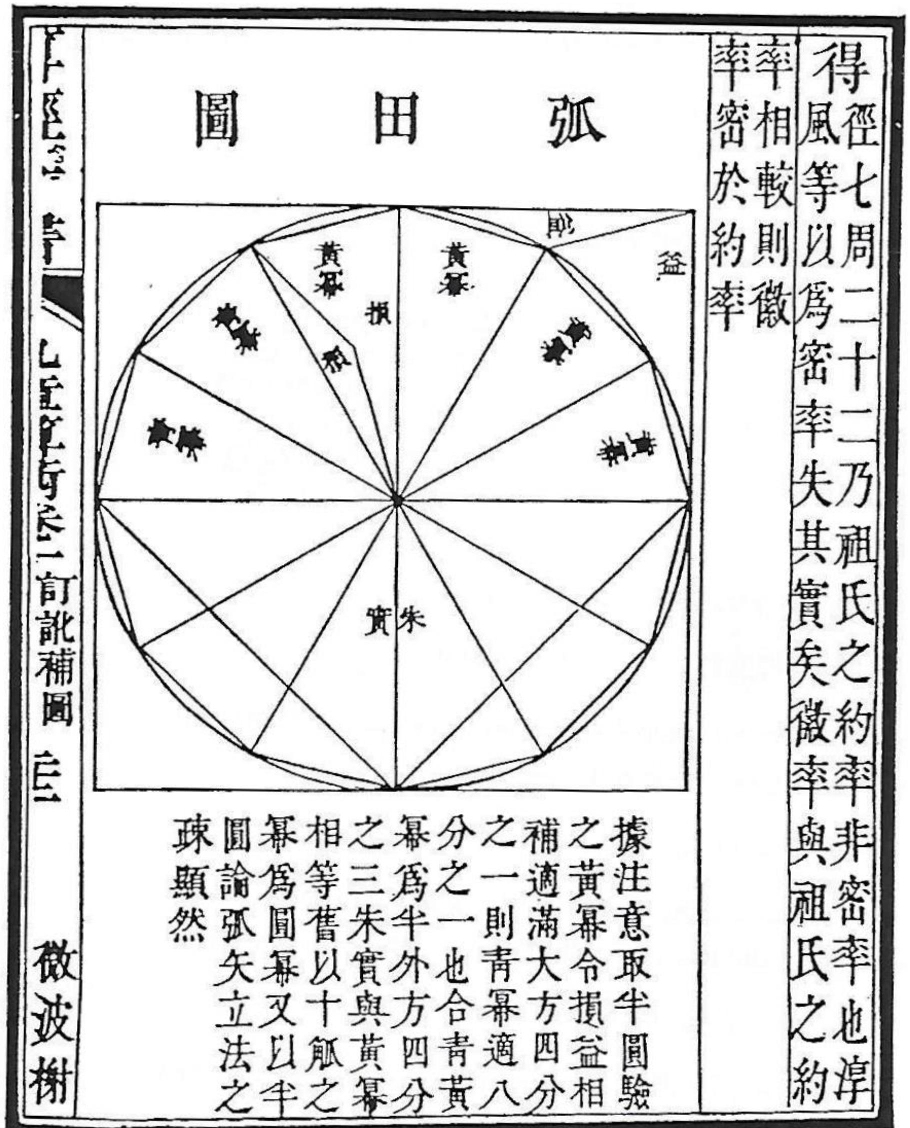
Archimedes: repeatedly double the number of sides:

$6 \rightarrow 12 \rightarrow 24 \rightarrow 48 \rightarrow 96$

result: $3\,{}^{10}/_{71} < \pi < 3\,{}^{1}/_{7}$

3.14084 3.14286

Chinese values for π



弧田圖

得徑七周二十二乃祖氏之約率非密率也淳
風等以爲密率失其實矣徽率與祖氏之約
率相較則徽率密於約率

據注意取半圓驗
之黃冪令損益相
補適滿大方四分
之一則青冪適八
分之一也合青黃
冪爲半外方四分
之三朱實與黃冪
相等舊以十觚之
冪爲圓冪又以半
圓論弧矢立法之
疎顯然

訂訛補圖

微波榭

← **Liu Hui (AD 263)**

$\pi = 3.14159$

(3072 sides)

Zu Chongzhi (AD 500)

$\pi = 3.1415926$

(24,576 sides)

and $\pi = {}^{355}/_{113}$

Ludolph van Ceulen (Dutch) (1540-1610)

1596: 20 d. p.

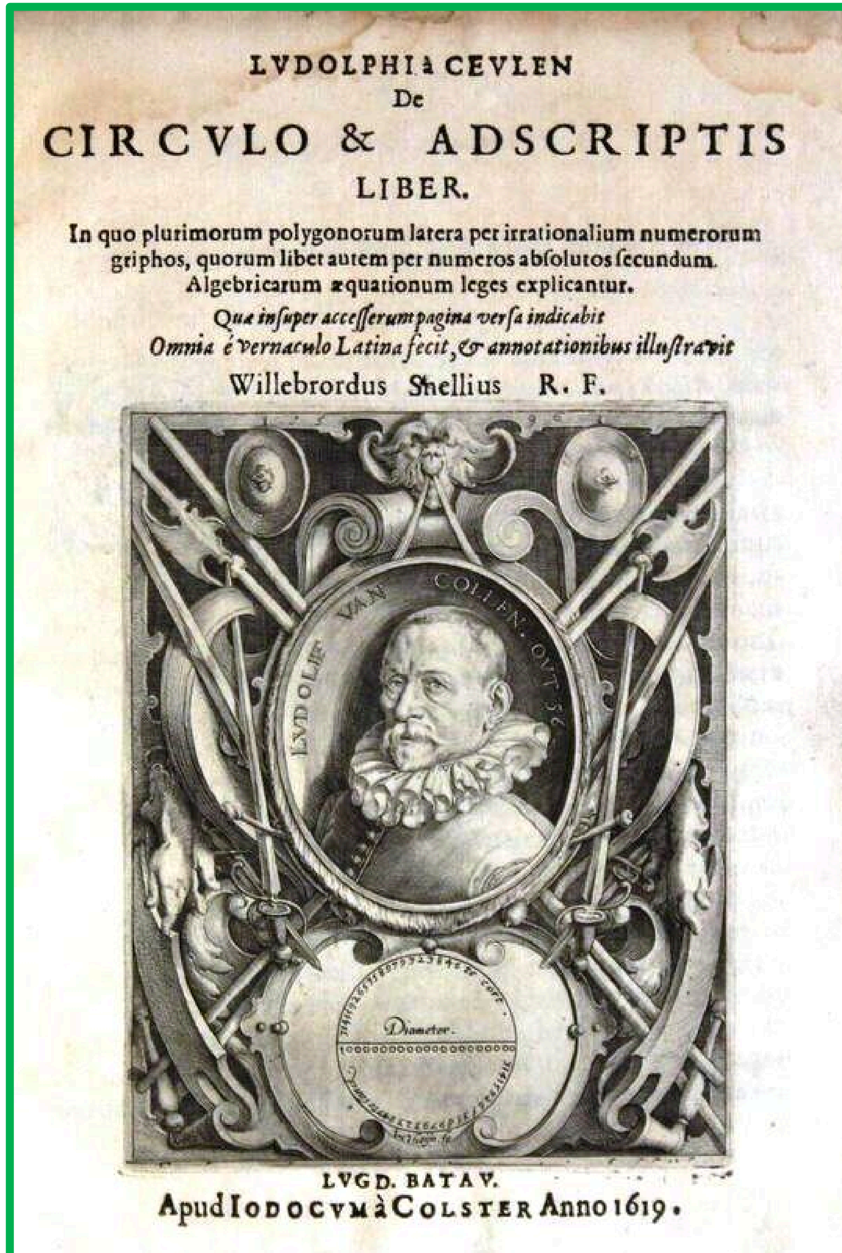
515,396,075,520

$= 60 \times 2^{33}$ sides →

1610: 35 d. p.

