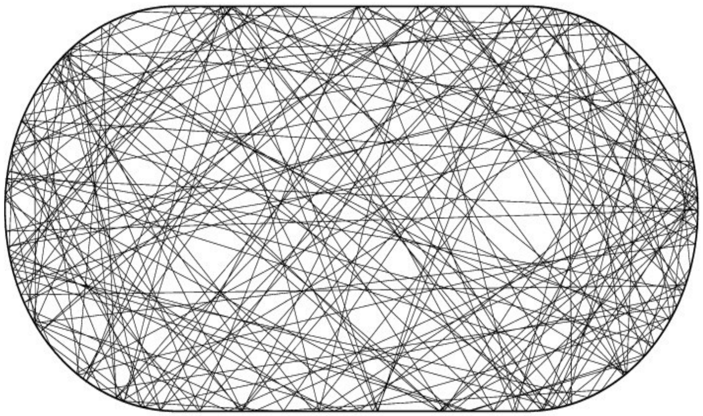
Chaos and the theory of change



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Is life predictable or unpredictable?

Can we tell what is going to happen

- In the next second?
- In the next hour?
- In the next year?

It is hard to predict anything, especially about the future.

Some things are predictable

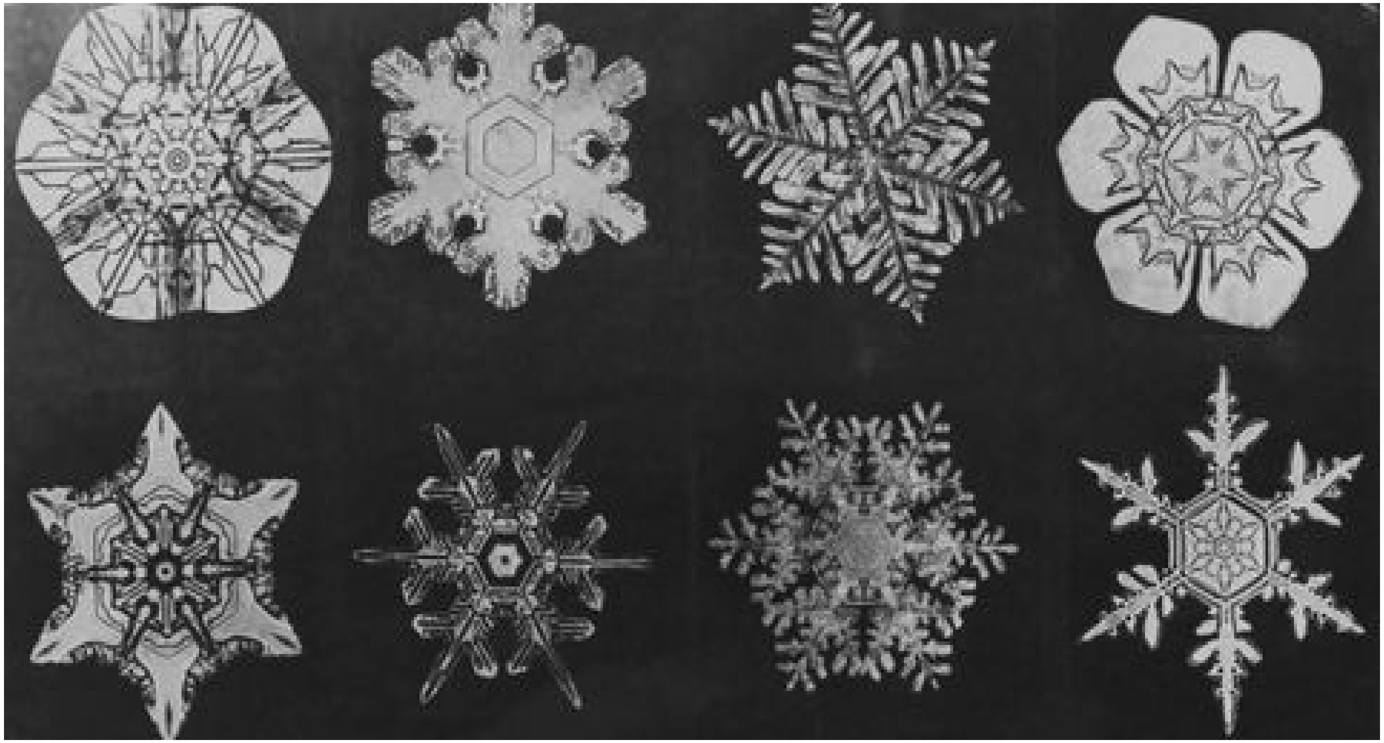
Others are not

Does nature have an underlying order and pattern?

First answer …. YES!

Science is the search for order and pattern in the universe

If we look we can see order and pattern all around us …..



Snow crystals

The animal world

Rock Folding

The motion of the planets

One of the first to realise this

Galileo (1564-1642)

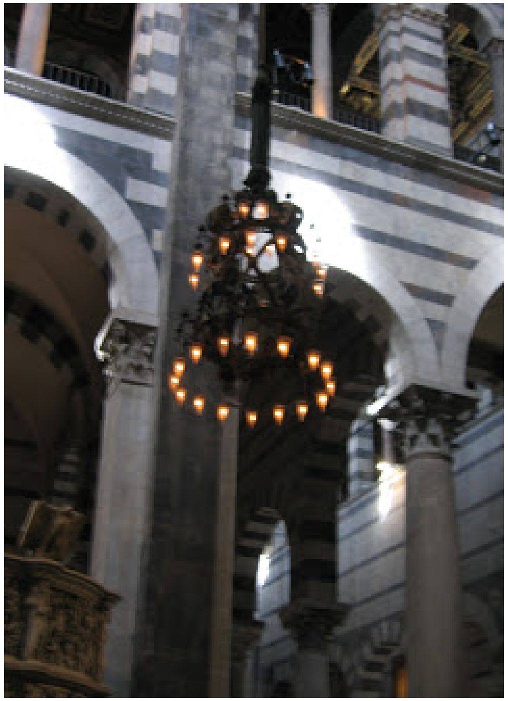
Pisa

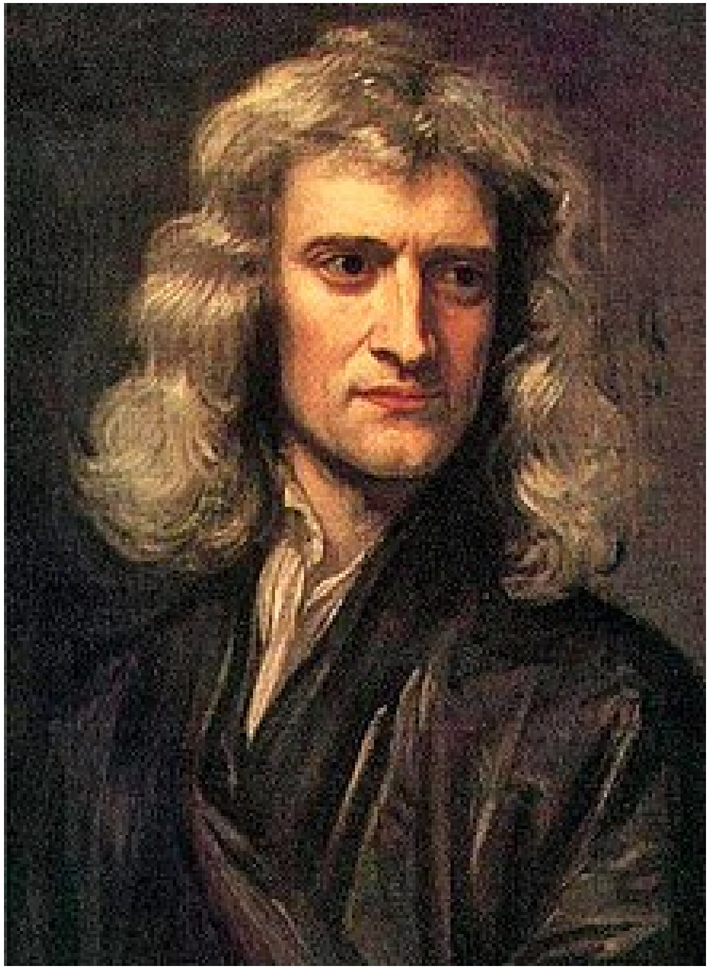
1581

Galileo watched a chandelier swing and realised that it was governed by predictable laws

Swing time was constant

- Regardless of how it was pushed
- Or where it was
- Or when





Isaac Newton (1643-1727)

1686

In the Principia Newton showed that this order and pattern could be expressed by using mathematics

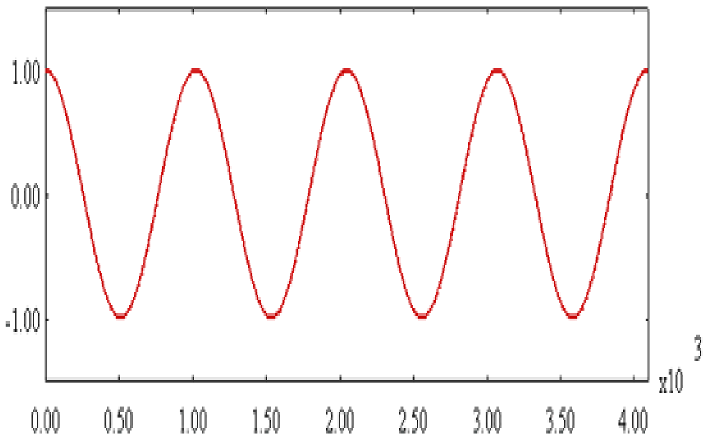
$$l\frac{d^2\theta}{dt^2} + k\frac{d\theta}{dt} + g\sin(\theta) = 0$$

Pendulum equation

Solution for small swings and negligible air resistance

is periodic and very predictable

$$\theta = A\cos\left(\sqrt{\frac{g}{l}}\,t\right).$$



Period

$$T = 2\pi\sqrt{l/g}$$

In perfect agreement with Galileo's observations

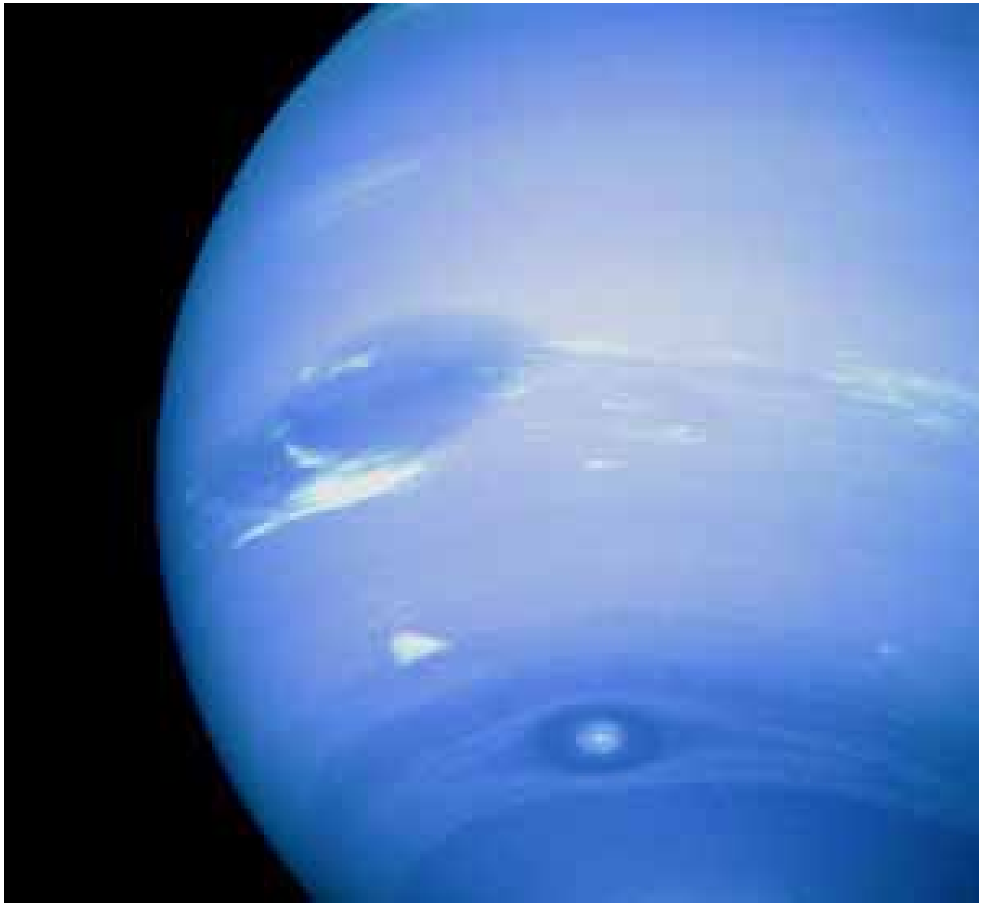
Key idea ....

