Gresham College

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Can You Do Mathematics In A Crowd?

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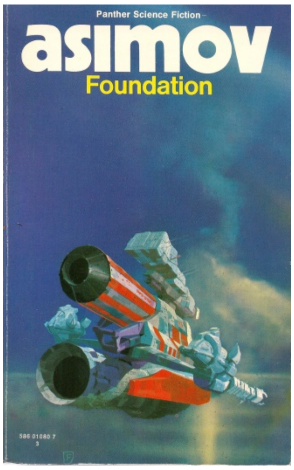
1. Introduction

At a recent maths conference that I attended, on the subject of the impacts of climate change on the human environment, I met a researcher who told me that their current project was to 'model Dorset'. When I pressed them on what they meant by this, they said that they were trying to model how the tourist industry and associated local businesses in Dorset would be affected by possible changes in the weather and the climate. This would involve trying to model how crowds of people would react to such events as an extreme weather situation. Naturally this proved to be a very challenging problem, but on the grounds that 'some model is better than no model at all' the researcher had come to the conference to present their findings and to seek advice. (For the record I was there to talk about the more mundane subject of peat bogs.) This project was an example of a common problem in applied mathematics, that is, how do you model the collective behaviour of a large group of individuals. If those individuals are people, what sort of rules can we come up with which mimic the way that free thinking individuals with free will can behave? This is far from an academic exercise. Many organisations rely for their operation on being able to control crowds safely and effectively. These include the police, the designers of sports stadia, London Underground, and even the Olympic Committee.

More generally if a system is governed by any set of rules, can we classify the *dynamical behaviour* that it can exhibit? It turns out that we can often make good predictions of this even if we don't know the precise rules by which it operates. This at least gives us some hope that we can make sense of the motion of a crowd, even if we can't control the behaviour of an individual. An early advocate of this idea was Isaac Asimov. We have already met Asimov in the lecture on robots, in which he foresaw some of the recent developments in machine learning. In his classic *Foundation Trilogy*, the mathematician Hari Seldon (along with Professor Moriarty, one of the greatest fictional mathematicians of all time) develops a branch of mathematics known as psychohistory which is a form of mathematical sociology and which can predict the behaviour of large crowds of people. It is so good at this that it could predict the (large scale) future. By applying it Hari Seldon saw the imminent fall of the Galactic Empire and also a way to return it to civilisation.



Whilst this is of course science fiction, it certainly contains a germ of truth. The ambition of this lecture is not as great as that of Asimov and Seldon, nor will it ever be, but I hope that I will be able to convince you that mathematics can still say something useful about the mass behaviour of large numbers of individuals.

2. Examples of Complexity and Collective Behaviour

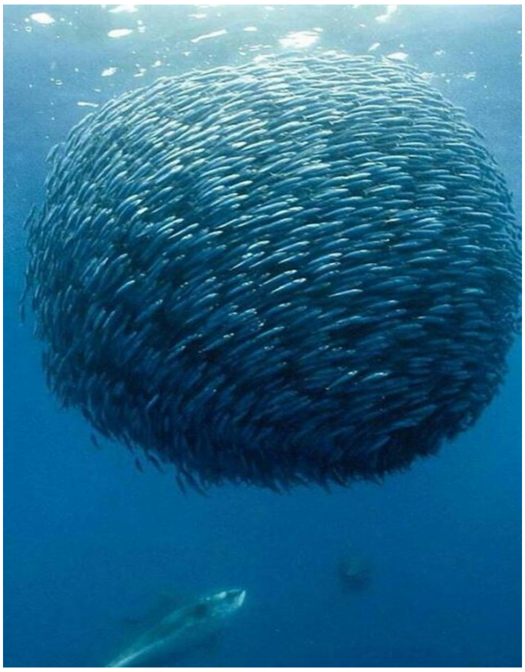
Anyone who has driven a car in heavy traffic will be aware of the fact that the traffic can seem to have a mind of its own, and that you have very limited ability to make decisions. A classic example of this occurs when you travel down a motorway and the traffic suddenly becomes much denser and slows down. After a period of driving slowly, the traffic starts to speed up and returns to normal. Then after a while of normal driving along the motorway, the whole process of slowing down and speeding up repeats itself, possibly many times. However, throughout all of the driving you see nothing at all which indicates why this is happening. What you have just experienced is a *phantom traffic jam*. A car slows down for some reason, as it does so the cars behind it slow down, and a shock wave passes down the line of the traffic as each car slows down. The car which originally slowed down will then speed up. But the damage has been done, and the shock wave continues to propagate backwards and lasts much longer than the original incident.

This shock wave is an example of an *emergent behaviour*. In this a large scale effect, or pattern, emerges from the *interaction* of all of the components of the system, rather than the individual behaviour of the components. Emergence is a defining property of complex systems where the collective behaviour is different from the sum of the parts of the system, and emerges from their interactions. Many natural systems exhibit emergent behaviour, indeed if they didn't they would be very hard to study. In particular, emergent behaviours, such as the shock waves in our traffic network, can often be much 'simpler' than the behaviour of the individual components. This simplicity allows us to describe them by straightforward mathematical rules, and we can also classify some of the most like forms of their behaviour. An example from the physical science is the laws of fluid motion, which emerge from the complex interaction of many millions of air molecules. Another is the pattern of a snowflake which emerges from the structure of the underlying crystal lattice of the ice as the water freezes.