4,611,686,018,427,387,904

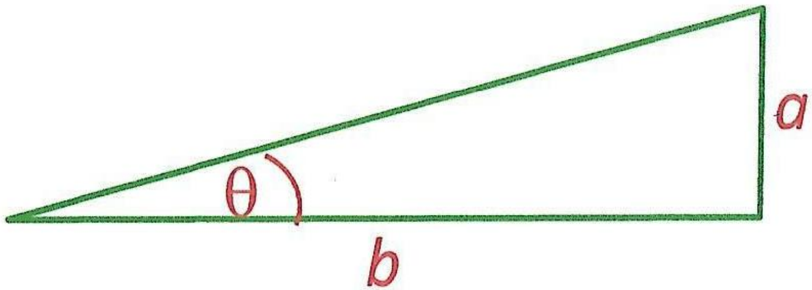
$= 2^{62}$ sides



LVDOLPHI à CEVLEN
De
CIRCVLO & ADSCRIPTIS
LIBER.
In quo plurimorum polygonorum latera per irrationalium numerorum griphos, quorum liber autem per numeros abſolutos ſecundum Algebricarum æquationum leges explicantur.
Quæ inſuper acceſſerum pagina verſa indicabit
Omnia é vernaculo Latina fecit, & annotationibus illuſtravit
Willebrordus Snellius R. F.

LVGD. BATAV.
Apud IODOCVM à COLSTER Anno 1619.

The $\tan^{-1}$ (or arctan) function

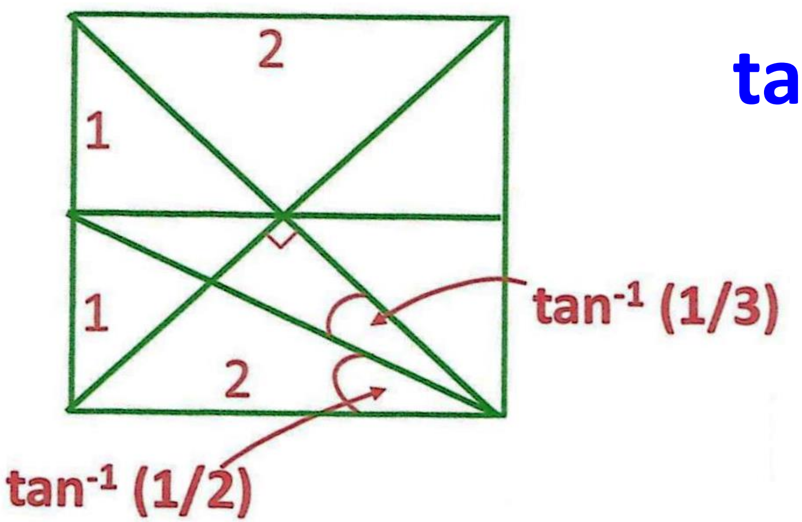


$\tan \theta = a/b$

$\theta = \tan^{-1}(a/b)$

$\tan \pi/4 = \tan 45° = 1$, so $\tan^{-1}(1) = \pi/4$

$\tan \pi/6 = \tan 30° = 1/\sqrt{3}$, so $\tan^{-1}(1/\sqrt{3}) = \pi/6$

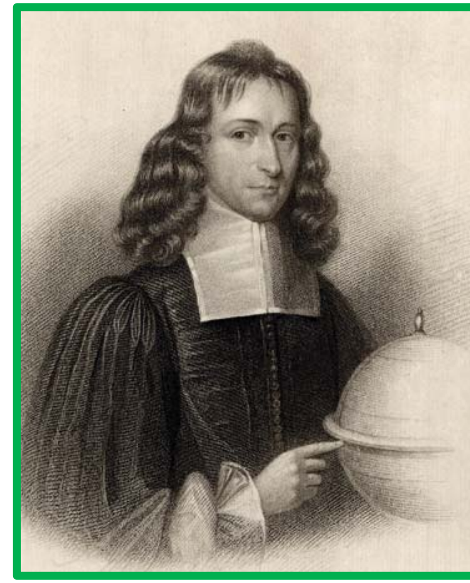


$\tan^{-1}(^1/_2) + \tan^{-1}(^1/_3) = \pi/4$

In general:

$$\boxed{\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)}$$

The series for $\tan^{-1} x$



$$\tan^{-1} x = x - {}^1/_3\, x^3 + {}^1/_5\, x^5 - {}^1/_7\, x^7 + \ldots$$

So, with $x = 1$ we have:

$$\pi/4 = 1 - {}^1/_3 + {}^1/_5 - {}^1/_7 + \ldots$$

But this converges extremely slowly: 300 terms give only two decimal places of π. Much better is:

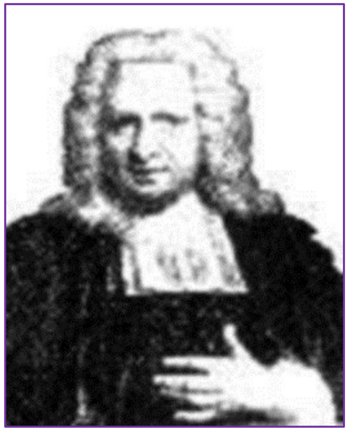
$$\pi/4 = \tan^{-1} ({}^1/_2) + \tan^{-1} ({}^1/_3)$$

$$= \{{}^1/_2 - {}^1/_3\, ({}^1/_2)^3 + {}^1/_5\, ({}^1/_2)^5 - {}^1/_7\, ({}^1/_2)^7 + \ldots\}$$

$$- \{{}^1/_3 - {}^1/_3\, ({}^1/_3)^3 + {}^1/_5\, ({}^1/_3)^5 - {}^1/_7\, ({}^1/_3)^7 + \ldots\},$$

which converges much faster.

Machin's $\tan^{-1}$ formula (1706)



$$\pi = 16 \tan^{-1} (1/5) - 4 \tan^{-1} (1/239)$$

$$= 16 \{ 1/5 - 1/3 (1/5)^3 + 1/5 (1/5)^5 - 1/7 (1/5)^7 + \ldots \}$$

$$- 4 \{ (1/239) - 1/3 (1/239)^3 + 1/5 (1/239)^5 - 1/7 (1/239)^7 + \ldots \}$$

Machin used this series to calculate π to 100 decimal places

John Machin was Gresham Professor of Astronomy from 1713 to 1751

William Jones introduces π (1706)

There are various other ways of finding the *Lengths*, or *Areas* of particular *Curve Lines*, or *Planes*, which may very much facilitate the Practice; as for Inſtance, in the *Circle*, the Diameter is to Circumference as 1 to