- Write down the equations describing a physical system
- Solve the equations
- Predict the future

Does this work?

$$\frac{d^2x}{dt^2} = -\frac{GMx}{|x|^3}$$

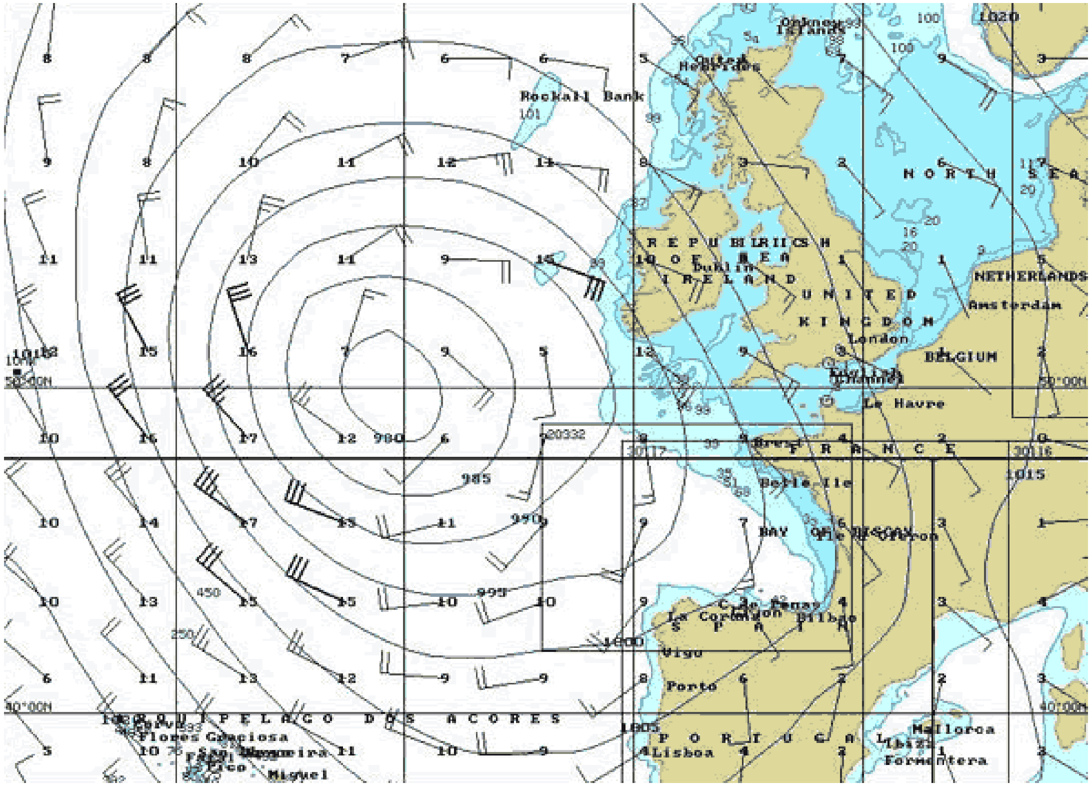
Newton's law of gravitation



Neptune: discovered by maths

$$u_t + u.\nabla u = -\nabla P + \frac{1}{Re}\nabla^2 u, \quad \nabla.u = 0.$$

Navier-Stokes equations



Weather forecasting under a week ahead

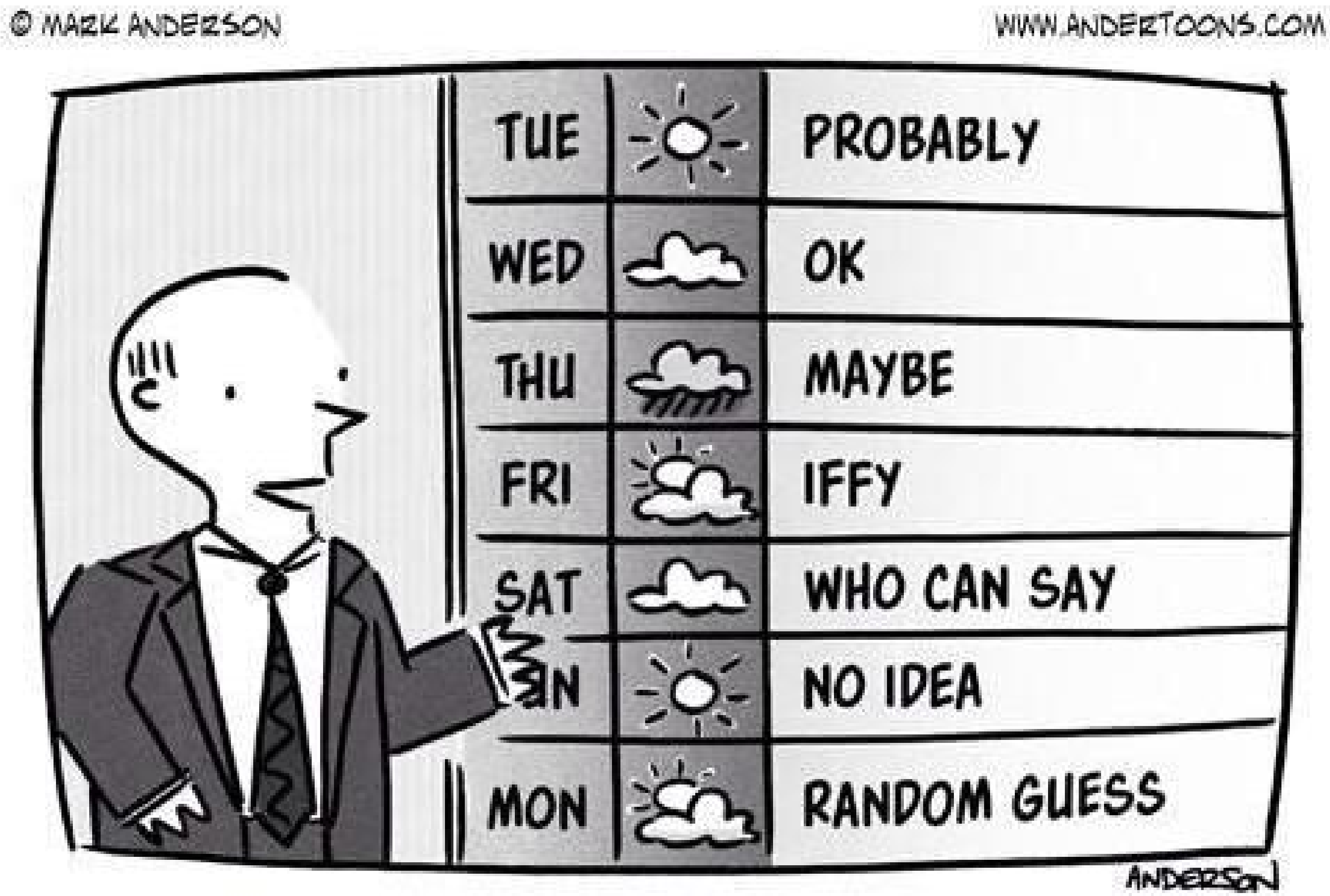
Laplace's Demon 1814

We may regard the present state of the universe as the effect of its past and the cause of its future. An intellect which at a certain moment would know all forces that set nature in motion, and all positions of all items of which nature is composed, if this intellect were also vast enough to submit these data to analysis, it would embrace in a single formula the movements of the greatest bodies of the universe and those of the tiniest atom; for such an intellect nothing would be uncertain and the future just like the past would be present before its eyes.

No room here for free will!

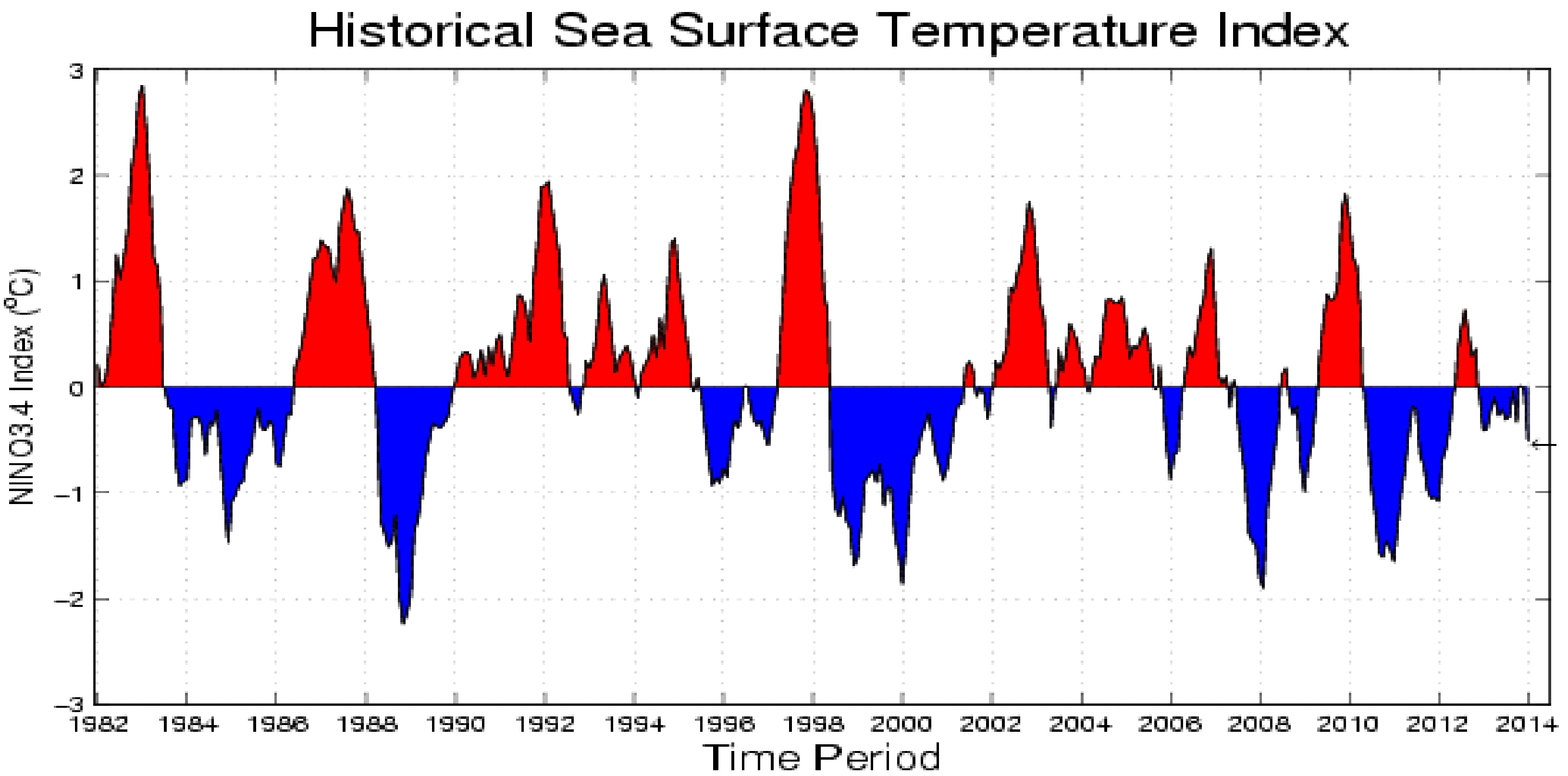
But ….Lots of natural and human events seem to be very unpredictable!!

Do you feel lucky?

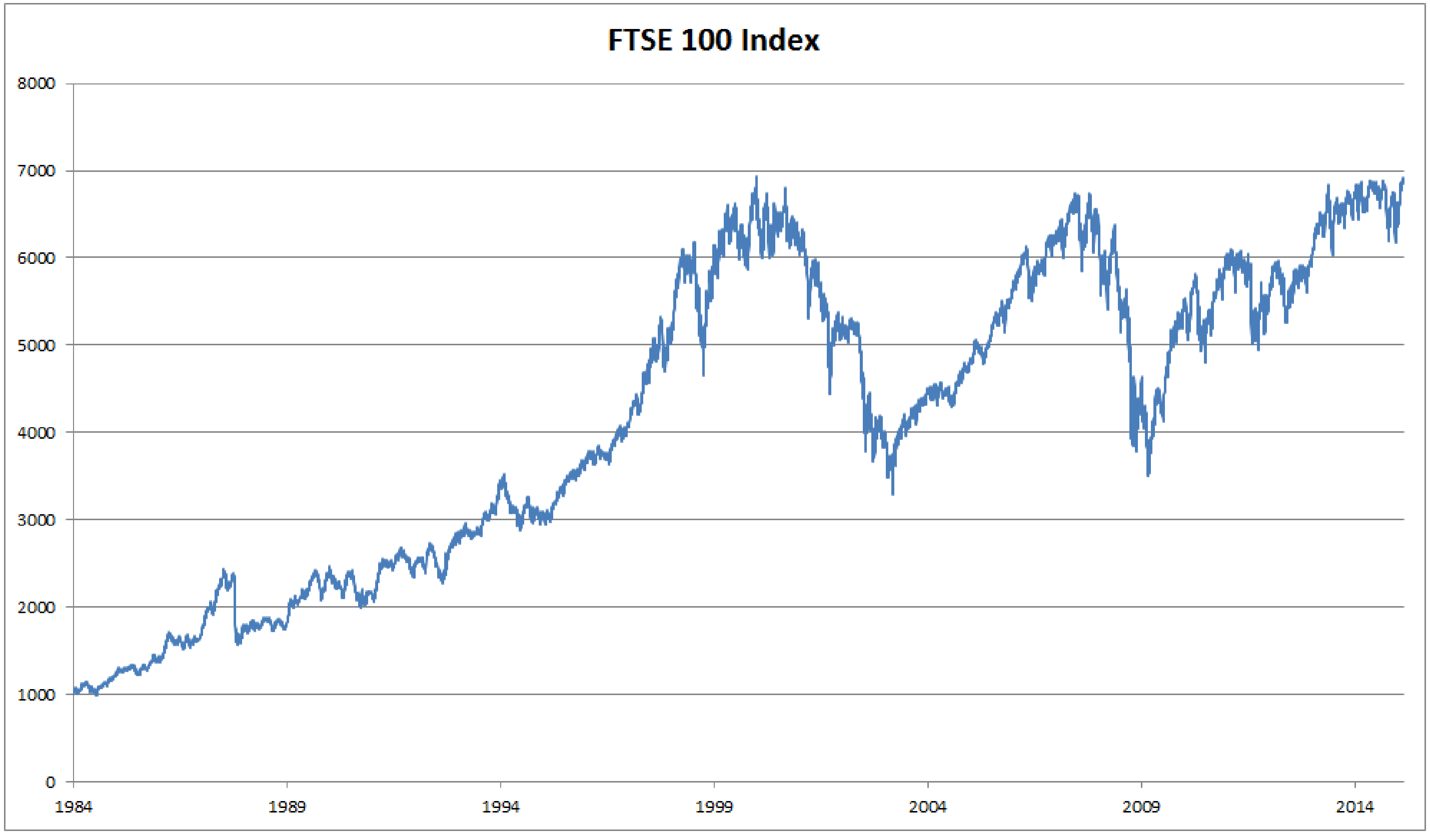


"And now the 7-day forecast..."