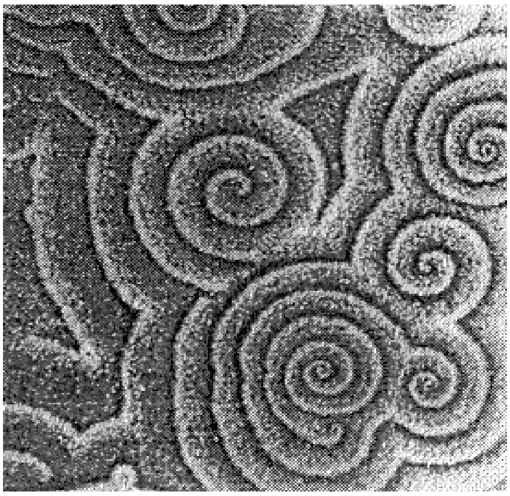
On the largest scale the lovely spiral structure of the galaxies emerges from the complex interactions of many, many stars under the action of gravitational forces.

We also see emergent patterns in biological systems. Indeed this is the real motivation for this lecture. Anyone who has watched a swirling flock of starlings (often called a *murmeration of starlings*) illustrated below, cannot fail to be impressed by the way that the flock often seems to behave as though it was a single coordinated organism.

Much of this behaviour has evolved to counter predators such as hawks. The flock, whilst made up of birds which are small, collectively behaves as though it were much larger. This often causes a predatory hawk to become confused and to break off the attack. Just as in the case of the traffic, the behaviour of the flock is governed by the way that the starlings interact with each other. Each starling has 'rules' which govern how it reacts to nearby birds, such as getting closer to some birds, staying further away from others, and aligning themselves with the overall flow. The combination of these individual rules is the emergence of the collective behaviour of the flock. We will look at how these rules operate in later sections. We see similar collective behaviour in shoals of fish, particularly when they are trying to avoid predators (see the fish ball illustrated below), termites, bees, herds of cows, and, of course, crowds of people in places such as railway stations and sports stadia.



Emergent behaviour can also be seen at a much smaller scale. Below is the pattern formed by the *Dictyostelium slime mould*. This elegant shape is called a *scroll wave* and it can also be seen in quite different systems, such as certain chemical reactions.



The process driving this pattern formation in slime mould is called *chemotaxis*. The individual slime mould cells of density u release chemicals of density v in response to certain stimuli. These chemicals then diffuse through the medium containing the cells. As they diffuse they spread out and can then be detected by other slime mould cells. The slime mould then moves in the direction of the increasing gradient of the chemical. The entire process can be given a mathematical formulation as follows which describes how the cell density u and the chemical density v evolve in time and space.

$$u_t = \nabla(k_1(u,v)\nabla u - k_2(u,v)u\nabla v) + k_3(u,v)$$
$$v_t = D_v \nabla^2 v + k_4(u,v) - k_5(u,v)v$$

These equations then have evolving solutions for u and v which give the patterns above. Personally I find it remarkable that the almost random motion of many different cells can produce patterns of such striking order, and indeed that mathematics can be applied in this way at all. It is as though the slime mould cells are behaving as if they were a single organism. On a slightly larger scale, it is a similar mechanism of following pheromones which governs the behaviour of termites, illustrated below.

Elegant patterns are also seen of course on the bodies of many animals, from leopards, zebras and tigers, to butterflies and beetles.

A first mathematical study of these was made in 1940s by the great Alan Turing (of computer and code breaking fame). They were found to be generated by similar reaction-diffusion mechanisms, driven by differential equations, as for the slime mould system. As in these systems, the reaction diffusion equations have solutions which display strongly ordered patterns which emerge from the chemical interactions in the animal's coat. It is then possible to analyse the types of patterns which arise, and to explain why spotted and striped coat patterns arise. These models are also predictive and lead to the important mathematical result that:

Spotty animals can have striped tails, but striped animals can't have spotty tails.

Details of how this process works can be found in the wonderful text book by Jim Murray [1] which goes in great depth into the way that many biological systems obey mathematical rules.

Much has been written about emergent behaviour, and extravagant claims have been made for it. It is claimed that order will generally emerge from complex systems random interactions if we leave them to themselves. My own studies of such complex systems indicate that this is far from the case, and that whilst simple ordered patterns can emerge, it is usually more common to see disorder instead. Certainly I find rather far-fetched the claims that consciousness is just an example of an emergent phenomenon.

However, in many complex systems patterns *do* emerge, and can be classified. The most common types of pattern, which emerge after a suitably long time, are described by the mathematics of dynamical systems. A very readable account of the mathematics behind this is given in the book [2] by Steve Strogatz. In particular we expect to see the following:

1. Patterns which are constants, with everything the same, or are slowly varying.
2. Periodic repeating patterns, such as stripes and spots and the waves in traffic.
3. Chaotic patterns, which look disordered but are in fact highly structured.