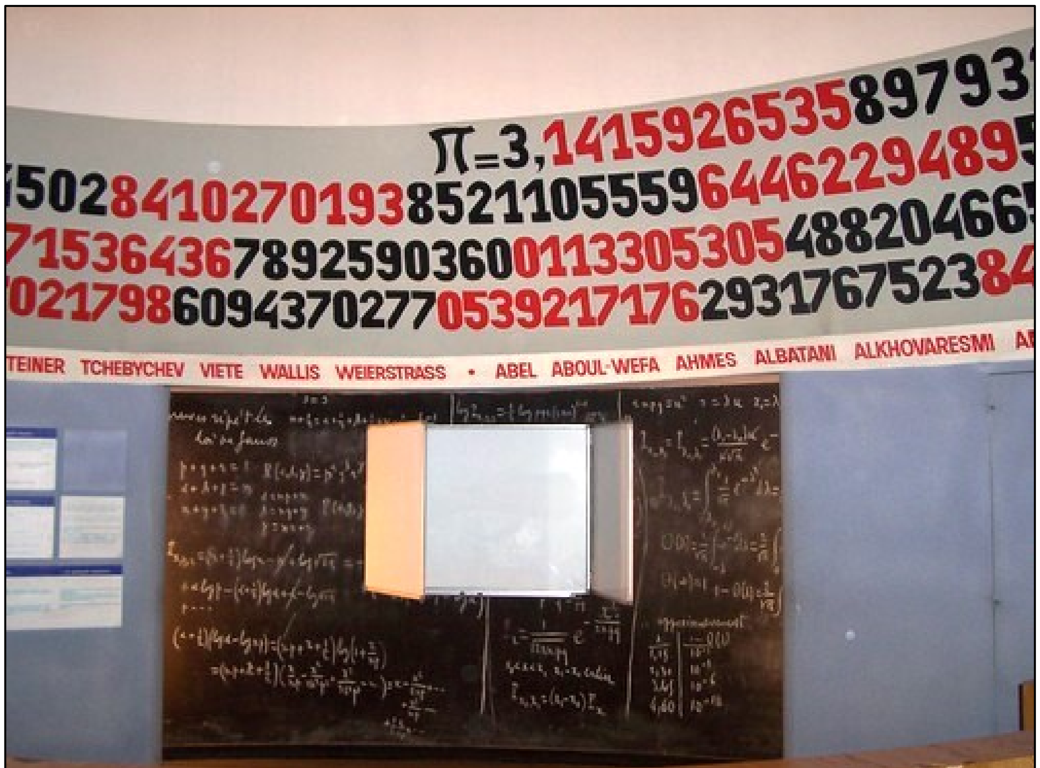
$$\overline{\frac{16}{5} - \frac{4}{239}} - \frac{1}{3}\,\overline{\frac{16}{5^3} - \frac{4}{239^3}} + \frac{1}{5}\,\overline{\frac{16}{5^5} - \frac{4}{239^5}} -, \text{\&c.} =$$

3.14159, &c. $= \pi$. This *Series* (among others for the ſame purpoſe, and drawn from the ſame Principle) I re-

Theref. *the* (*Radius is to* $\frac{1}{2}$ *Periphery, or*) *Diameter is to the Periphery, as* 1,000, &c. *to* 3.141592653.58979323 84.6264338327.9502884197.1693993751.0582097494.4592307816.4062862089.9862803482.5342117067.9 +, True to above a 100 Places; as Computed by the Accurate and Ready Pen of the Truly Ingenious Mr. *John Machin*: Purely as an Inſtance of the Vaſt advan-

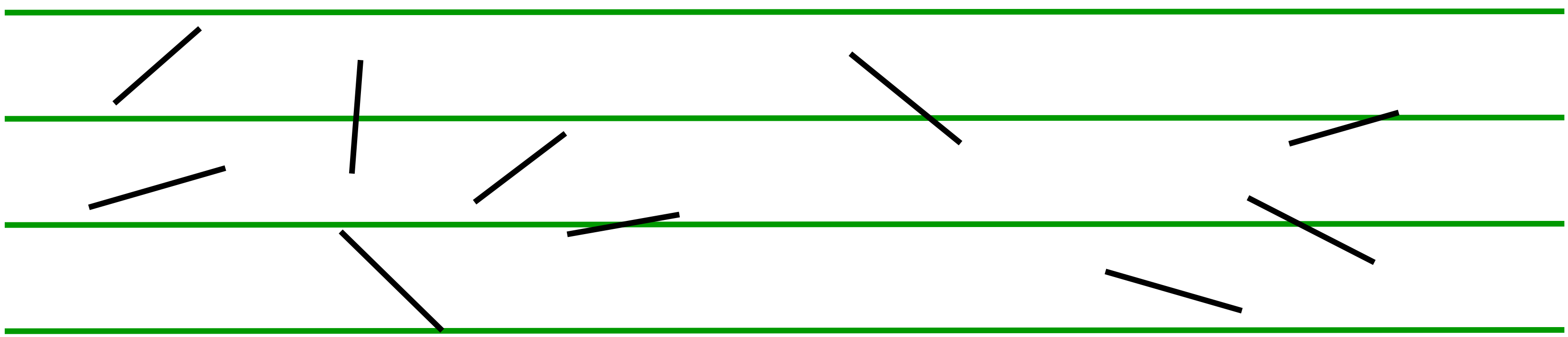
William Shanks's value of π

In 1873, using Machin's formula, William Shanks calculated π to 707 decimal places, which appear in the 'π-room' in the Palais de la Découverte in Paris.



Unfortunately, . . .

Buffon's needle experiment (1777)



L **= length of needle**
D **= distance between lines**

Probability of crossing a line = $2/\pi \times L/D$

Here:

$^{5}/_{10} = 2/\pi \times {}^{4}/_{5}$, so $\pi = {}^{16}/_{5} = 3.2$

Legislating for π

State legislature of Indiana (1897)

Edwin J. Goodman M.D. proposed:

A bill for an act introducing a new mathematical truth and offered as a contribution to education to be used only in the State of Indiana free of cost by paying any royalties whatever on the same, provided it is accepted and adopted by the official action of the legislature in 1897.

The bill went to the House Committee on Swamp Lands (!)

and then to the Committee on Education, where it was passed:

Be it enacted by the General Assembly of the State of Indiana:

It has been found that the circular area is to the quadrant of the circumference, as the area of an equilateral rectangle is to the square on one side. The diameter employed as the linear unit according to the present rule in computing the circle's area is *entirely wrong ...*

The bill then went to the Committee on Temperance (!), where it was noticed (and stopped) by Mathematics professor C. A. Waldo (Purdue)

Some weird results

Ramanujan (1914):

$$\frac{1}{\pi} = \frac{\sqrt{8}}{9801} \sum_{n=0}^{\infty} \frac{(4n)!}{(n!)^4} \times \frac{1103 + 26390n}{396^{4n}}$$

The Chudnovsky brothers (1989):

$$\frac{1}{\pi} = 12 \sum_{n=0}^{\infty} (-1)^n \frac{(6n)!}{(n!)^3 (3n)!} \times \frac{13591409 + 545140134n}{(640320)^{n+1/2}}$$

Bailey, Borwein and Plouffe (1995):

$$\pi = \sum_{n=0}^{\infty} \frac{1}{16^n} \left(\frac{4}{8n+1} - \frac{2}{8n+4} - \frac{1}{8n+5} - \frac{1}{8n+6} \right)$$

Circling the earth

The earth's circumference is about 25,000 miles
(132 million feet)

Tie a string tightly around the earth. Then extend the string by just 2π (≈ 6.3) feet, and prop it up equally *all around the earth*.

How high above the ground is the string?

Circling the earth

The earth's circumference is about 25,000 miles. Tie a string tightly around the earth, extend the string by 2π (≈ 6.3) feet, and prop it up equally *all around the earth*. How high above the ground is the string?

If the earth's radius is r,
then the original string has length $2\pi r$.

When we extend the string by 2π,
the new circumference is $2\pi r + 2\pi = 2\pi\,(r + 1)$,
so the new radius is $r + 1$ – so
everywhere the string is now *one foot* off the ground.

e: the ‘exponential number’

e = 2.7182818284590452360287471352…

The letter *e* first appeared in an unpublished paper of Euler around 1727, and in a letter of 1731.

It first appeared in print in 1736 in his *Mechanica*:

‘where *e* denotes the number whose hyperbolic logarithm is 1’.