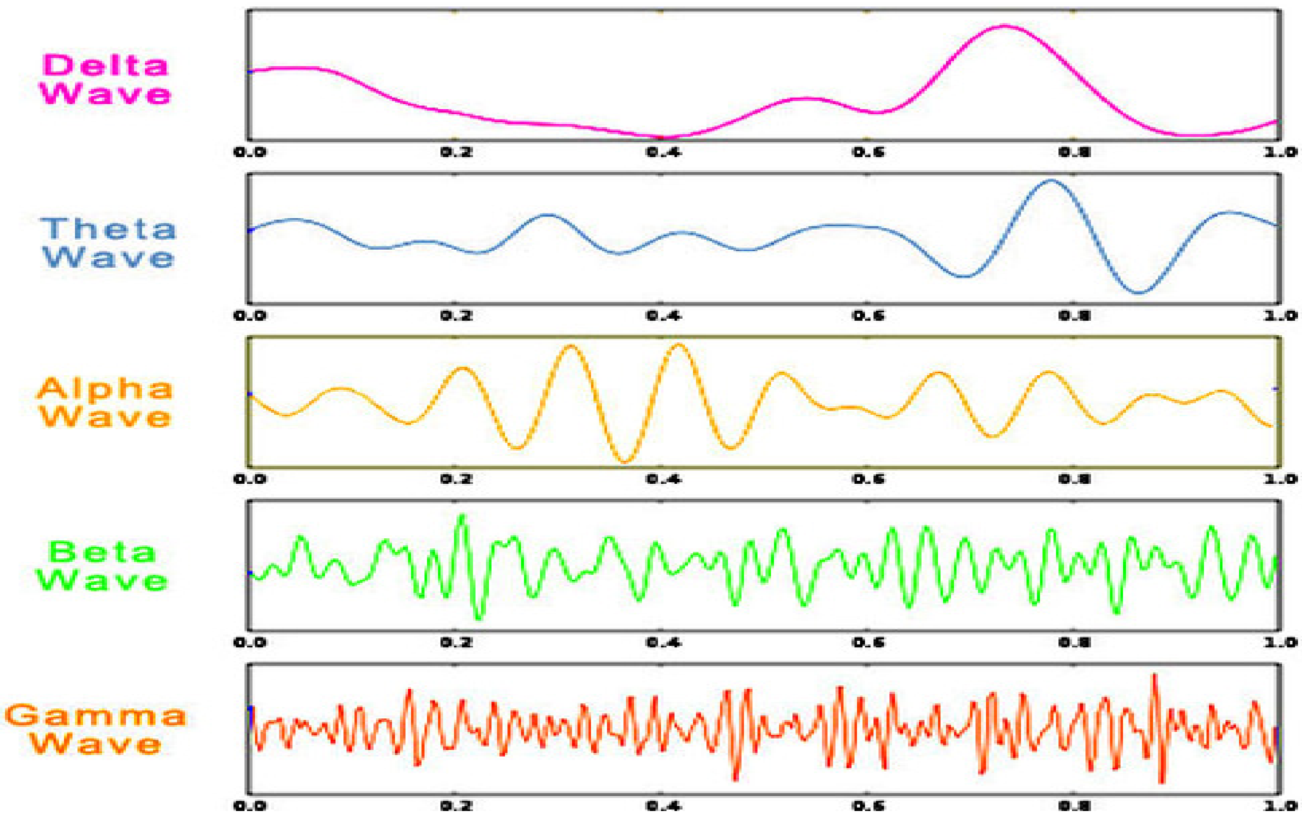
Weather, after a week



El Nino Southern Ocean Index



The FTSE Index



Brain waves (EEG)

My dog

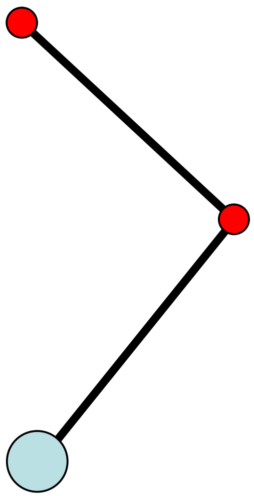
Does this complex behaviour arise because nature is really complicated and unexplainable

Or …….

does it arise naturally in systems governed by Newton's laws???

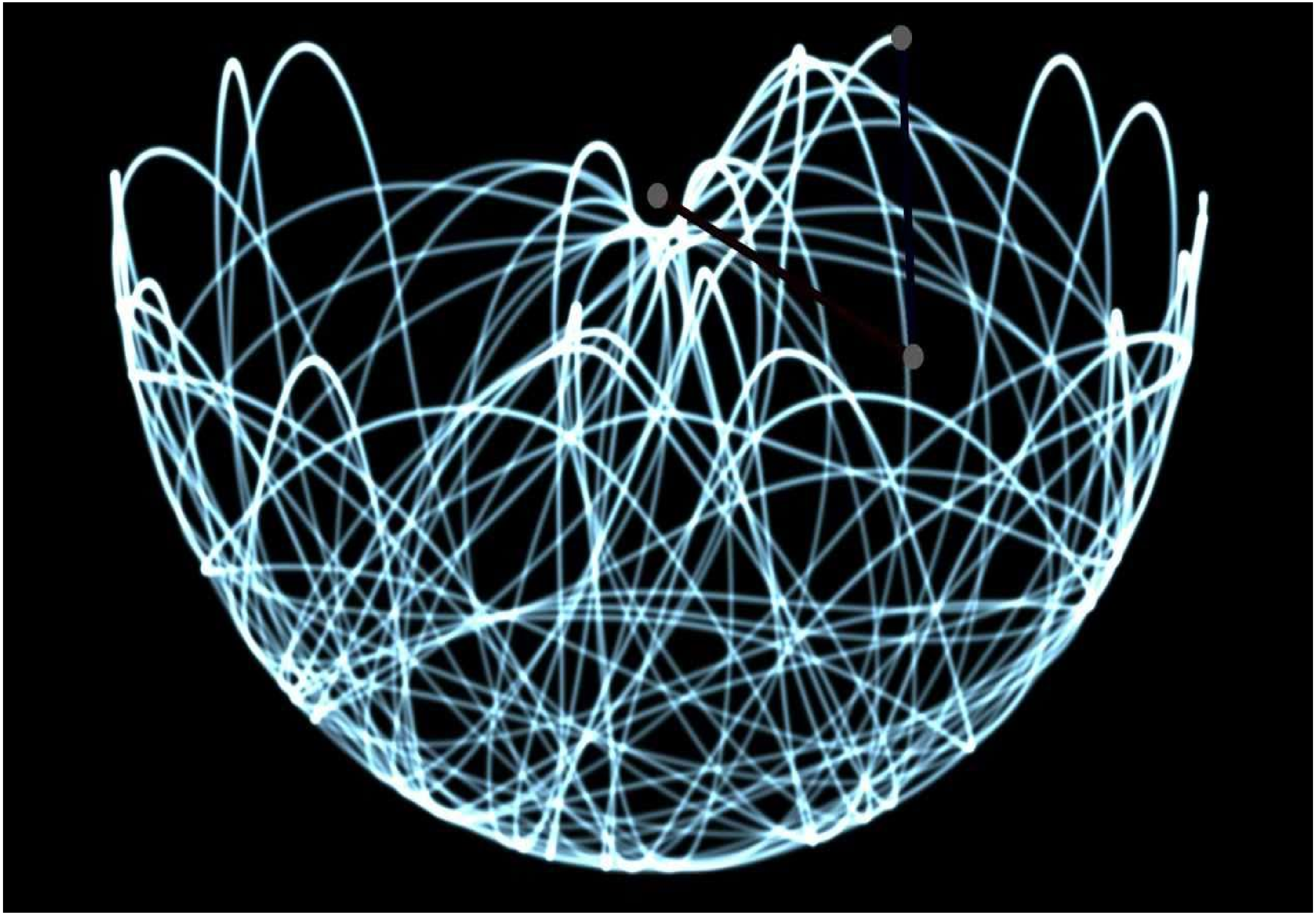
A simple mechanical example:

The Double Pendulum



Motion can be

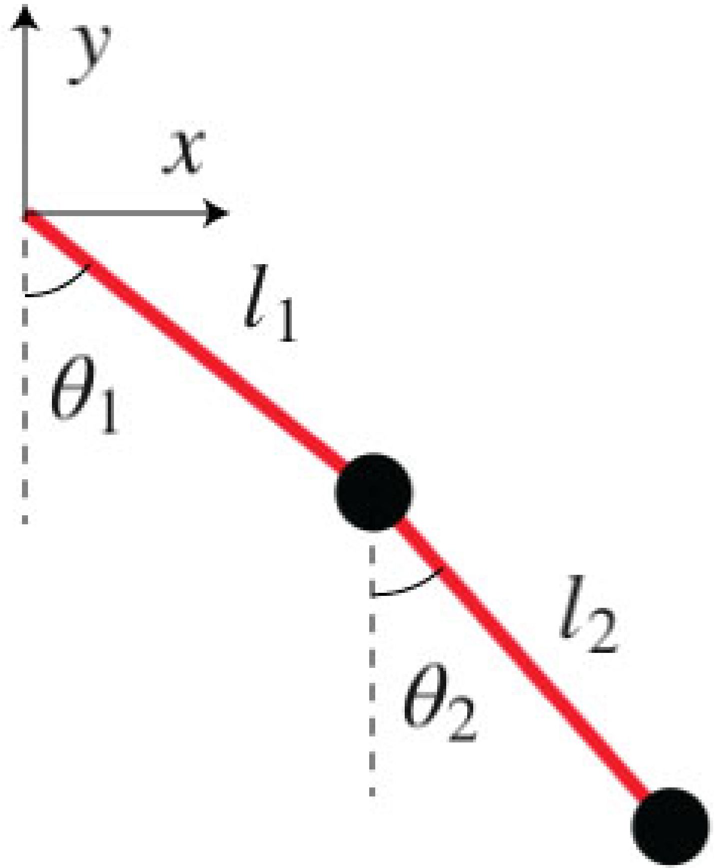
- Periodic in phase : predictable
- Periodic out of phase : predictable
- Chaotic : unpredictable



Newton's laws apply to the double pendulum!

θ_1 Angle of top part

θ_2 Angle of bottom part



Motion is described by a coupled pair of second order nonlinear ordinary differential equations

$$(m_1 + m_2)l_1\ddot{\theta}_1 + m_2 l_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) + m_2 l_2 \dot{\theta}_2^2 \sin(\theta_1 - \theta_2) + g(m_1 + m_2)\sin(\theta_1) = 0$$

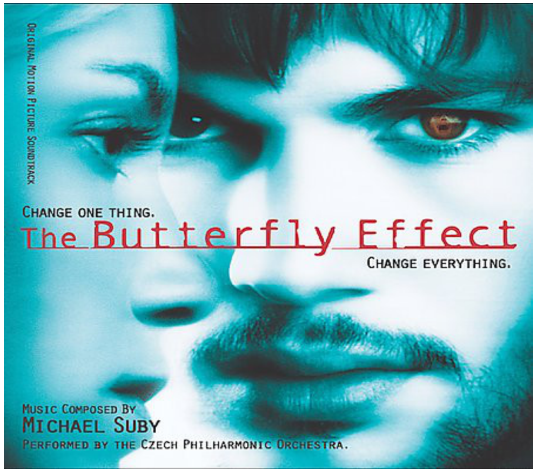
$$m_2 l_2 \ddot{\theta}_2 + m_2 l_1 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) - m_2 l_1 \dot{\theta_1}^2 \sin(\theta_1 - \theta_2) + m_2 g \sin(\theta_2) = 0$$

- Small swings: Can be solved exactly to give in phase and out of phase solutions
- Large swings: Solved numerically. Solutions are chaotic

Chaos

Chaotic motion is complex, irregular and otherwise unpredictable behaviour which arises from a ‘simple’ system which can be exactly described by ‘simple’ mathematical laws.