We will now look at some of the ways that mathematical models and algorithms can reproduce some of the patterns that we have observed above in complex systems. In particular study how the emergent behaviour demonstrated in the last section actually arises.

3. Cellular Automata and Agent Based Models

3.1 Overview

There are powerful mathematical methods for both simulating and understanding collective behaviour. These include *cellular automata* (CAs) and *Agent Based Models* or ABMs. In a CA cells in the automaton can change their state (but not their position) as a result of simple rules, usually described by algebraic formulae. In the more sophisticated ABMs the agents are the individuals that we want to simulate, for example the starlings, cars, people or even businesses. For each such individual we then devise a set of rules for how that agent interacts both with its general environment and all of the other agents. These systems can be very complex, with many thousands of agents all described by nonlinear ordinary differential equations. The agents can both change their state and change their position as a result of the action of these equations. However, they can also be simple, with a limited number of agents, all in a line, and interacting according to simple rules. What is remarkable is that we see two, almost contradictory, types of behaviour. In case of many agents with complex interactions we can see strikingly ordered patterns emerging. In contrast, even simple systems of interacting agents can produce very rich and complex patterns of behaviour.

3.2 Cellular Automata

We will start by looking at the behaviour of a *cellular automaton (CA)* and at some of the systems which they can simulate. CAs are widely used in physics, chemistry and biology to model many types of natural phenomena, such as phase changes in materials and infection mechanisms. They also turn out to have pleasing applications in recreational mathematics including knitting. Cellular automata can display a wide variety of the dynamical behaviour which we described in the previous section. The concept of the CA was developed in the 1940s by Stanislaw Ulam and John von Neumann (who we met in the lecture on machine learning) while they were at the Los Alamos National Laboratory. In the 1970s Conway's Game of Life, a two-dimensional cellular automaton, led to an interest in them which expanded beyond academia and into recreational maths after it was popularised in Scientific American. In the 1980s, Stephen Wolfram (of Mathematica fame) engaged in a systematic study of one-dimensional cellular automata. Wolfram published *A New Kind of Science* [3] in 2002, claiming that cellular automata have applications in many fields of science. These include computer processors and cryptography.

In a cellular automaton we have a series of generations that are represented by numbers in cells. We then have a simple rule that allows us to compute the next generation. Remarkably even simple rules can, as we will see, generate complex patterns. It is suspected that this may be similar to the mechanisms behind some of the natural phenomena, which we looked at in the previous section.

In the CA the cells may be arranged in a line (one-dimensional systems) or in a square or even cubic lattice (two and three dimensional). We consider a set of one-dimensional examples first. Here is a typical starting grid of cells for a one-dimensional CA.

a	b	c	d	e
A	B	C	D	E

In this table, the lower case letters refer to the cells in the first generation of the CA and the upper case to the second generation of the CA. The values of the upper case letters are then given by a rule, applied to the lower case letters so that the first generation evolves into the second. A simple example might be to take an average in each cell away from the boundary. So that, for example

$$B = (a + b + c)/3, \qquad C = (b + c + d)/3, \qquad D = (c + d + e)/3.$$

We can also keep the end values fixed so that $A = a$ and $E = e$.

These rules are then applied repeatedly from one generation to the next to give an evolutionary pattern. What is interesting about this case is that, regardless of the values of the digits b, c, d, the pattern which emerges is always the same and is a linear function interpolating a and e. So that if we have a = 1 and e = 5, then the final pattern is always

1 2 3 4 5.

As an example of a different type of behaviour which can arise in a CA, we consider a line of *ones and zeros* only in the first generation such as

1 1 0 1 0 1 1 0 0 1 0 1 0 1 1 0 0 0 1 0 1 1 1 0

Again, we will think of this as the *first generation* of the systems that we want and use this first generation to generate a second and future generations. A simple rule is as follows

- Keep the first digit as 1 and the last digit as 0
- Replace each digit by the **Exclusive Or** of the two digits on either side

Here the Exclusive Or (ExOr) of the two digits a and b is defined by

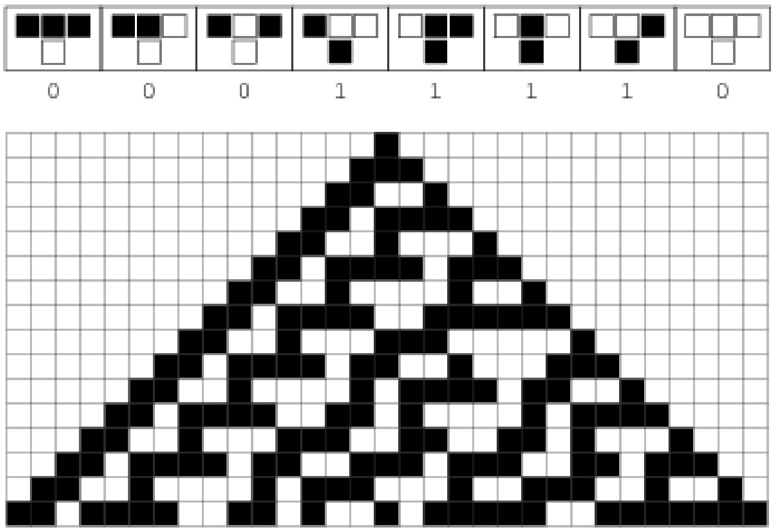
a ExOr b = 1 if a and b differ, and a ExOr b = 0 if they are the same.