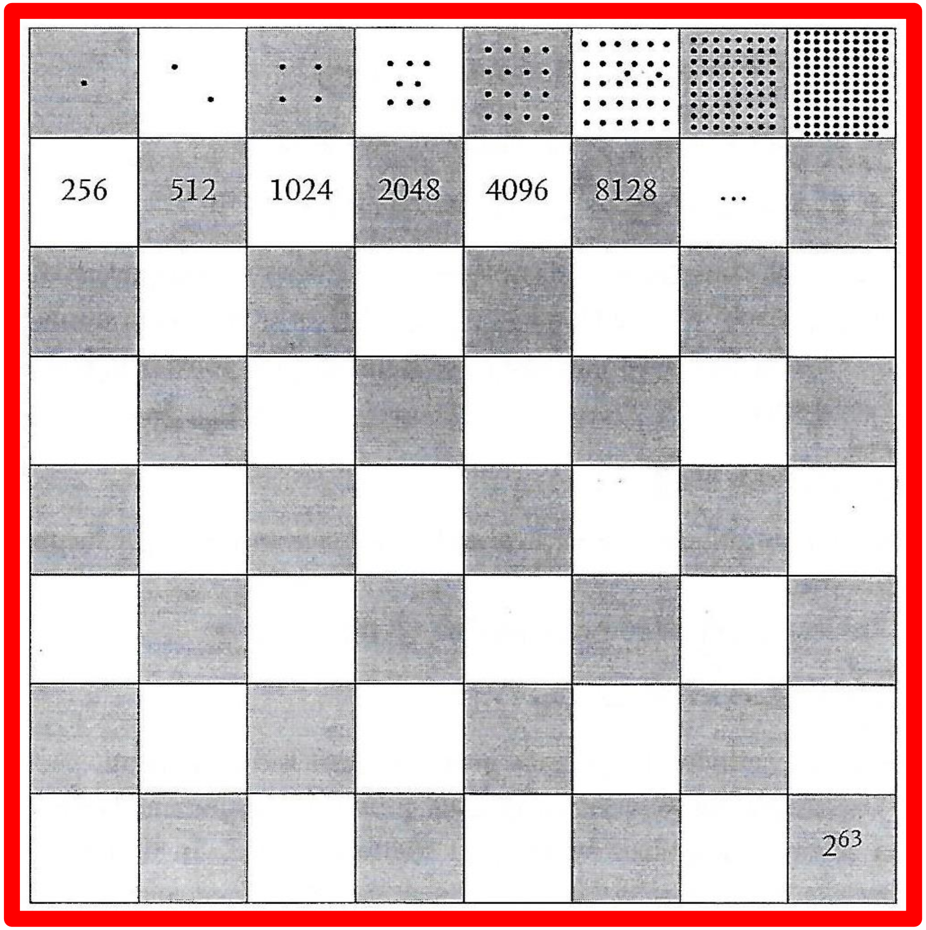
> **Corollarium II.**
>
> 171. Quanquam autem in iſta aequatione ipſa potentia *p* non ineſt, tamen eius directio a qua relatio elementorum *dx* et *dy* pendet, adhuc ſuper eſt. Data igitur directione potentiae punctum in quouis loco ſollicitantis, et ipſa curua in qua punctum mouetur, poterit ex his ſolis datis determinari puncti celeritas in quouis loco. Erit enim $\frac{dc}{c} = \frac{dyds}{rdx}$ ſeu $c = e^{\int \frac{dyds}{rdx}}$ vbi *e* denotat numerum, cuius logarithmus hyperbolicus eſt 1.

A chessboard problem

The wealthy king of a certain country was so impressed by the new game of chess that he offered the wise man who invented it any reward he wished.

The wise man replied:
‘My prize is for you to give me
1 grain of wheat for the first square of the chessboard,
2 grains for the second square,
4 grains for the third square,
and so on, doubling the number of grains on each successive square until the chessboard is filled.’

A chessboard problem



Total number of grains:

$$1 + 2 + 2^2 + 2^3 + \ldots + 2^{63}$$

$$= 2^{64} - 1$$

$$= 18{,}446{,}744{,}073{,}709{,}551{,}615$$

– enough wheat to form a pile the size of Mount Everest.

Exponential growth

polynomial growth

n : 1, 2, 3, 4, 5, . . .

n^2 : 1, 4, 9, 16, 25, . . .

n^3 : 1, 8, 27, 64, 125, . . .

exponential growth

2^n : 1, 2, 4, 8, 16, 32, . . .

3^n : 1, 3, 9, 27, 81, 243, . . .

	$n = 10$	$n = 30$	$n = 50$
polynomial			
n	0.00001 seconds	0.00003 seconds	0.00005 seconds
n^2	0.0001 seconds	0.0009 seconds	0.0025 seconds
n^3	0.001 seconds	0.027 seconds	0.125 seconds
exponential			
2^n	0.001 seconds	17.9 minutes	35.7 years
3^n	0.059 seconds	6.5 years	2.3×10^{10} years

In the long run, exponential growth usually exceeds polynomial growth.

An interest-ing problem

Jakob Bernoulli asked:

How much is earned when compound interest is calculated yearly? twice a year? . . . , every day? *n* times a year? continuously?

Invest £1 at 100% interest per year:

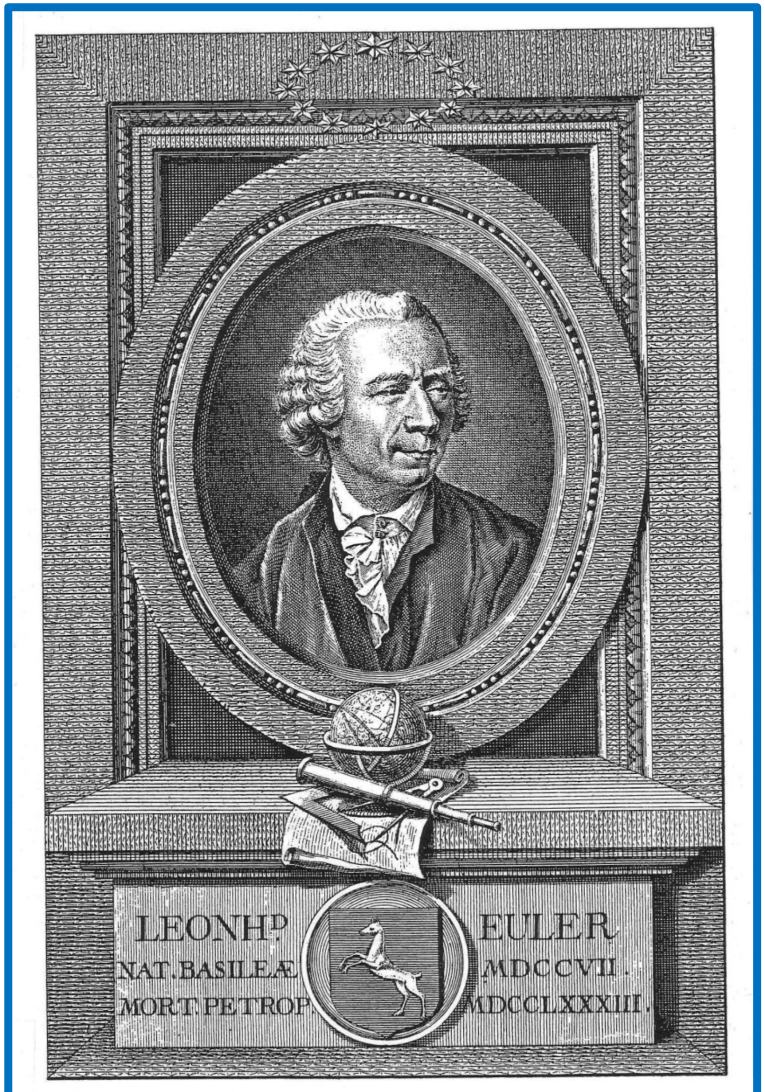
twice a year: $£1 \rightarrow £(1 + {}^{1}/_{2}) = £1.50 \rightarrow £(1 + {}^{1}/_{2})^2 = £2.25$