The Butterfly Effect



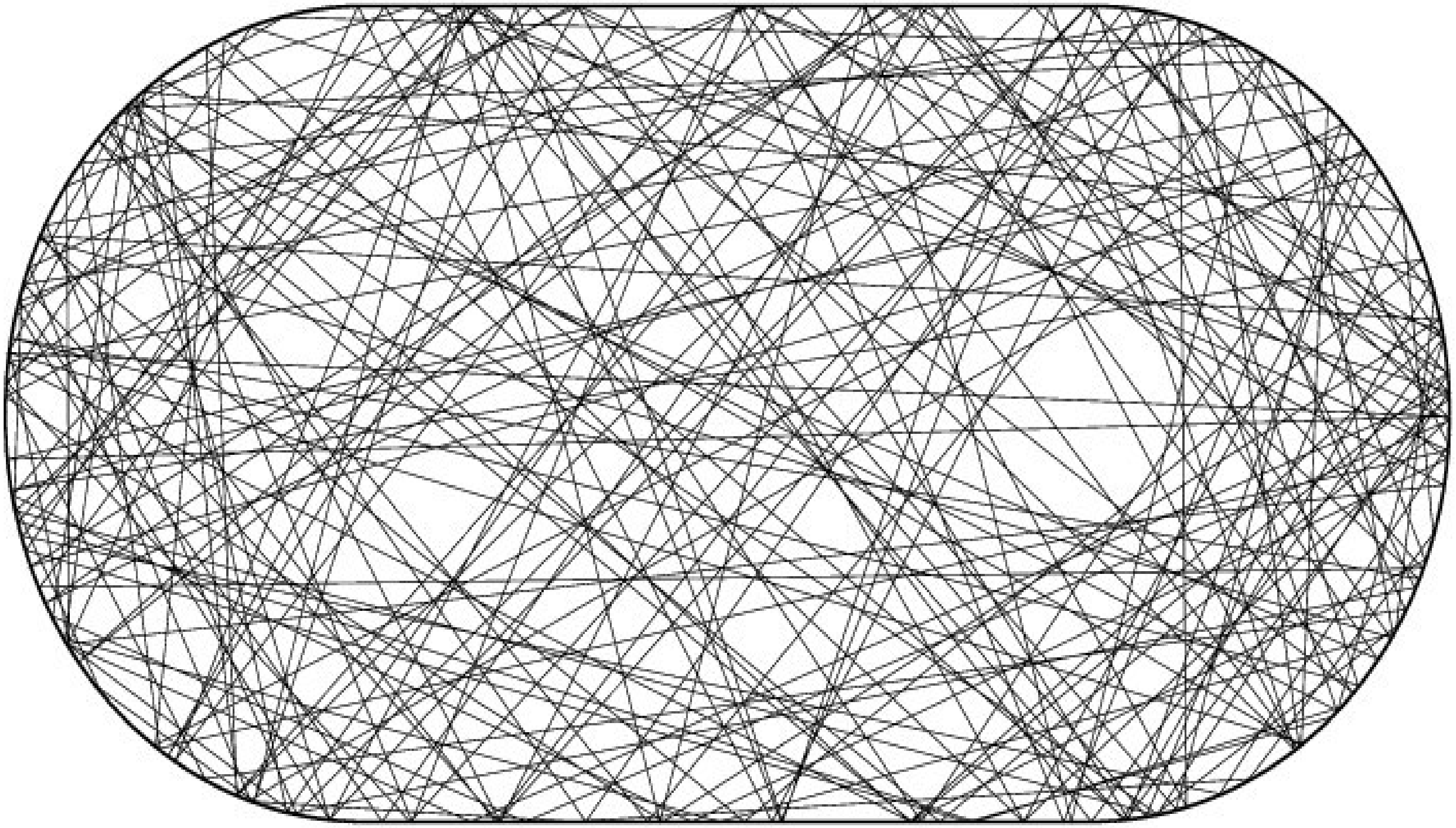
Two states which are very close to each other initially evolve in very different ways.

Sensitivity to initial conditions

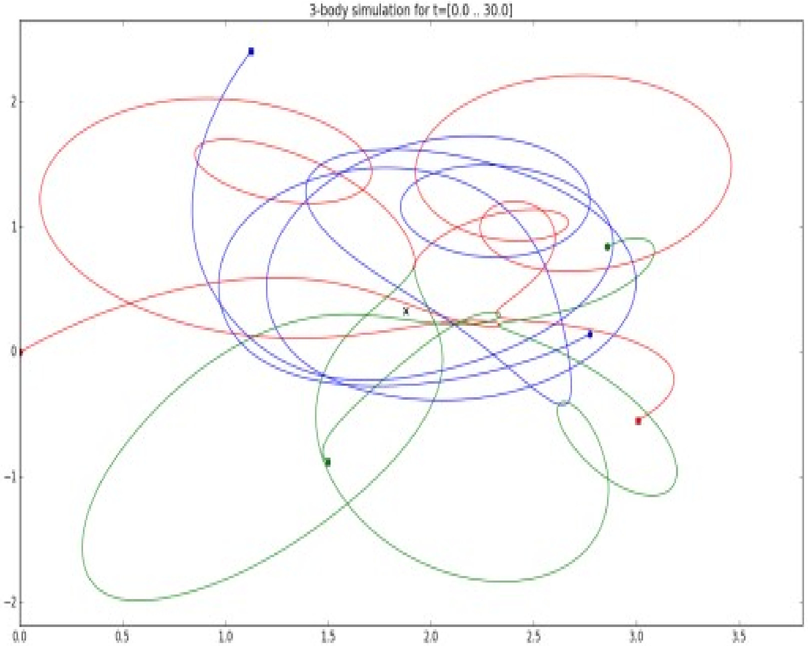
A partial answer to Laplace's Demon

St Mary Redcliffe (Bristol): Chaotic water pendulum

Chaotic Billiards



A short history of chaos theory

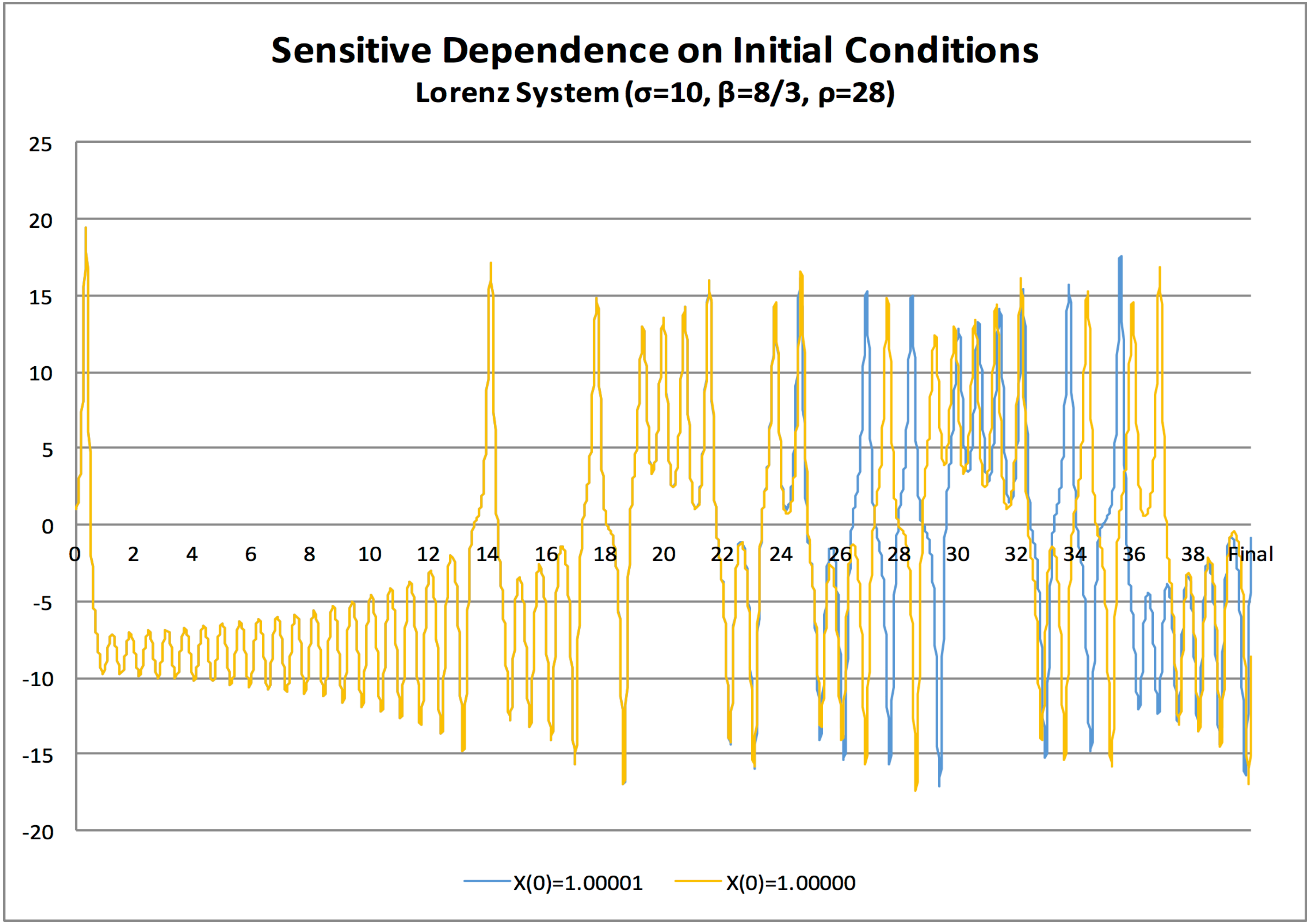


Poincare: discoverer of chaos in the three body problem

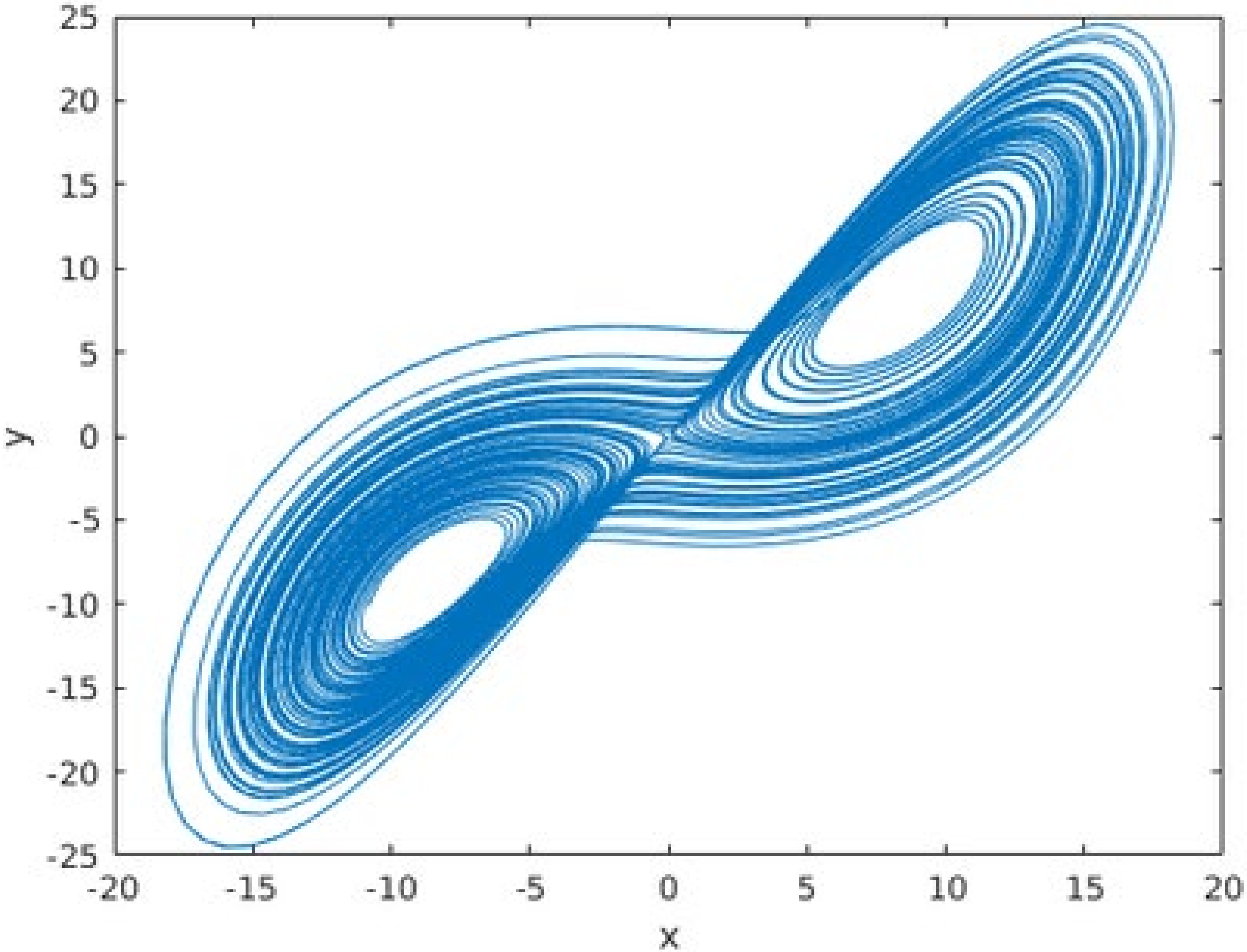
Meteorologist Lorenz (1960s) was studying convection in the atmosphere and derived the Lorenz equations:

$$\frac{dx}{dt} = \sigma(y - x),$$
$$\frac{dy}{dt} = x(\rho - z) - y,$$
$$\frac{dz}{dt} = xy - \beta z.$$

Computer studies showed the existence of chaotic solutions. This came as a complete surprise!