If we apply this rule to the above first generation and proceed from one generation to the next then we get the following table

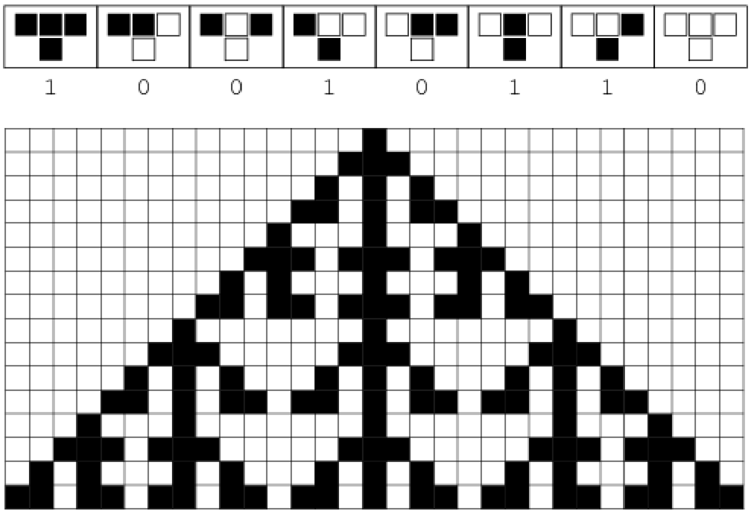
1 1 0 1 0 1 1 0 0 1 0 1 0 1 1 0 0 0 1 0 1 1 1 0
1 1 0 0 01 0 1 1 00 0 01 1 1 0 1 0 0 1 01 0
1 1 1 0 10 0 1 1 1 00 1 1 01 0 01 1 0 00 0
1 0 1 0 01 1 1 0 1 1 1 1 1 00 1 11 1 1 00 0
10 0 1 1 1 0 1 0 000 0 11 1 1 0 0 01 1 0 0

It is clear from this example that even a simple rule can lead to a complex pattern. Wolfram gave a series of different rules for one-dimensional CAs many of which generated remarkably complicated patterns. Here are the results of applying Rule 30 and Rule 150 which work on 1s (black) and 0s (white). I hope that these pictures are self-explanatory. (The first generation starts at the top of each figure.)

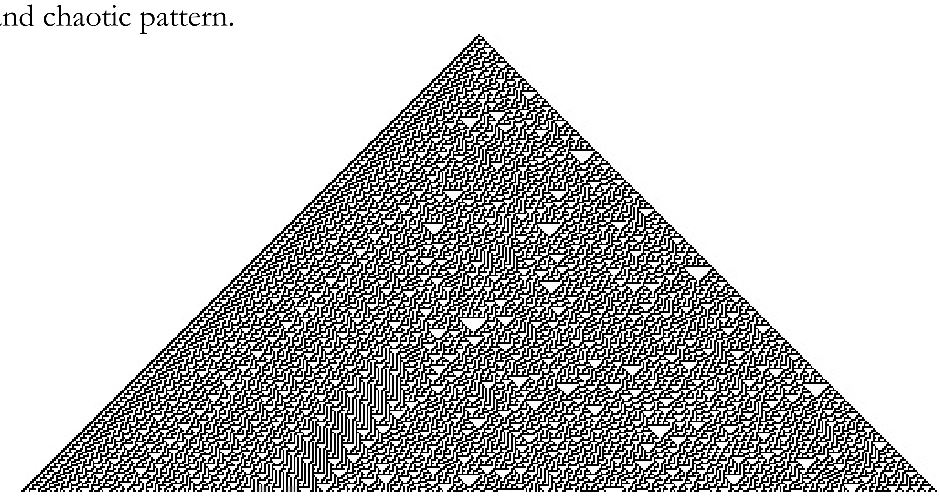
rule 30



rule 150

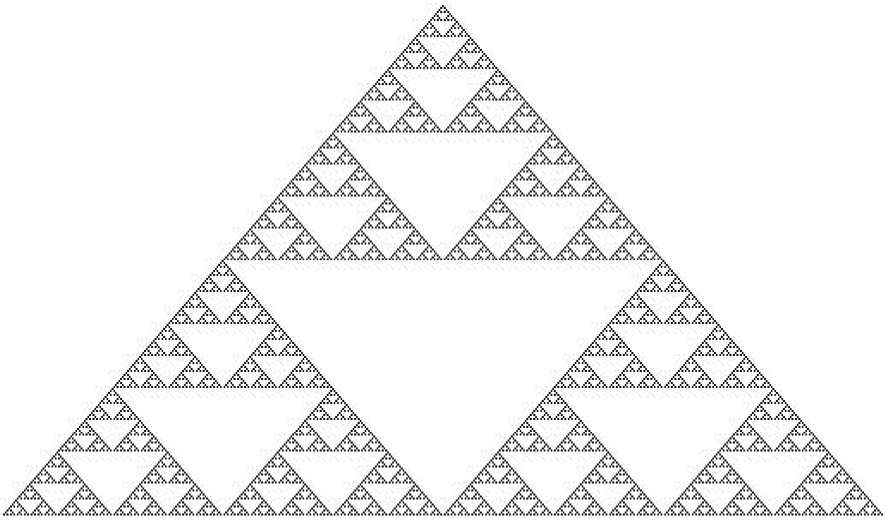


If we continue to apply Rule 30 then we get the following pattern, which is remarkable for its complex structure. Note that on the left it appears to be quite regular, but this regularity vanishes as we move to the right, where we see a disordered and chaotic pattern.