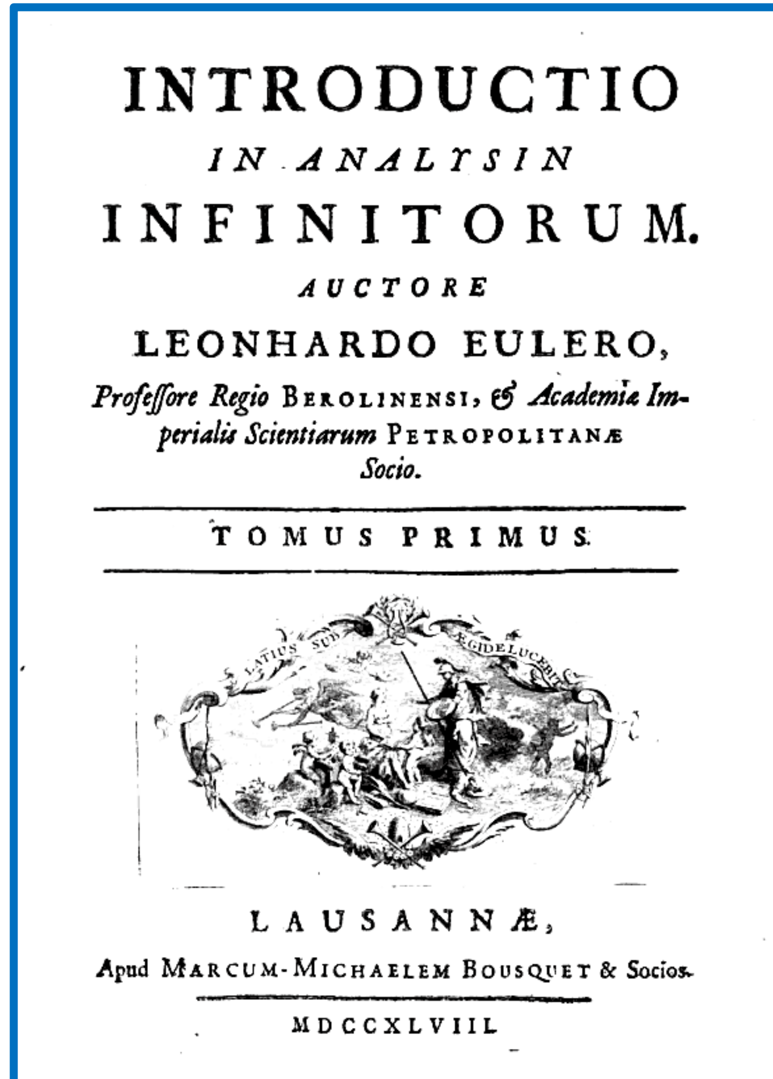
***n* times a year:** $£1 \rightarrow £(1 + {}^{1}/_{n}) \rightarrow £\,(1 + {}^{1}/_{n})^2 \rightarrow \ldots \rightarrow £\,(1 + {}^{1}/_{n})^n$

period:	year	half-year	quarter-year	two months	month
final amount:	2.00000	2.25000	2.44141	2.52153	2.61304
period:	week	day	hour	minute	second
final amount:	2.69260	2.71417	2.71813	2.71828	2.71828

As *n* increases, these numbers tend to the limiting value of *e*.

Leonhard Euler (1707-1783)





INTRODUCTIO
IN ANALYSIN
INFINITORUM.
AUCTORE
LEONHARDO EULERO,
Professore Regio BEROLINENSI, & *Academiæ Imperialis Scientiarum* PETROPOLITANÆ *Socio.*

TOMUS PRIMUS.

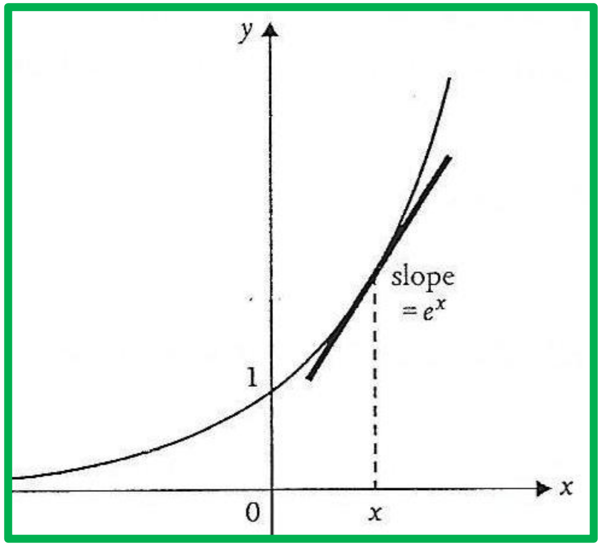
LAUSANNÆ,
Apud MARCUM-MICHAELEM BOUSQUET & Socios.

MDCCXLVIII.

Some properties of *e*

- $e = \lim_{n\to\infty}(1 + 1/n)^n$
- $e^x = \lim_{n\to\infty}(1 + x/n)^n$
- $e = 1 + {}^1/_{1!} + {}^1/_{2!} + {}^1/_{3!} + {}^1/_{4!} + \ldots$
- $e^x = 1 + x/1! + x^2/2! + x^3/3! + \ldots$
- The slope of the curve $y = e^x$ at the value x is $y = e^x$: →

$$dy/dx = y$$



Exponential growth

as predicted by Thomas Malthus (1798)

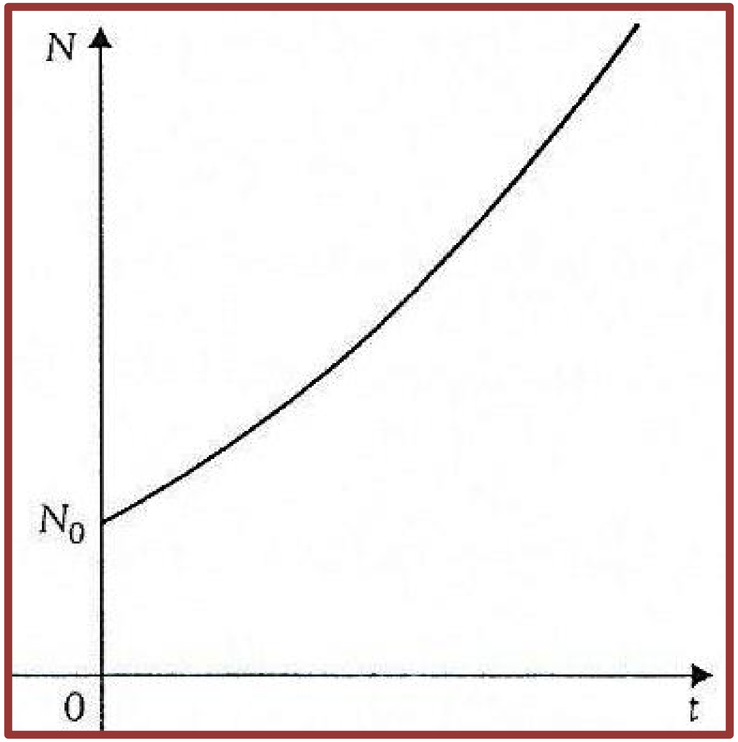
If $N(t)$, the size of a population at time t, grows at a fixed rate k proportional to its size, then

$$dN/dt = kN ,$$

which can be solved to give

$$N = N_0 e^{kt} ,$$

where N_0 is the initial population.



Similarly for the decay of radium, the cooling of a cup of tea, etc.


HIERONYMI CARDANI, PRÆSTANTISSIMI MATHEMATICI, PHILOSOPHI, AC MEDICI,
ARTIS MAGNÆ,
SIVE DE REGVLIS ALGEBRAICIS,
Lib. unus. Qui & totius operis de Arithmetica, quod
OPVS PERFECTVM
inscripsit, est in ordine Decimus.

HAbes in hoc libro, studiose Lector, Regulas Algebraicas (Itali, de la Cossa uocant) nouis adinuentionibus, ac demonstrationibus ab Authore ita locupletatas, ut pro pauculis antea uulgò tritis, iam septuaginta euaserint. Neq; solum, ubi unus numerus alteri, aut duo uni, uerum etiam, ubi duo duobus, aut tres uni æquales fuerint, nodum explicant. Hunc aūt librum ideo seorsim edere placuit, ut hoc abstrusissimo, & planè inexhausto totius Arithmeticæ thesauro in lucem eruto, & quasi in theatro quodam omnibus ad spectandum exposito, Lectores incitarētur, ut reliquos Operis Perfecti libros, qui per Tomos edentur, tanto auidius amplectantur, ac minore fastidio perdiscant.

Girolamo Cardano (1545)

Divide 10 into two parts whose product is 40.