Sensitive Dependence on Initial Conditions
Lorenz System (σ=10, β=8/3, ρ=28)
25
20
15
10
5
0
-5
-10
-15
-20
0
2
4
6
8
10
12
14
16
18
20
22
24
26
28
30
32
34
36
38
Final
X(0)=1.00001
X(0)=1.00000



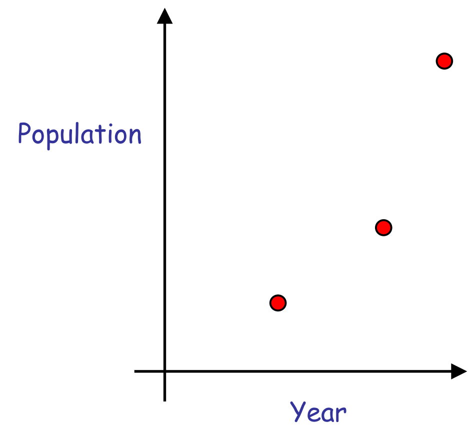
Lorenz attractor

A mathematical theory of chaos

The problems of being a town planner

Can we predict the population of a town?

$$x_n$$



Population of the town in the year n

Can we relate this years population: x_n

To next years population: x_{n+1}

Malthus:

$$x_{n+1} = a x_n$$

Birthrate/Deathrate

- a = 1 … population stays constant
- a > 1 … population increases
- a < 1 … population decreases