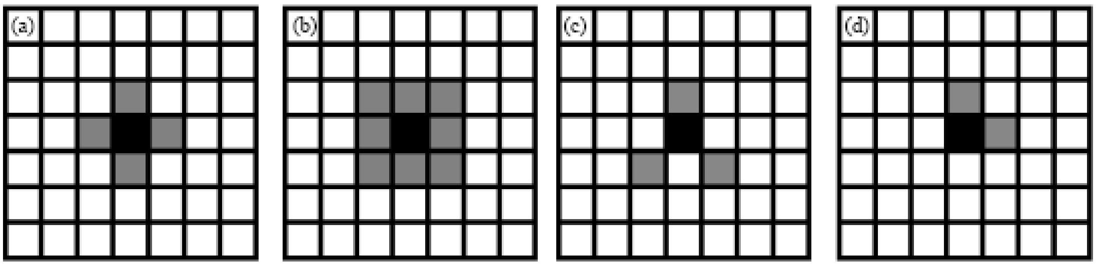
A classification of cellular automata was given by Wolfram, and the various patterns follow the description of possible dynamic states that we gave at the end of the last section. Wolfram described automata in which patterns generally stabilize into homogeneity, automata in which patterns evolve into oscillating structures, automata in which patterns evolve in a seemingly chaotic fashion, and automata in which patterns become extremely complex and may last for a long time, with stable local structures.

We can see all of this structure in the figure above which is created on a grid in which the cells of successive generations are offset by one half square from those below, and the evolutionary rule is that the cell value in the next generation is the ExOr of the cell values on either side. This is exactly the pattern if you colour the Odd values of Pascal's triangle black, and the Even values white. The pattern has a very rich structure, and is in fact a *fractal* called the *Sierpinski Gasket* [4], with the same structure repeated at smaller and smaller scales.



Remarkably, we see very similar patterns in nature, for example on shells. Here is an example of a *Conus Textile* shell with a very similar pattern to that of Wolfram's Rule 30 and presumably, is driven by a similar rule.

We can extend the above idea to that of a **two dimensional Cellular Automaton**. In these we take numbers in a two dimensional grid as a first generation. All of these values are then replaced in the next generation according to a set of simple rules. Two-dimensional CAs go back to Von Neumann who worked with them in the 1940s and are often used to model disease and infection processes. In general a set of several cells in the first generation are used to produce the next. The choice of which cells are used to update the next generation depends on the type of CA used. Here are some different examples.

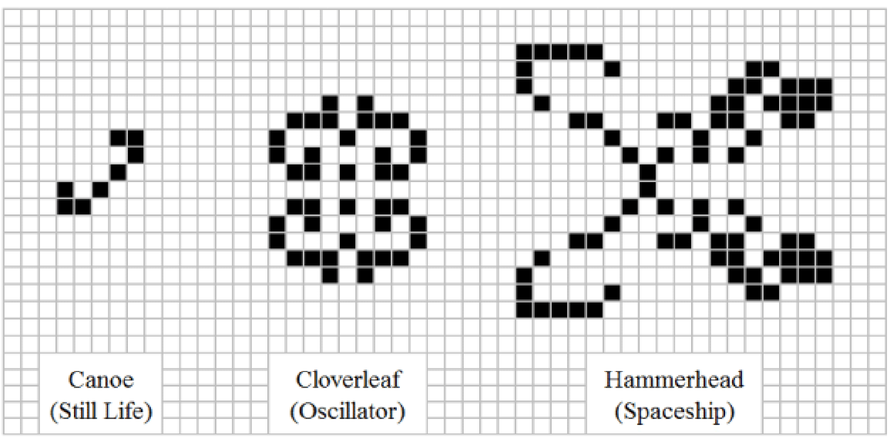


Neumann Moore Cole Smith

The **Game of Life** is an example of such a *two-dimensional CA* which was invented by John Conway and is supposed to produce patterns which resemble living organisms. The Game of Life is played on an infinite two-dimensional grid of square "cells", each of which is in one of two possible states, *alive* (black) or *dead* (white). Every cell interacts, using a Moore grid as above, with its eight *neighbours* which are the cells that are horizontally, vertically, or diagonally adjacent. At each step in time, the following transitions occur:

1. *Any live cell with fewer than two live neighbours dies, as if caused by underpopulation.*
2. *Any live cell with two or three live neighbours lives on to the next generation.*
3. *Any live cell with more than three live neighbours dies, as if by overpopulation.*
4. *Any dead cell with exactly three live neighbours becomes a live cell, as if by reproduction.*