If the parts are x and $10 - x$, then

$x(10 - x) = 40$,

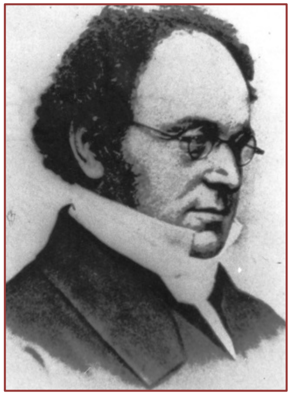
or $x^2 - 10x + 40 = 0$,

with solutions

$x = 5 + \sqrt{-15}$ and $x = 5 - \sqrt{-15}$.

Nevertheless we will operate, putting aside the mental tortures involved . . . So progresses arithmetic subtlety the end of which is as refined as it is useless.

The imaginary number √−1



Augustus De Morgan:

We have shown the symbol √−a to be void of meaning, or rather self-contradictory and absurd.

George Airy:

I have not the smallest confidence in any result which is essentially obtained by the use of imaginary symbols.

Gottfried Leibniz:

The imaginary numbers are a wonderful flight of God's spirit: they are almost an amphibian between being and not being.

Complex numbers

For many purposes our ordinary numbers are enough.
But let's now allow this object called $\sqrt{-1}$
(or i, as Euler named it).
We can then calculate as follows:

Addition: $(a + bi) + (c + di) = (a + c) + (b + d)i$

For example: $(1 + 3i) + (2 + i) = 3 + 4i$

Multiplication:

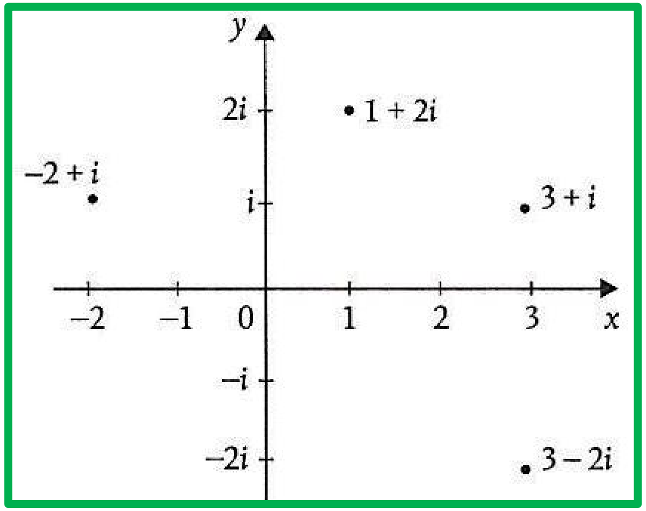
$$(a + bi) \times (c + di) = ac + adi + bci + bdi^2$$
$$= (ac - bd) + (ad + bc)i$$

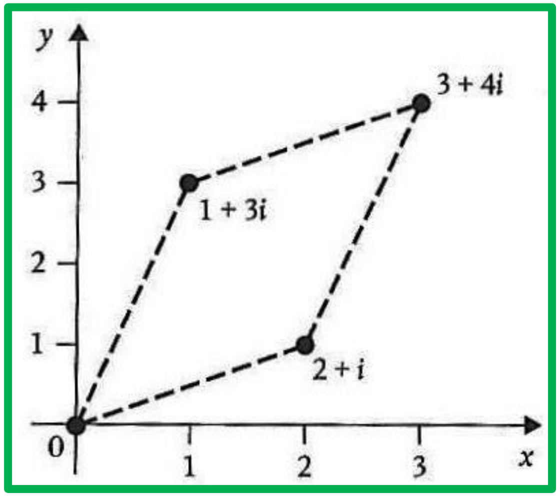
For example: $(1 + 3i) \times (2 + i) = (2 - 3) + (1 + 6)i = -1 + 7i$

The complex plane

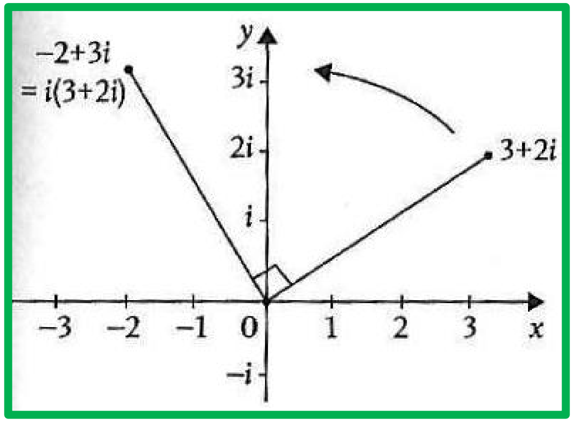
[Caspar Wessel (1799), and later, Gauss and Argand]

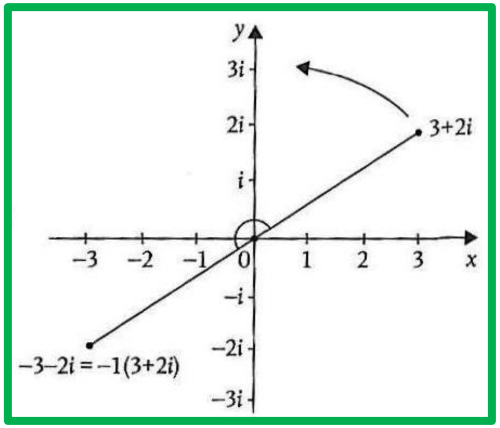
Represent complex numbers *a* + *bi* by points (*a*, *b*) in the plane:



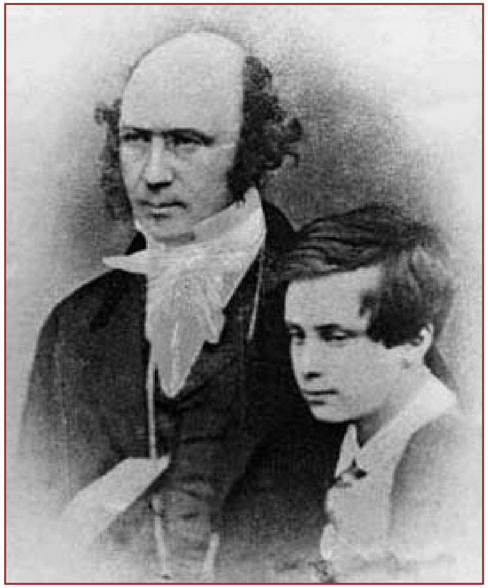


We can also multiply by *i* or *i*^{2} by rotating through 90° or 180°





William Rowan Hamilton



Define complex numbers as **pairs** (a, b) which combine as follows:

$$(a, b) + (c, d) = (a + c, b + d)$$

$$(a, b) \times (c, d) = (ac - bd, ad + bc)$$

How about three dimensions: $a + bi + cj$, with $i^2 = j^2 = -1$?

$$(a + bi + cj) + (d + ei + fj) \quad \text{OK}$$

$$(a + bi + cj) \times (d + ei + fj) = ??? \qquad [i \times j = ???]$$

Every morning on my coming down to breakfast, your brother William Edwin and your yourself used to ask me. ‘Well, Papa, can you multiply triplets?’ Whereto I was always obliged to reply with a sad shake of the head, ‘No, I can only add and subtract them.’