Predictable behaviour $$x_n = a^n x_0$$

Problem … if a > 1 population eventually runs out of resources

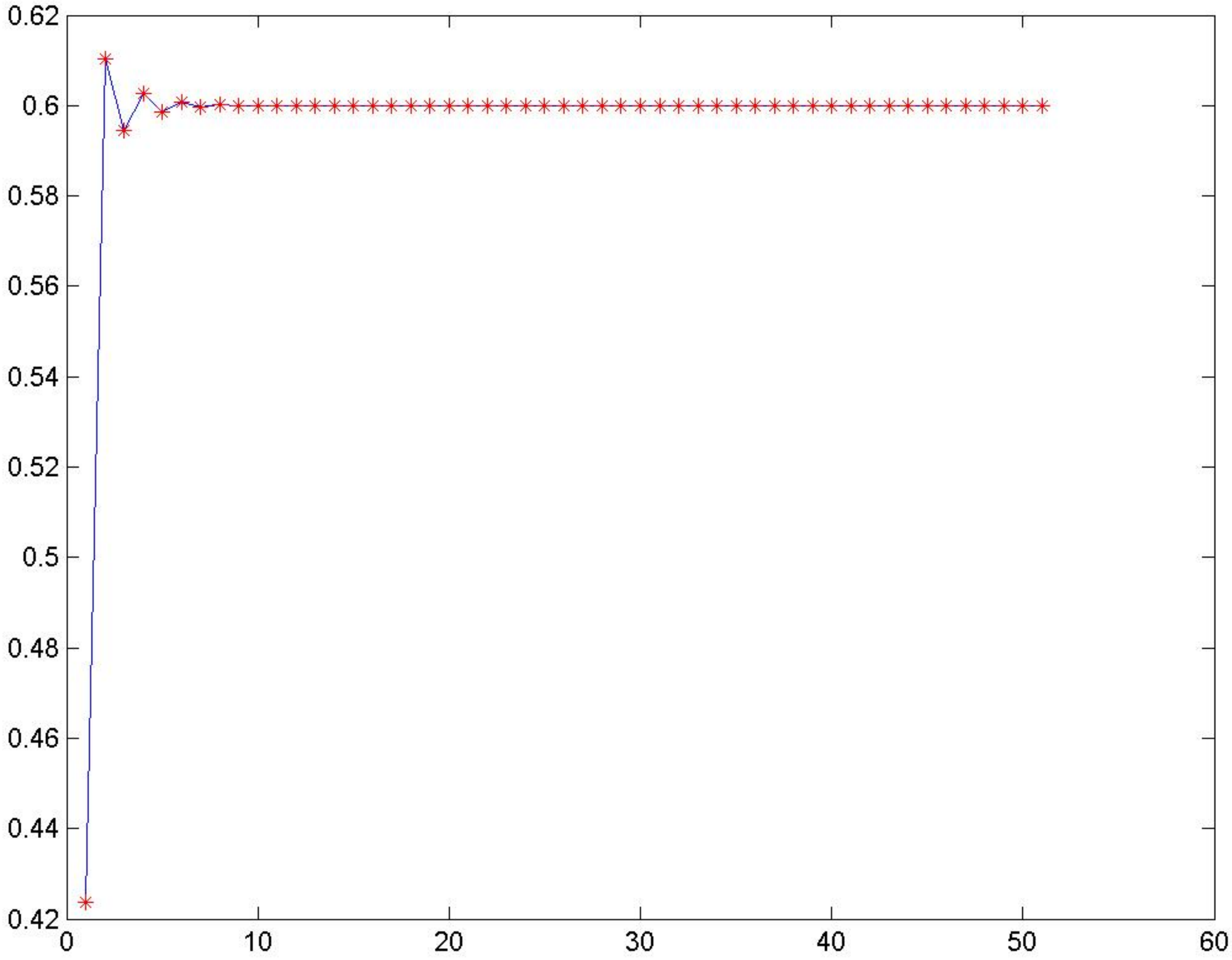
Improved model: Logistic map

$$x_{n+1} = a x_n (M - x_n)$$

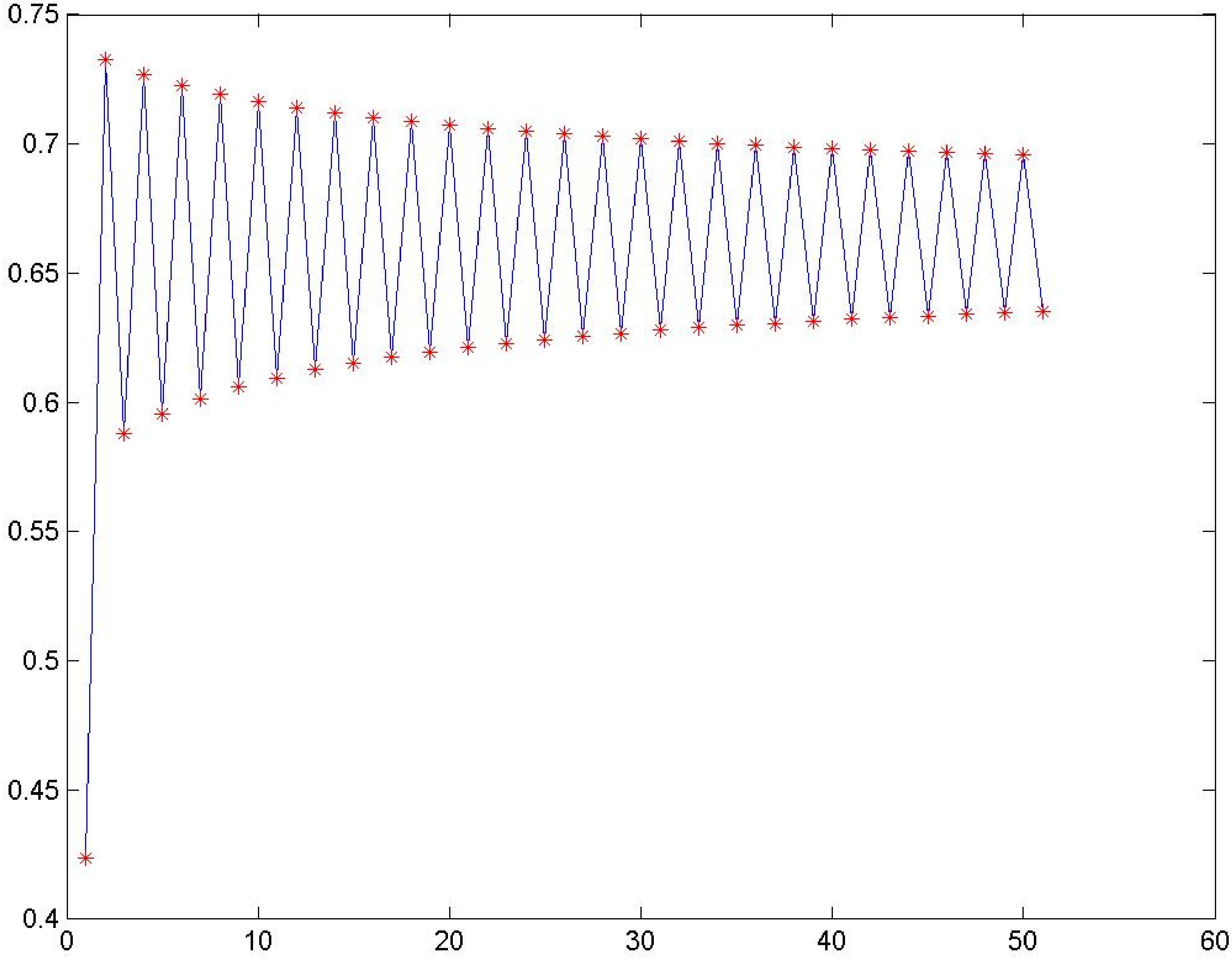
Maximum population

Rescale to give the classical logistic map

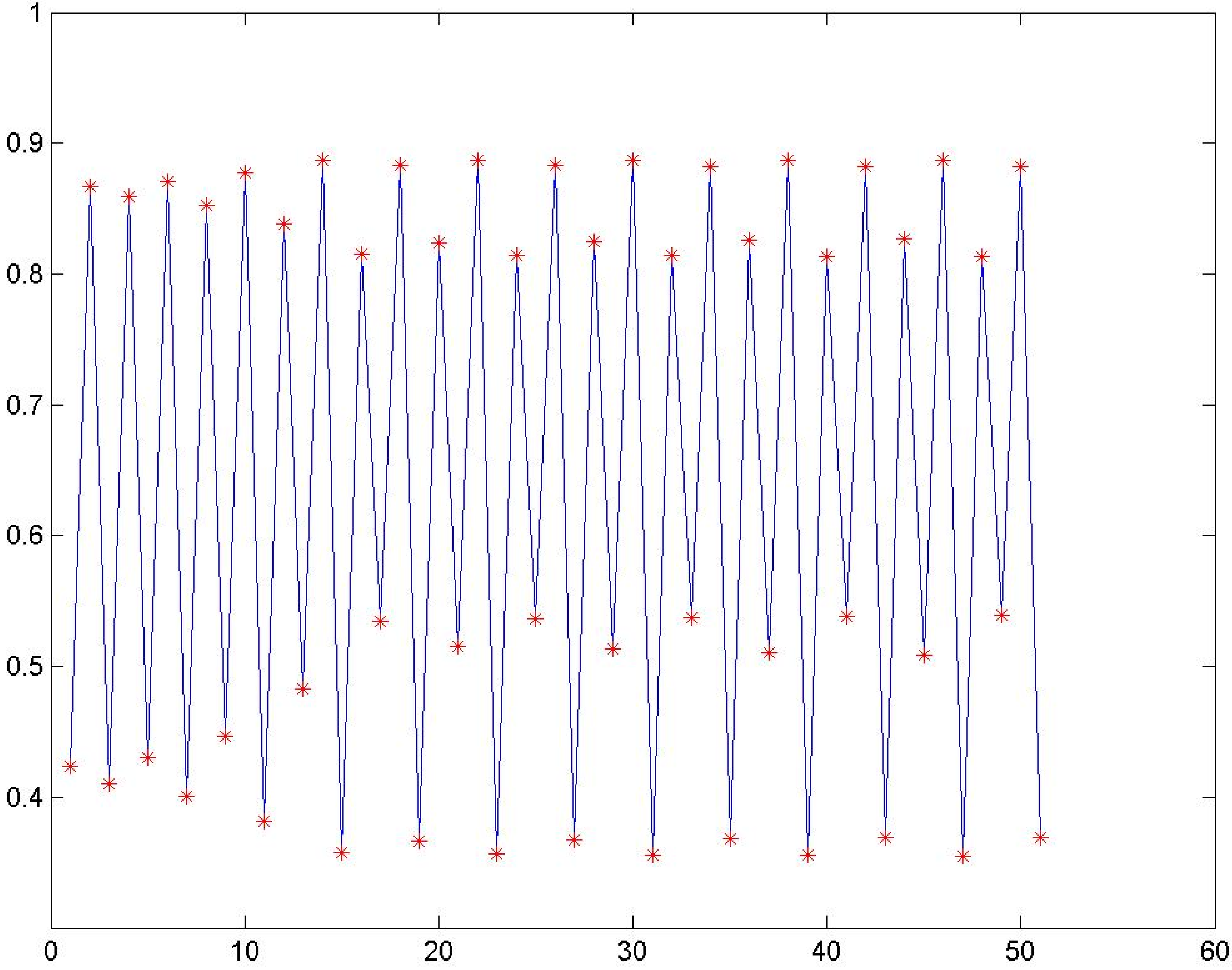
$$x_{n+1} = r\, x_n(1 - x_n)$$



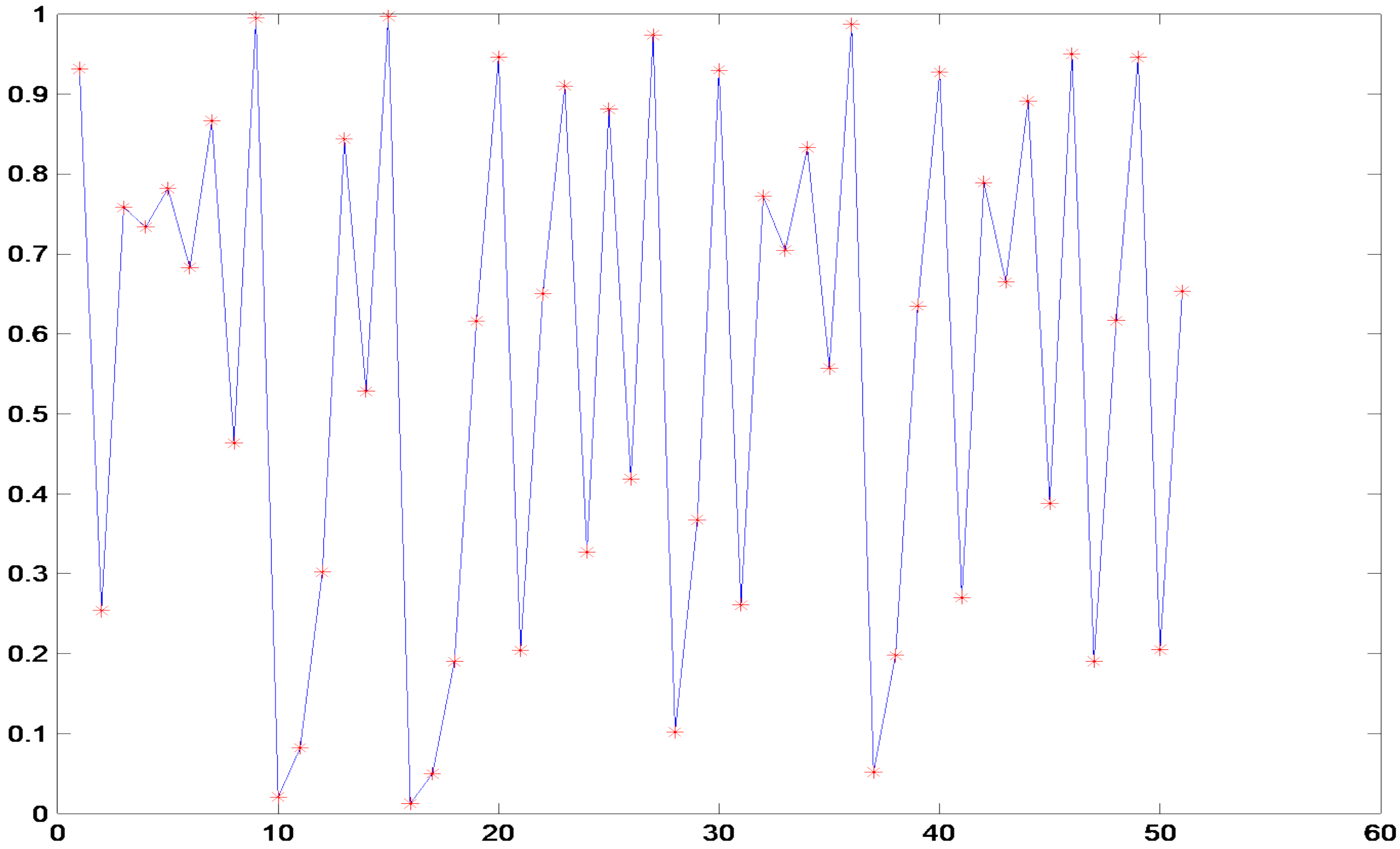
r = 2 Fixed point



r = 3.2 Two points



r = 3.55 8 points



r = 4 Chaos

The anatomy of chaos when r = 4

$$x_n = \frac{1}{2}(1 - \cos(\theta_n))$$

$$x_{n+1} = 4x_n(1 - x_n) = \frac{1}{2}(1 - \cos(2\theta_n))$$

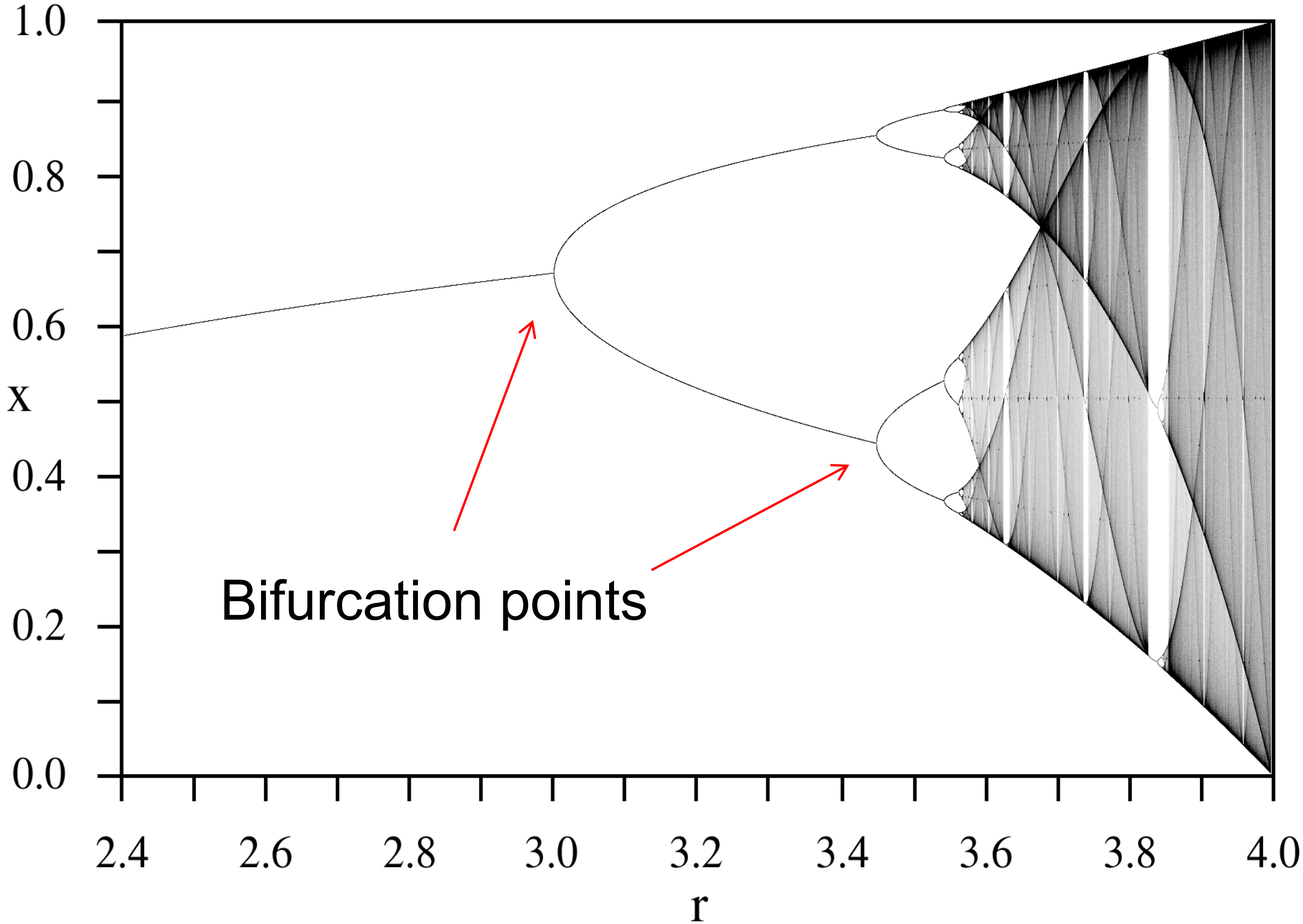
$$\theta_{n+1} = 2\ \theta_n$$

$$x_n = \frac{1}{2}(1 + \cos(2^n\ \theta_0))$$

$$x_n = \frac{1}{2}(1 + \cos(2^n\ \theta_0))$$

Solution stays bounded

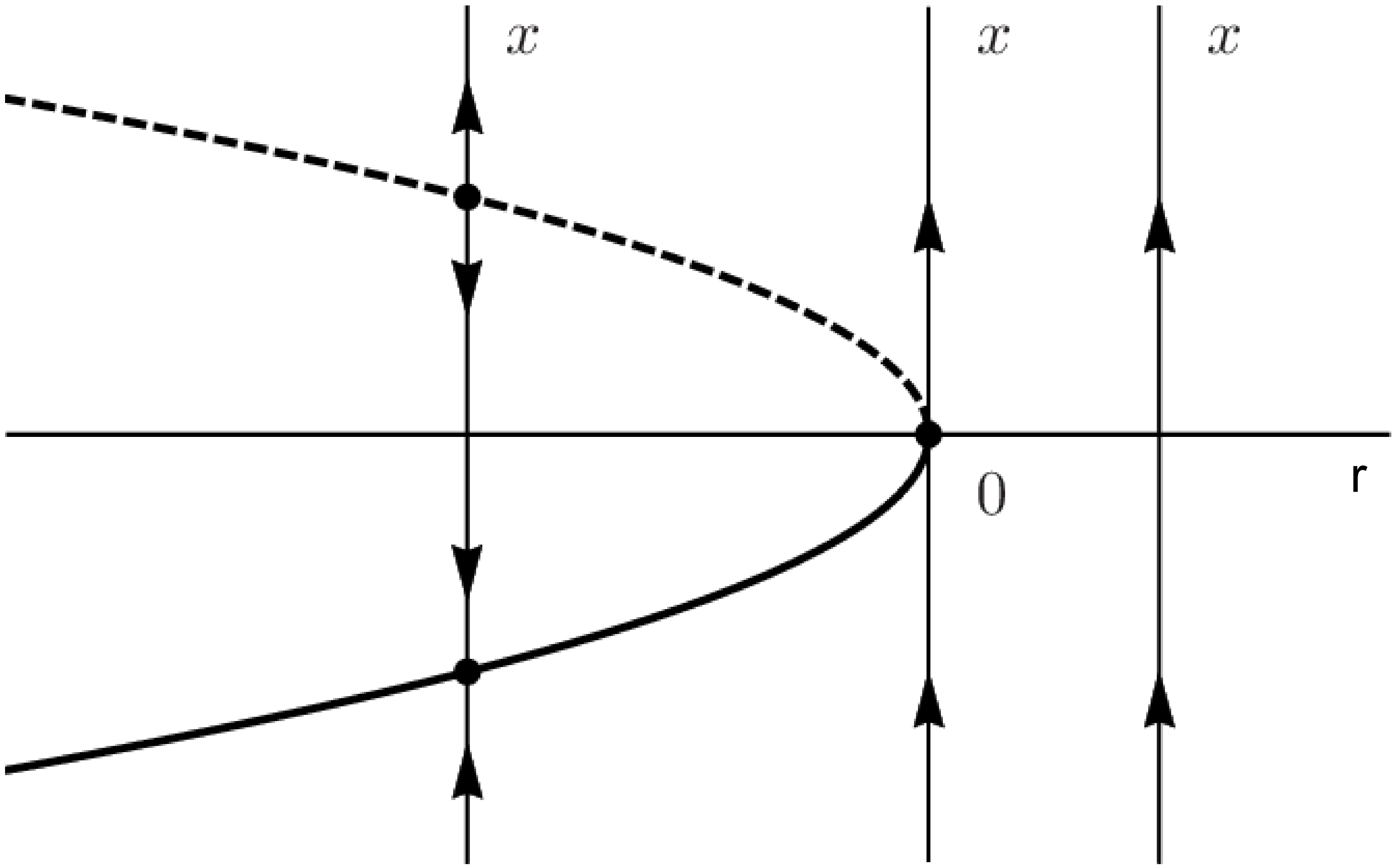
Nearby solutions separate exponentially fast



Bifurcation diagram showing the changes

Tipping points

Bifurcations can lead to sudden changes



$$\frac{dx}{dt} = r + x^2$$

So, what’s the use of chaos?