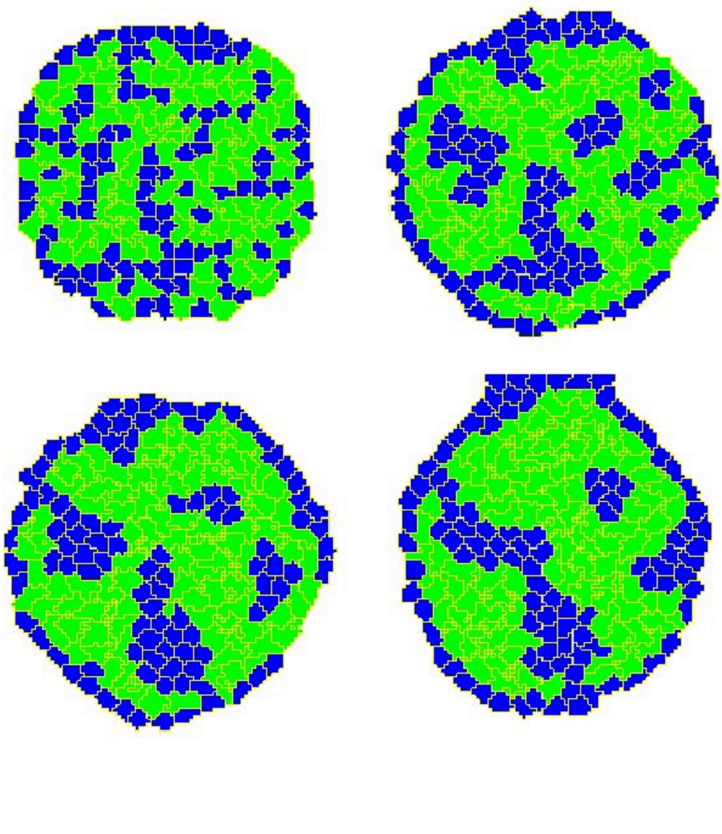
The initial pattern constitutes the *seed* of the system. The first generation is created by applying the above rules, simultaneously, to every cell in the seed. As with the one-dimensional CA the rules continue to be applied repeatedly to create further generations. The game of Life was first brought to the public attention in the October 1970 issue of *Scientific American*, in Martin Gardner's Mathematical Games column. It is characterised by having many exotic patterns. Pictured below are three of the stable patterns which are either fixed in time or periodically repeat.



More complex and time evolving patterns are also possible. One of these is the 'glider' which as its name implies glides across the grid, and the 'Gosper glider gun' which produces an unending set of gliders.

Claims have been made that the Game of Life mimics real life processes (and even that the whole of humanity might just be the agents of some CA being played by extra-terrestrial beings!). Whilst it can be shown that it is essentially equivalent to a Turing machine, and thus this is 'theoretically possible' it is extremely far-fetched. Indeed, John Conway designed the game of life by looking for the simplest possible set of rules that would lead to complex, evolving, patterns. More examples of games devised by Conway can be found in the book [5].

However, it is certainly true that two (and indeed three) dimensional CAs are used very effectively to model the action of bacteria, disease and infection. In particular we can model biological cells by the cells in the grid, and also model bacteria by other cells. Just as in the Game of Life we can look at biological cells and bacteria as being alive or dead. Such models were the original motivation for the work of Von Neumann and are used predictively to study many types of disease propagation. In the figure below, you can see a simulation of how the cells of an immune system (green) fight a bacterial infection (blue).



3.3 Agent based models

Agent Based Models (ABMs) are an extension of the idea of the cellular automaton. In an ABM we have a number of agents which (as in a CA) interact with other agents, and which often move under the effects of this interaction. Individual agents are typically characterized as rational and are presumed to be acting in what they perceive as their own interests, such as reproduction, economic benefit, or social status, ABM agents may also learn from their experiences. Most agent-based models are composed of:

1. Numerous agents specified at various scales (typically referred to as agent-granularity).

2. A set of decision-making rules for each agent.
3. A set of learning rules for each agent.
4. A space in which the agents can move/operate and an environment in which they can interact.

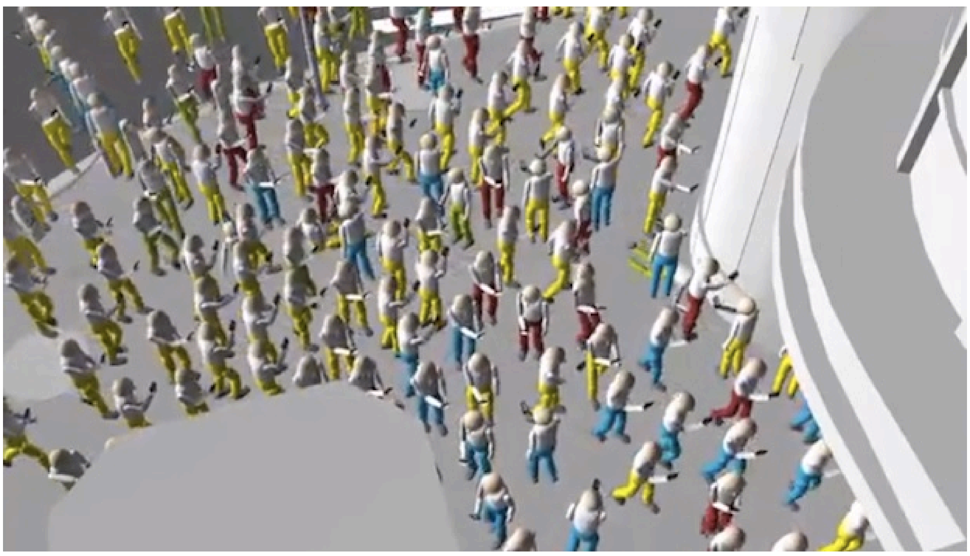
Typically, the agents obey nonlinear ordinary differential equations and the resulting ABMs are much too complex to be solved by hand. Instead they are usually the subject of heavy duty computing. The benefits of using such an ABM is that it is easy to change the parameters for the individual agents, and thus we can test by the computer simulation how changes in the behaviour patterns of individual agents will affect the system's emerging overall behaviour.