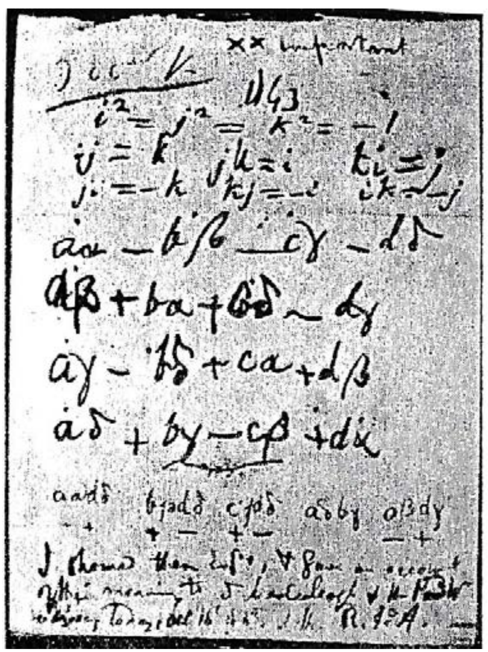
Hamilton's quaternions

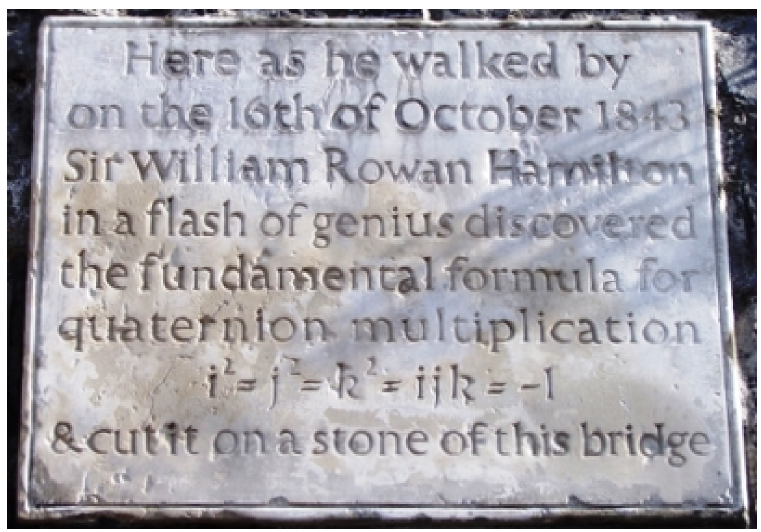
a + bi + cj + dk,

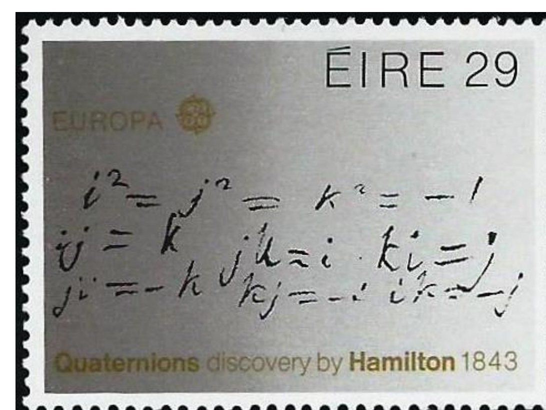
where $i^2 = j^2 = k^2 = -1$

and $ijk = -1$

$ij = -ji,\ jk = -kj,\ ki = -ik,$

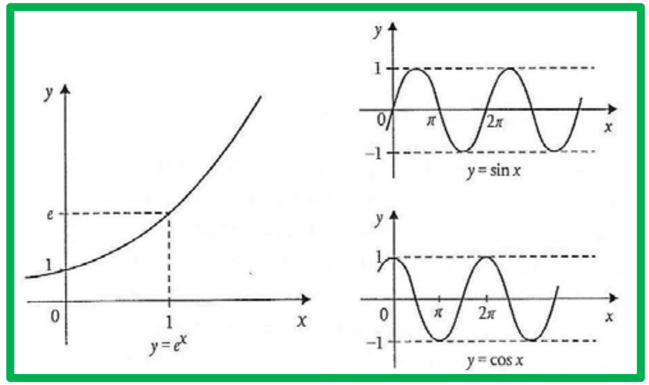






Euler's identity

$$e^{ix} = \cos x + i \sin x$$



$$e^x = 1 + x/1! + x^2/2! + x^3/3! + x^4/4! + x^5/5! + \ldots$$

$$\cos x = 1 - x^2/2! + x^4/4! - \ldots$$

$$\sin x = x - x^3/3! + x^5/5! - \ldots$$

so

$$e^{ix} = 1 + ix/1! + (ix)^2/2! + (ix)^3/3! + (ix)^4/4! + (ix)^5/5! + \ldots$$

$$= (1 - x^2/2! + x^4/4! - \ldots) + i\,(x - x^3/3! + x^5/5! - \ldots)$$

$$= \cos x + i \sin x$$

$$\boxed{e^{ix} = \cos x + i \sin x}$$

Euler's identity: $e^{iv} = \cos v + i \sin v$

138. Ponatur denuo in formulis §. 133, Arcus z infinite parvus, & sit n numerus infinite magnus i, ut iz obtineat valorem finitum v. Erit ergo $nz = v$; & $z = \frac{v}{i}$, unde $\mathit{fin.}\, z = \frac{v}{i}$ & $\mathit{cof.}\, z = 1$; his substitutis fit $\mathit{cof.}\, v =$

$$\frac{\left(1 + \frac{v\sqrt{-1}}{i}\right)^i + \left(1 - \frac{v\sqrt{-1}}{i}\right)^i}{2};$$ atque $\mathit{fin.}\, v =$

$$\frac{\left(1 + \frac{v\sqrt{-1}}{i}\right)^i - \left(1 - \frac{v\sqrt{-1}}{i}\right)^i}{2\sqrt{-1}}.$$ In Capite autem praecedente vidimus esse $\left(1 + \frac{z}{i}\right)^i = e^z$, denotante e basin Logarithmorum hyperbolicorum: scripto ergo pro z partim $+v\sqrt{-1}$ partim $-v\sqrt{-1}$ erit $\mathit{cof.}\, v =$

$$\frac{e^{+v\sqrt{-1}} + e^{-v\sqrt{-1}}}{2}$$ & $\mathit{fin.}\, v = \frac{e^{+v\sqrt{-1}} - e^{-v\sqrt{-1}}}{2\sqrt{-1}}$.

Ex quibus intelligitur quomodo quantitates exponentiales imaginariae ad Sinus & Cosinus Arcuum realium reducantur. Erit vero $e^{+v\sqrt{-1}} = \mathit{cof.}\, v + \sqrt{-1}.\mathit{fin.}\, v$ & $e^{-v\sqrt{-1}} = \mathit{cof.}\, v - \sqrt{-1}.\mathit{fin.}\, v$.

Euler's equation:

In Euler's identity, $e^{ix} = \cos x + i \sin x$, put $x = \pi$ (= 180°):

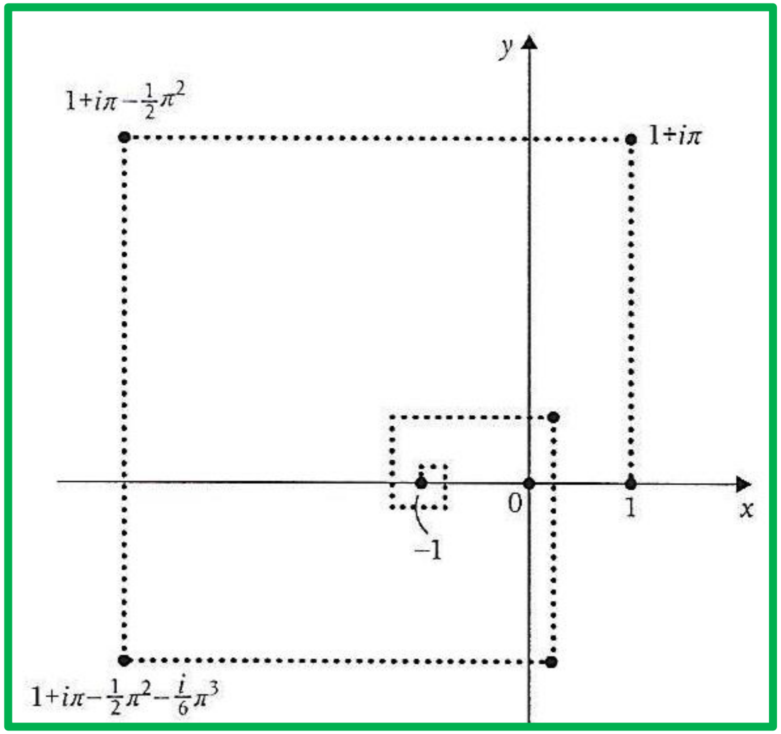
since $\cos \pi = -1$ and $\sin \pi = 0$, $e^{i\pi} = -1$ (or $e^{i\pi} + 1 = 0$).