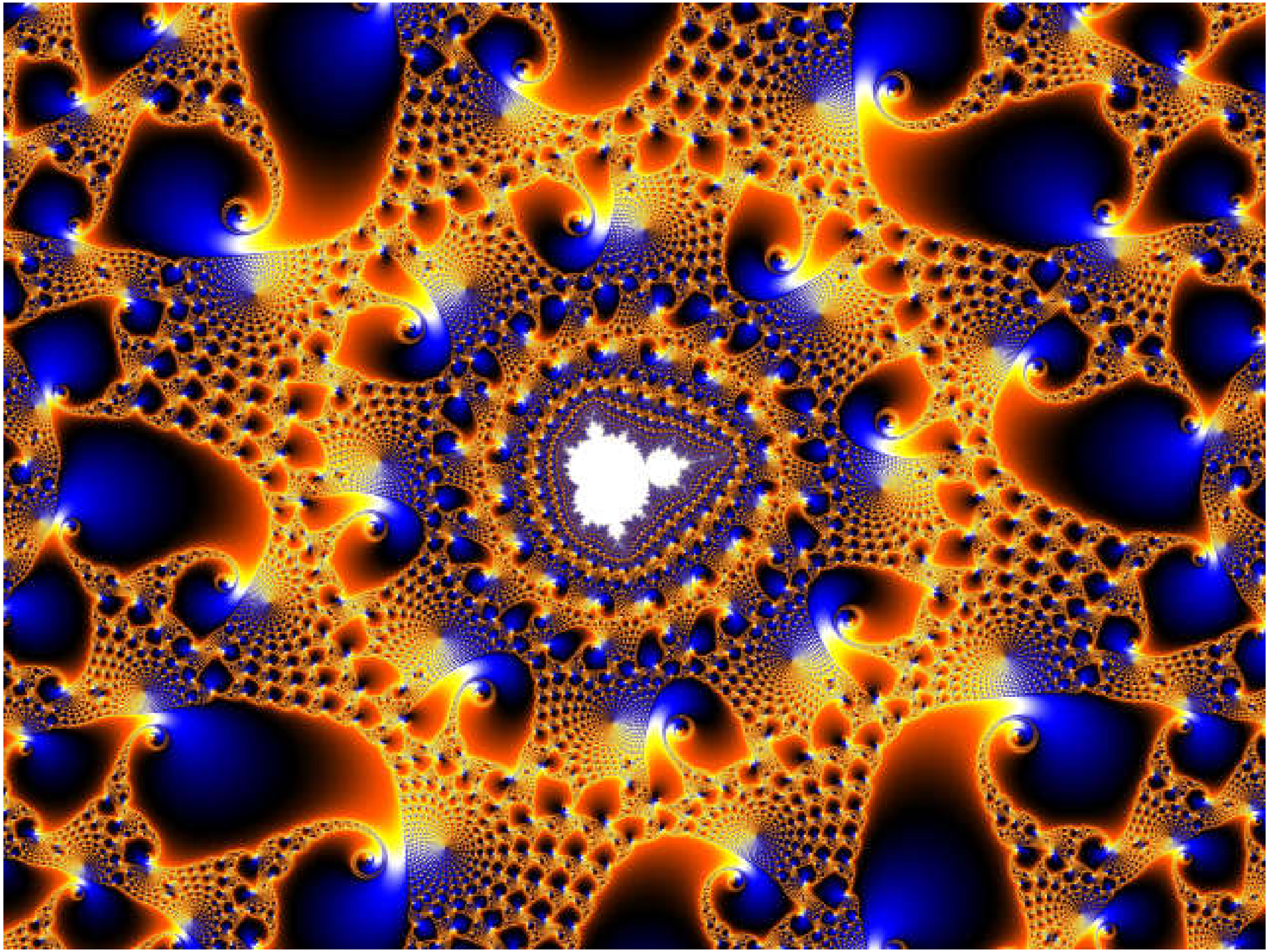
Chaos: Simple rules can give complex patterns