ABMs are now used widely in simulating systems in biology, sociology, economics and even management. We will now look at using them in two particular examples.

4. Modelling a crowd of people and a flock of birds using an ABM

When I first visited Japan I was giving a tour of lectures on the general subject of complexity. My hosts were very attentive to me, and suggested that I visit the Shibuya metro station. This has the 'distinction' of being (so it is claimed), the busiest metro station in the world. We sat in a coffee shop above the station where it was possible to observe the crowds below. Here is what we could see.

What we are looking at in this picture is a *scramble crossing*. When the traffic lights become red, thousands of people coming from many different directions cross at the intersection. It is a truly awe inspiring sight. Somehow it all works, and they get to their directions without colliding in the middle. So, how does a scramble crossing work in practice? We can discover this by using an ABM in which the agents are the people in the crowd and they all interact with each other, as illustrated in the simulation below.



Some of the fundamental work on the dynamics of crowds has been done by the Swiss mathematician/sociologist Dirk Helbing and his group [6]. Helbing makes the following assumptions about the motion of people in a crowd:

1. They have an overall goal, which may be to exit a building or to follow signage.

2. They cannot walk through walls or other solid obstacles.

3. They have an (individual/cultural) view on how close they may want to be to strangers.

4. They will want to be close to family or friends.

5. They will have a certain amount of randomness in their movements.

The crowd is then modelled as a set of agents satisfying differential equations for their movement and what Helbing called a *social force model* for their interactions. In Helbing's model [6], the position of the (agents) people in the crowd are denoted by r_i with velocity v_i. The people are then subjected to three forces. The first is a global attractive force f_i giving their intended motion. This is typically towards a target position p_i at a desired speed v*_i (which may evolve with time if the person is in a hurry for example). There is also a repulsive/attractive force f_{i,j} to avoid/come closer other people, and a similar repulsive force f_{i,B} to avoid obstacles. The first of these satisfy the differential equations:

$$dr_i/dt = v_i, \qquad dv_i/dt = f_i + f_{i,j} + f_{i,B}$$

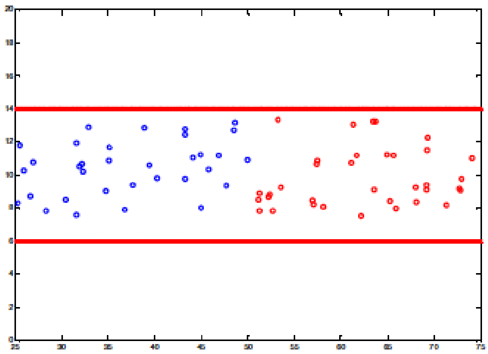
Global intent

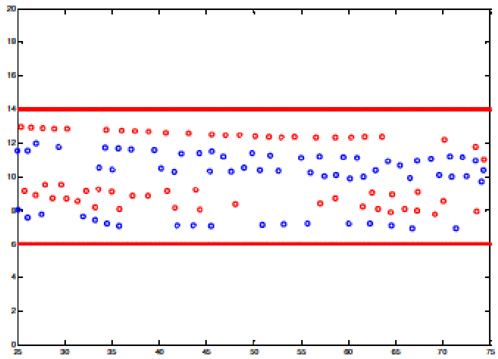
$$f_i = \frac{1}{\tau_i}\left[v_i^* \frac{p_i - r_i}{||p_i - r_i||} - v_i\right]$$

The social force f_{i,j} is rather more complex to model and must take individual preferences into account. It requires information on the size and personal space preferences of each agent, how well they can see and interact with other members of the crowd, so that pedestrians take more notice of people in front of them than behind them. Similar factors are used in the construction of the collision avoidance force f_{i,B}. These expressions are rather complex and I won't attempt to give them in detail here. More details are given in [6].