But $e^x = 1 + x/1! + x^2/2! + x^3/3! + x^4/4! + x^5/5! + \ldots$

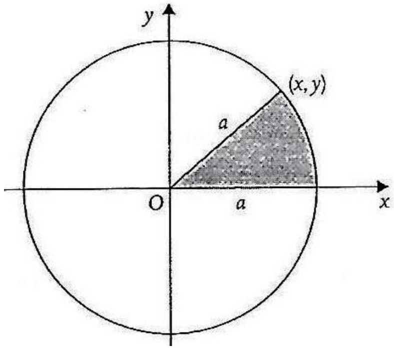
So $e^{i\pi} = 1 + i\pi/1! + (i\pi)^2/2!$

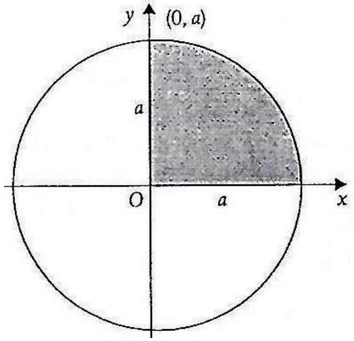
$+ (i\pi)^3/3! + (i\pi)^4/4! + \ldots$

$= 1 + i\pi - \pi^2/2 - i\pi^3/6 + \pi^4/24 + \ldots = -1$



A near-miss by Johann Bernoulli





Bernoulli found the area A of the sector on the right to be

$$A = (a^2/4i) \times \log \{(x + iy)/(x - iy)\}.$$

Euler then put $x = 0$:

$$A = (a^2/4i) \times \log (-1) \;=\; \pi a^2/4 .$$

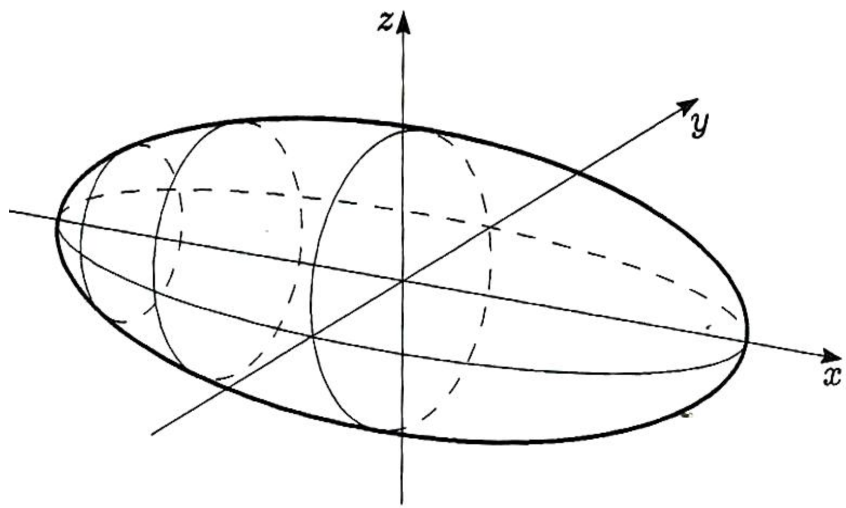
So $\log (-1) \;=\; i\pi$

Taking exponentials then gives Euler's equation in the form $e^{i\pi} = -1$

Another near-miss: Roger Cotes

Roger Cotes introduced radian measure for angles, and worked with Newton on the *Principia Mathematica*.

He also investigated the surface area of an ellipsoid, and found two expressions for it involving logarithms and trigonometry, and both involving an angle φ.



Equating these, he found that $\log(\cos\varphi + i\sin\varphi) = i\varphi$.

Taking exponentials then gives Euler's identity

Who discovered 'Euler's equation'?

Bernoulli/Euler: $i\pi = \log(-1)$

Roger Cotes: $\log(\cos\varphi + i\sin\varphi) = i\varphi$

Euler seems never to have written down $e^{i\pi} + 1 = 0$
explicitly – though he surely realised
that it follows from his identity, $e^{ix} = \cos x + i\sin x$.

We don't know who first stated it explicitly
– though there's an early appearance in 1813-14.

Most people attribute it to Euler,
to honour this great mathematical pioneer.

Who discovered ‘Euler’s equation’?

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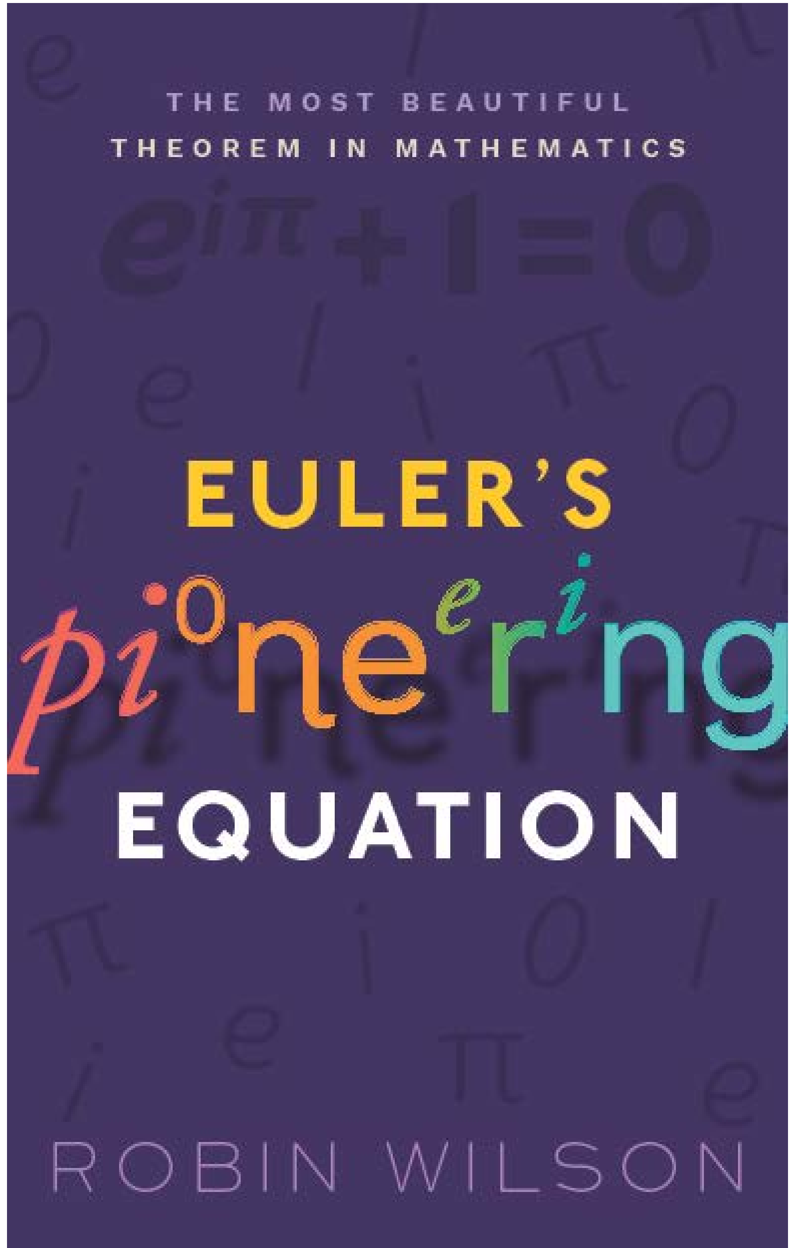
THE MOST BEAUTIFUL
THEOREM IN MATHEMATICS
$e^{i\pi}+1=0$
EULER'S
pioneering
EQUATION
ROBIN WILSON