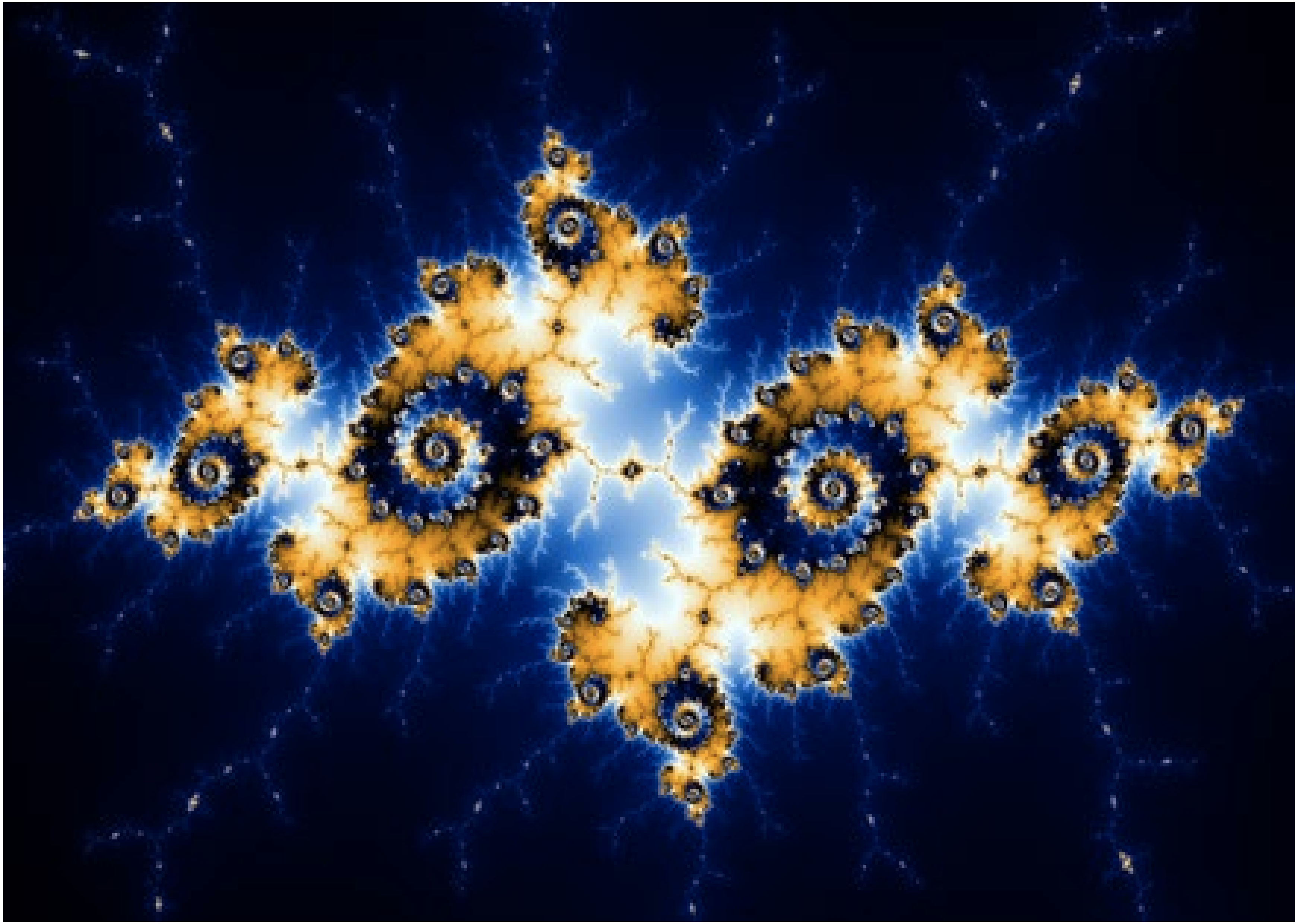
Useful for computer graphics

Barnsley fractal fern

Fractal mountain

Computer art: The Mandlebrot set





Engineering Design

Car exhausts

Car suspensions

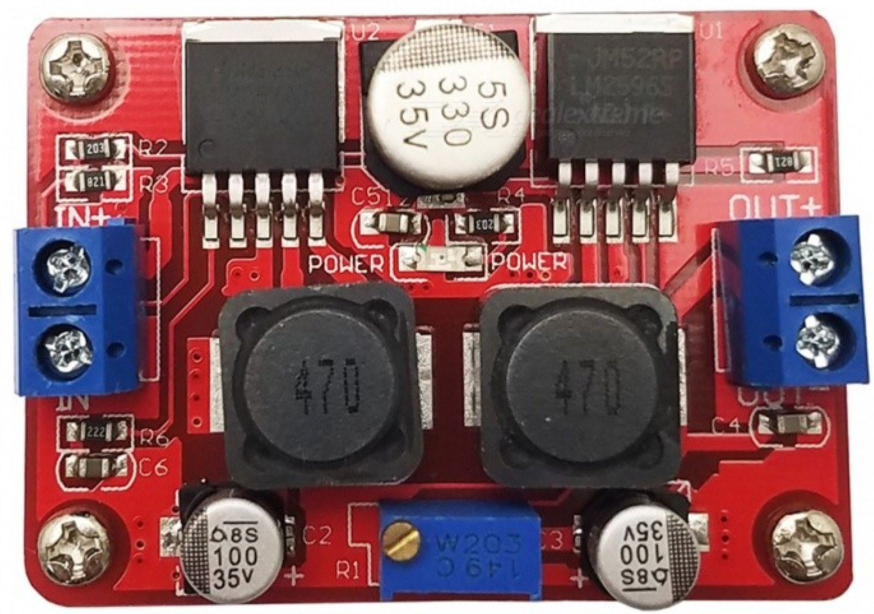
Boiler tubes

DC-DC Power supply systems

Friction brakes

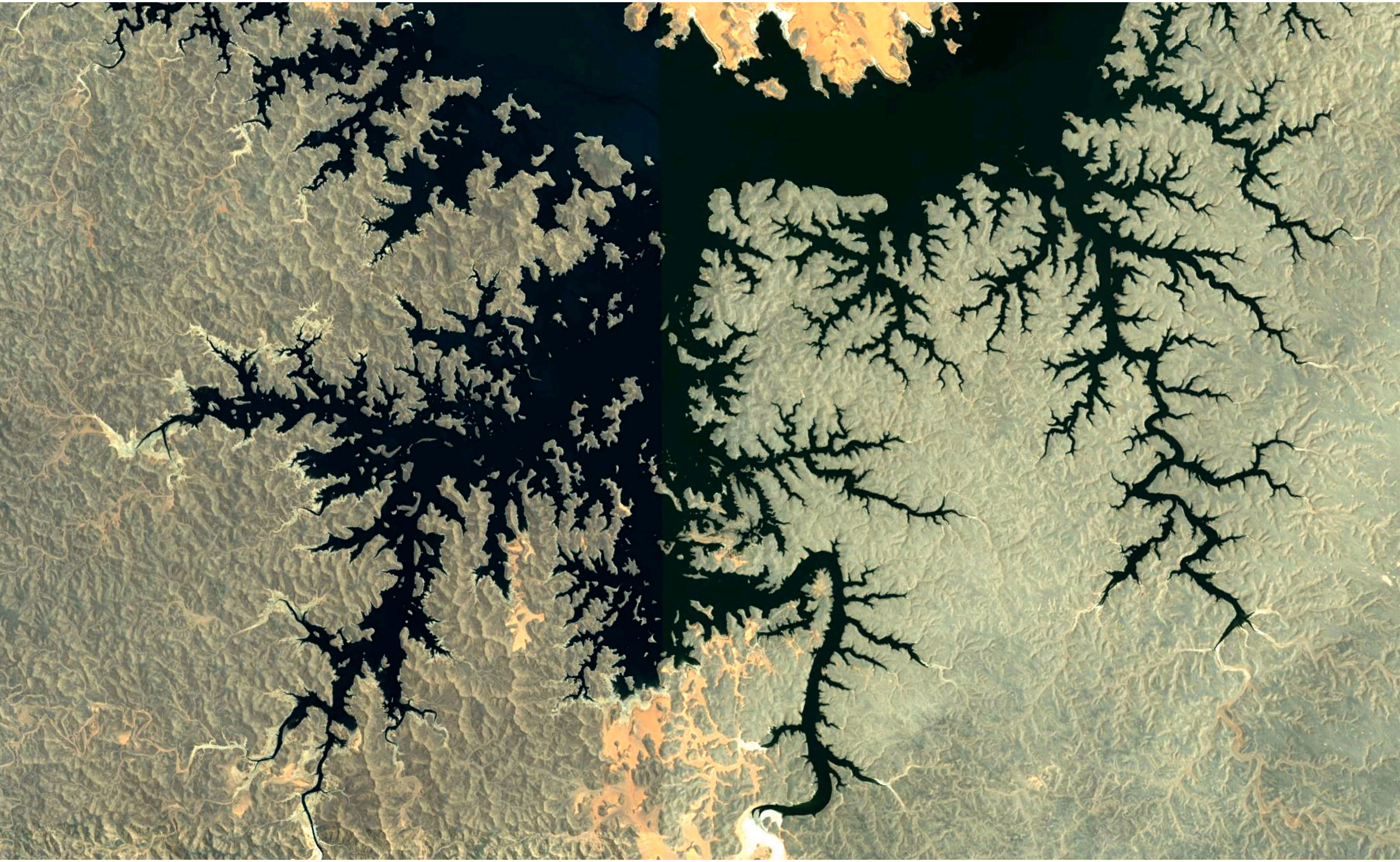
WiFi

Microwave cookers



New understanding of nature

Turbulence



River delta

Motion of the asteroids and the future of humankind

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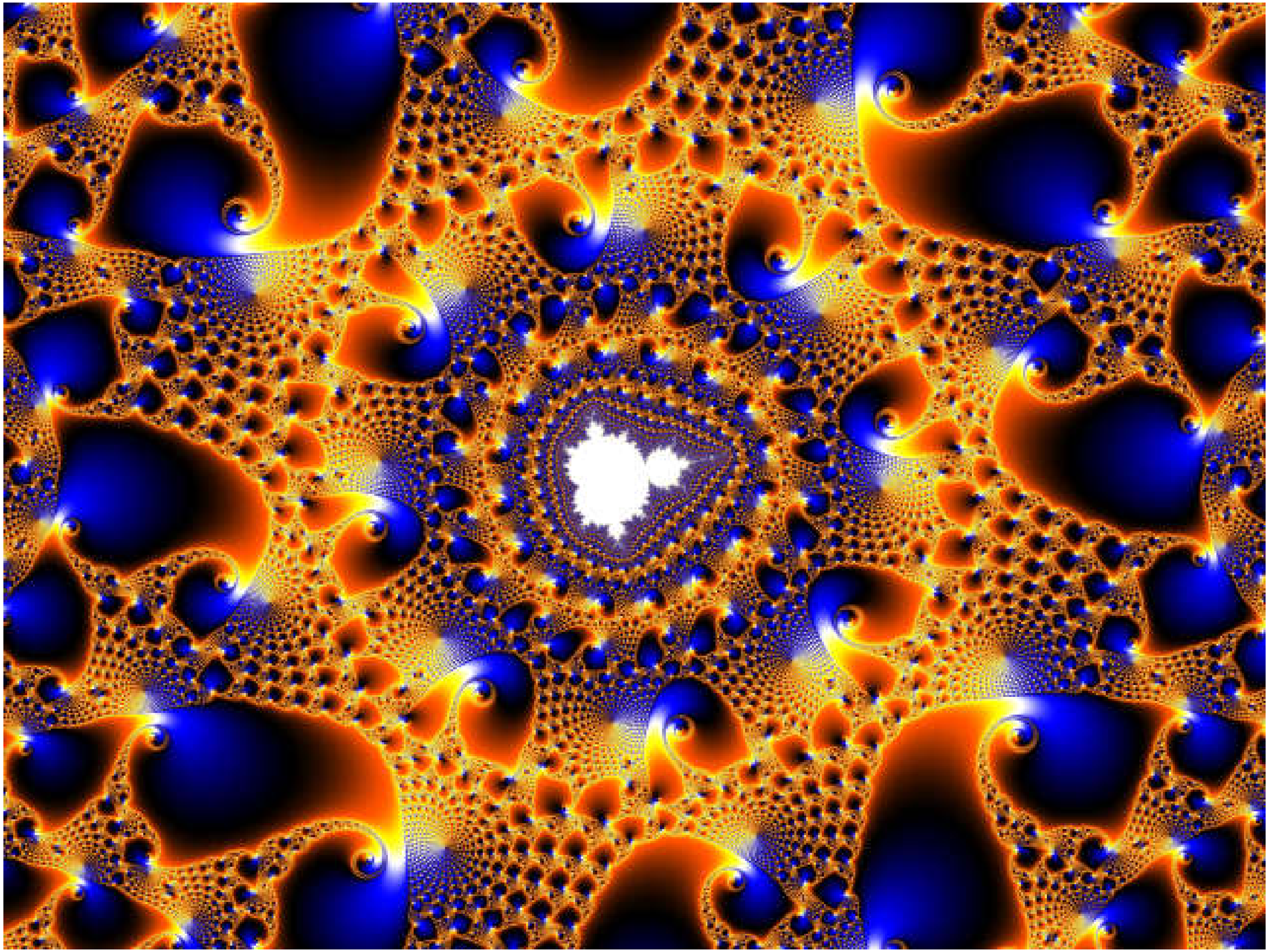

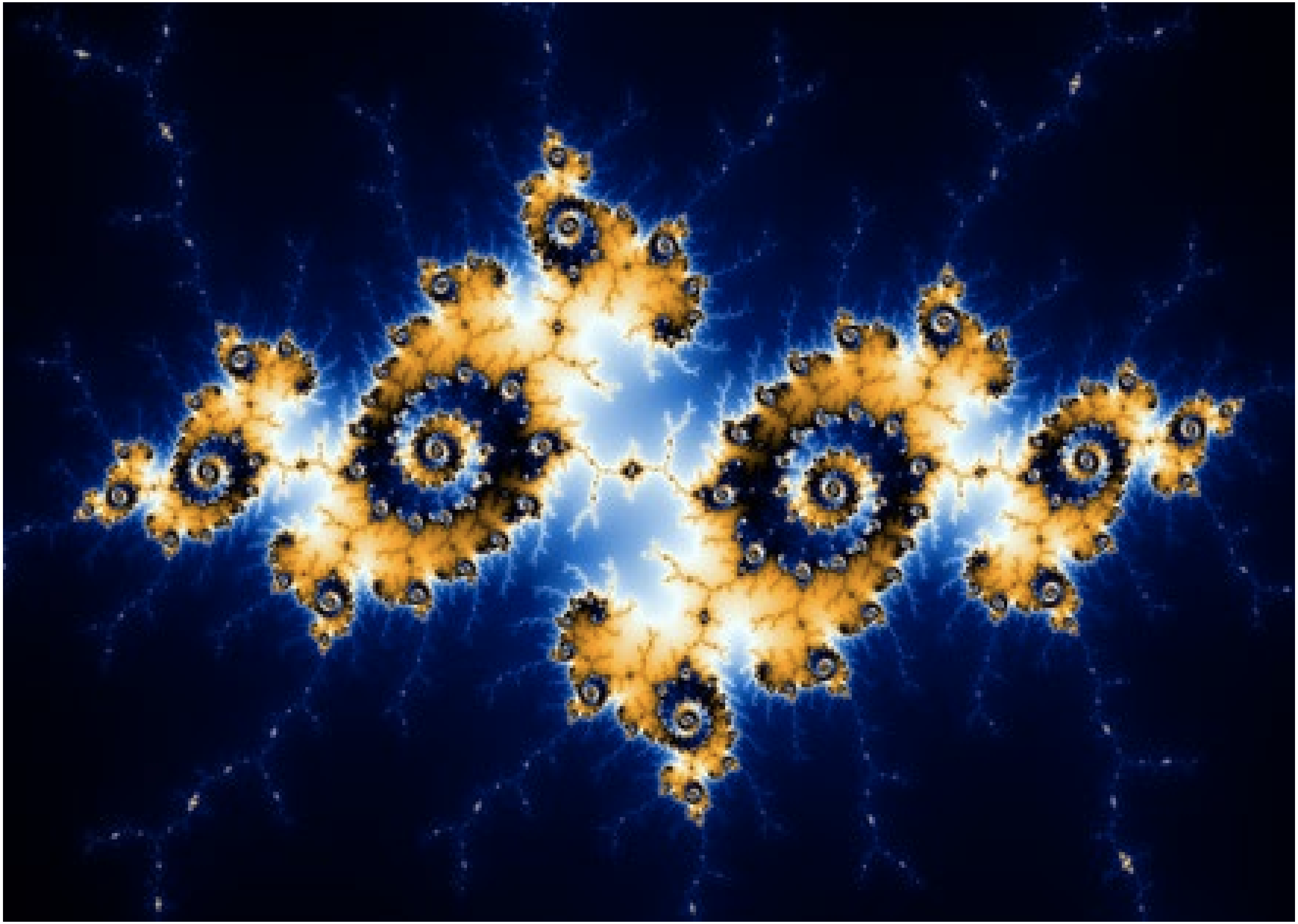

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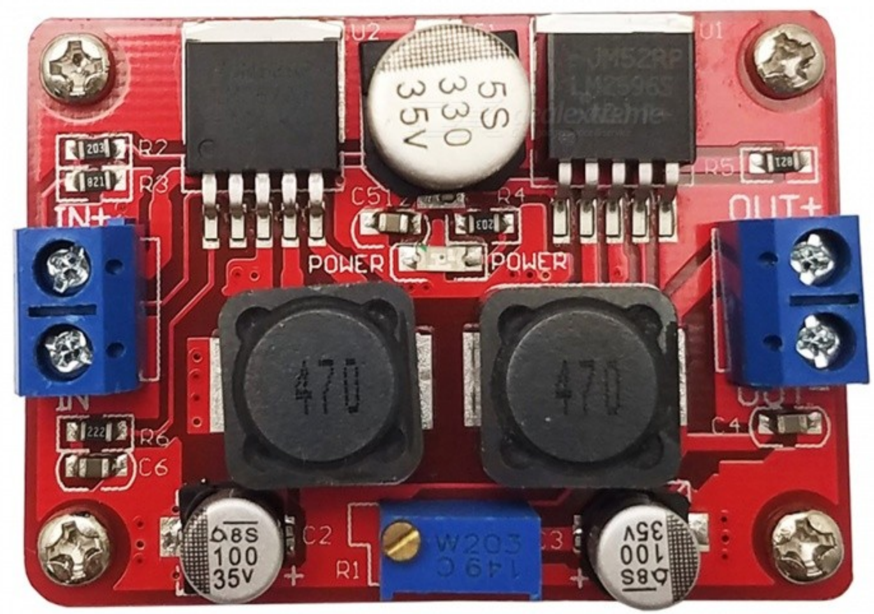

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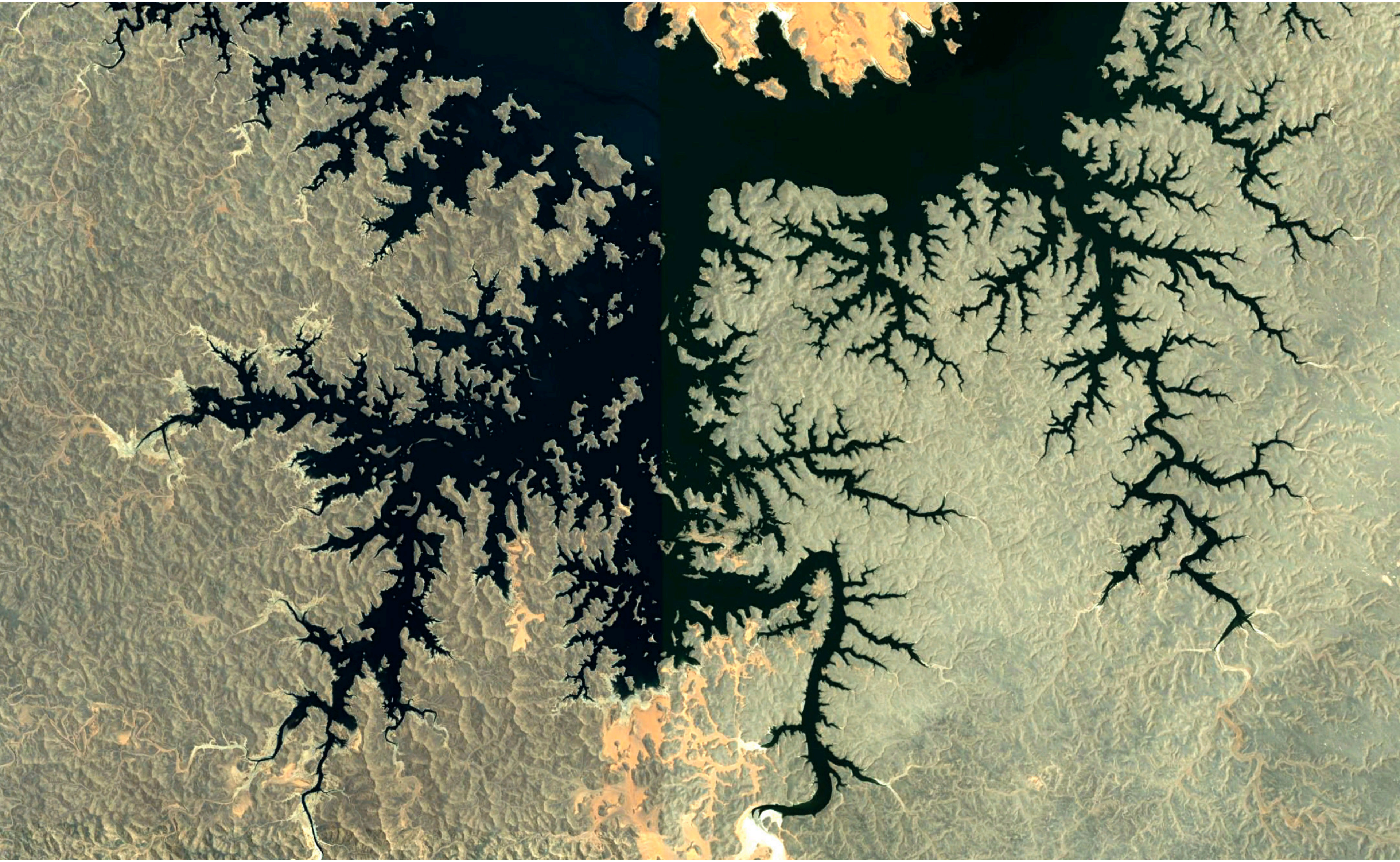

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Chaos is the science of the 21st Century!