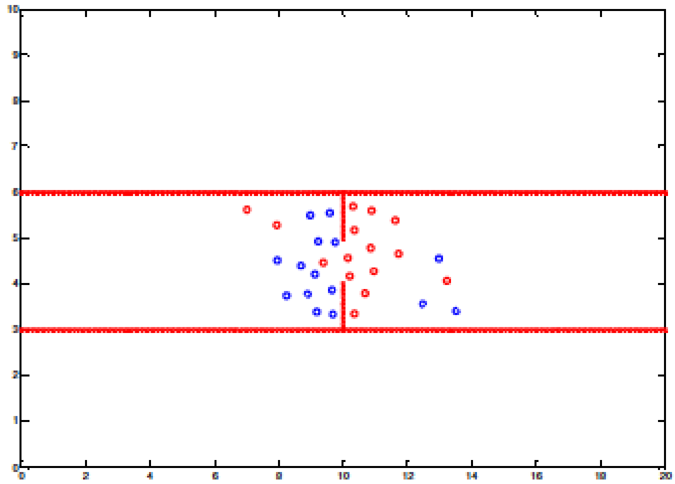
Although we are dealing with that most complex of all things, namely human behaviour, the crowd models actually do quite a good job in predicting the actual behaviour of crowds. They have now been well tested in experiments, including one memorable experiment which involved photographing undergraduates in various crowded situations, behaving as naturally as they could, given that they were wearing different coloured hats for easy identification. See reports of some of these experiments in [7].

As an example of this we consider a common example of two large groups of pedestrians moving in opposite directions in a corridor (this often happens in a train station for example as the passengers of a train disembark, to be met by another group trying to embark). Below we see the simulation of this using the ABM social force model. On the left we see the distribution before the pedestrians meet, and on the right, the pattern after the two groups have interacted. As you can see, the ABM has predicted that an emergent structure has arisen, with the pedestrians organising themselves into lanes. Next time you are in a similar situation have a look and see how true this is.





I have used these crowd models in the simulation, and design, of the lecture theatres at Bath. In an earlier design of one of our lecture theatres we were faced with the prospect of over 300 students exiting a lecture theatre through a single entrance, whilst at the same time the same number of students are trying to get in. In the simulation below you can see the results, in which the two groups of students are coloured red and blue. The outcome of these computer experiments was that the door arrangements at Bath were impractical, and a new door was provided for the exiting students, which greatly eased the congestion.



Models of crowds are now finding widespread applications. In particular they have been used to model the Olympic stadium and many other sports stadia. They are also used by the London Underground and even by the organisers of pilgrimages. The objective of these models is to find safe ways of moving large numbers of people around in both normal situations and also in emergency situations such as a fire of a terrorist attack. A recent and very profitable use of these has been in the movie industry. For example if a director wants to simulate a huge battle (of humans, or maybe trolls, orcs and elves) then it makes sense to use an agent based model to recreate the action. The models of traffic used to study the M25 are similar in many ways to the crowd models, with the obvious changes to account for the physics of the motion of a car, and the constraints of the road network.

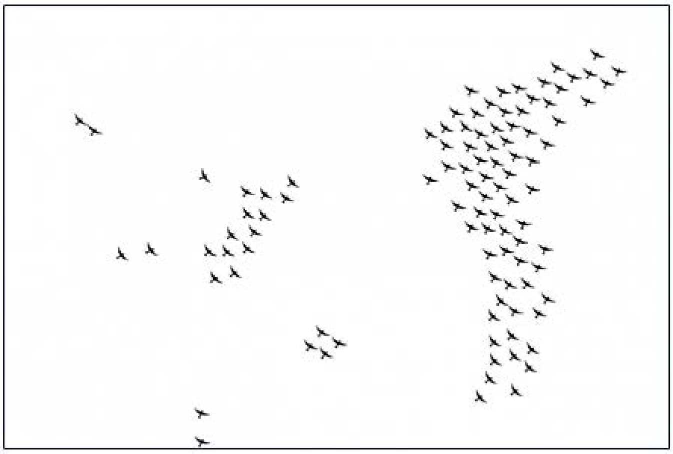
It is possible to model a **flock of birds** (and indeed a shoal of fish) in a similar way, however there are differences. One obvious one is that birds and fish move in three dimensions (they fly or swim), and also that they have less of a long term goal or a wish to follow signage. They also want to avoid predators. In this case of modelling a flock, the individual agents are the birds and these agents obey a set of three rules.

1. *Alignment*: Each bird aligns its flight with the average flight direction of the local flock.

2. *Cohesion:* Each bird moves towards the average position of the local flock.

3. *Separation:* Each bird tries to avoid local overcrowding, and predators.

These three rules, in varying strengths, combine to give emergent complex behaviours which can be simulated using an ABM. The rules are similar in many ways to those for crowds but have the additional features of the three-dimensional motion. The best models also include the effect of the motion of the air around the birds. If this is all included, the resulting models closely resemble actual flock behaviour.



There is now furious activity in improving these models and applying them to many other animal species. More details on flocking models can be found, for example in [8].

5. Where next

As I said in the introduction, one of the problems mathematicians are faced with is that of 'modelling Dorset'. In a future lecture I will look at how maths tells us what future cities may be like. Modelling Dorset is similar, but on a larger scale.

All we have to do is to identify the agents, which in this case are the many and varied local industries, together with the local population and the tourists, factor in the weather (and the effects of climate change), and predictions of the economy and the effects of Brexit, construct the rules of interaction, light the blue touch paper and hope that we get an answer which means something useful. Well, we will try at least. Watch this space.

6. References

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