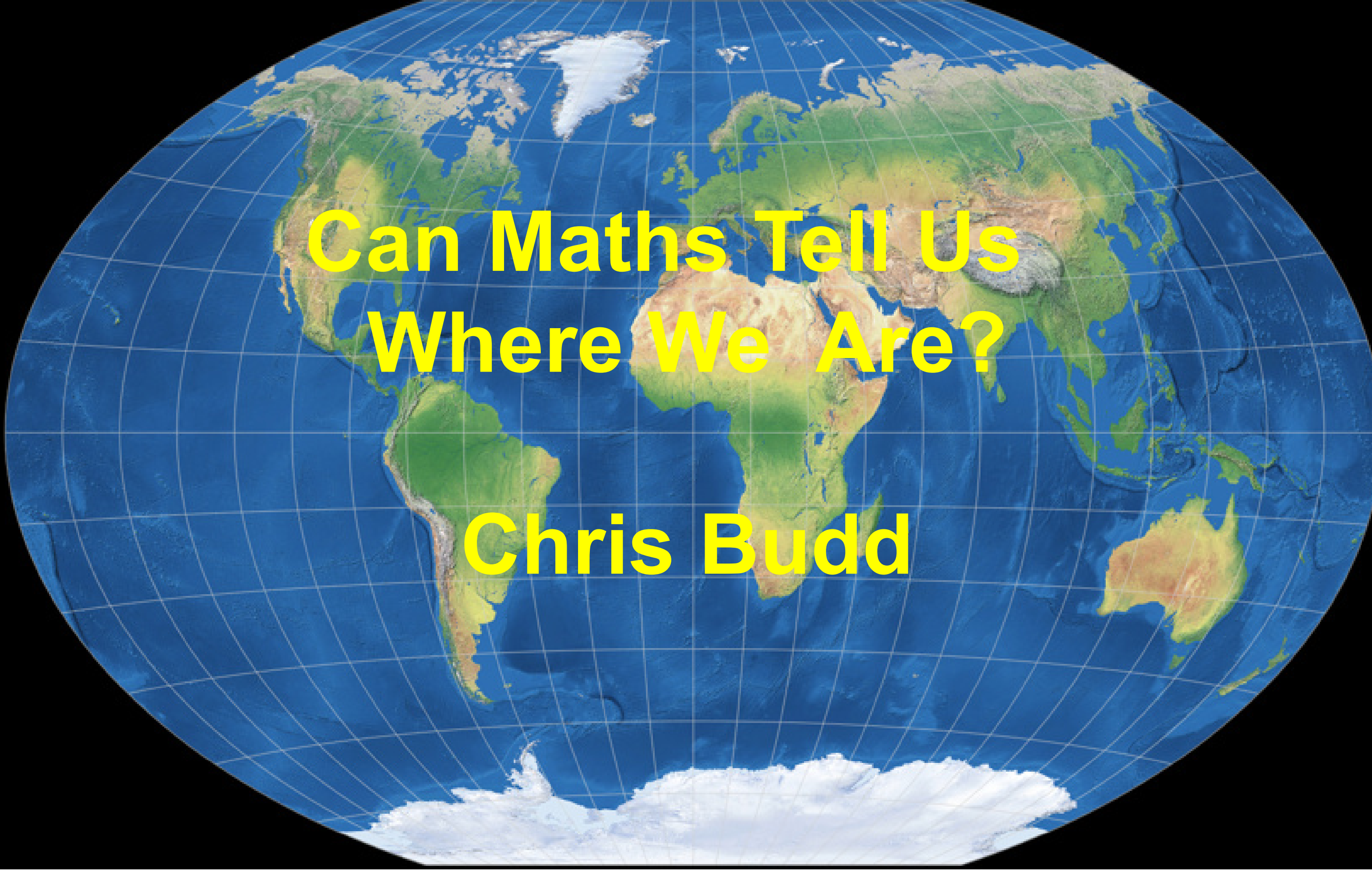
Can Maths Tell Us Where We Are?

Chris Budd



Gresham College

IMI

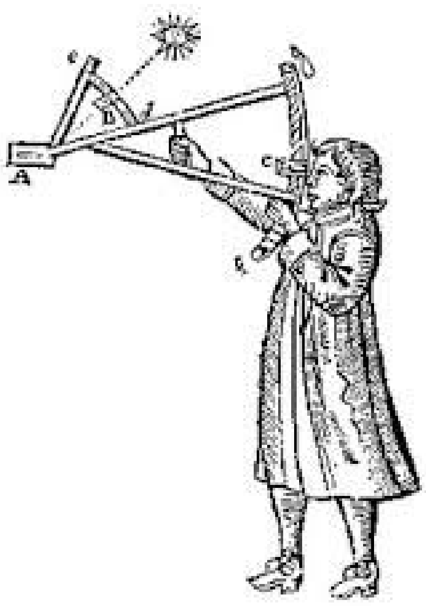
University of Bath

Ballad of Gresham College v. 26

The College will the whole world measure
Which most impossible conclude,
And Navigation make a pleasure
By fynding out the Longitude.
Every Tarpaulin shall then with ease,
Sayle any ship to the Antipodes.

Anon (circa 1660)

Knowing where you are and finding where to go safely is a hugely important part of human civilisation



Early navigation: dead reckoning

Later: Navigation by the stars

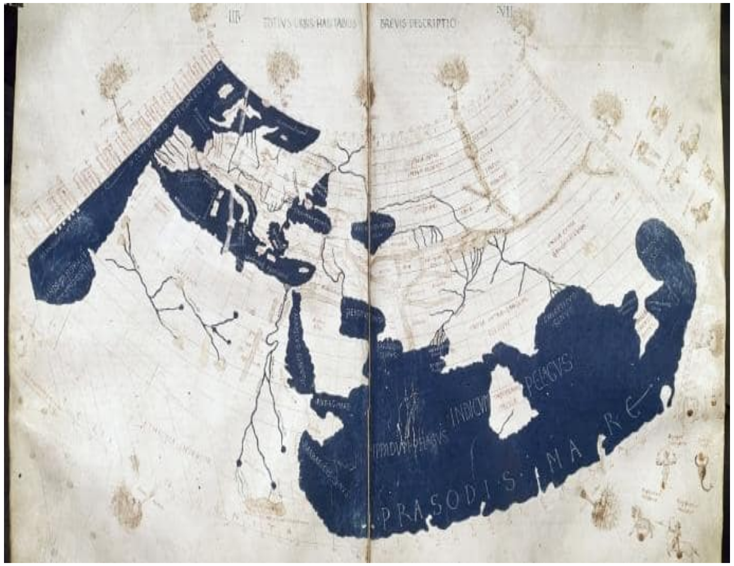
Now: Electronic means such as GPS

Mathematics has stimulated great advances in navigation

The need to advance navigation has stimulated mathematics

Geometry is crucial!

To know where you are you first need a map



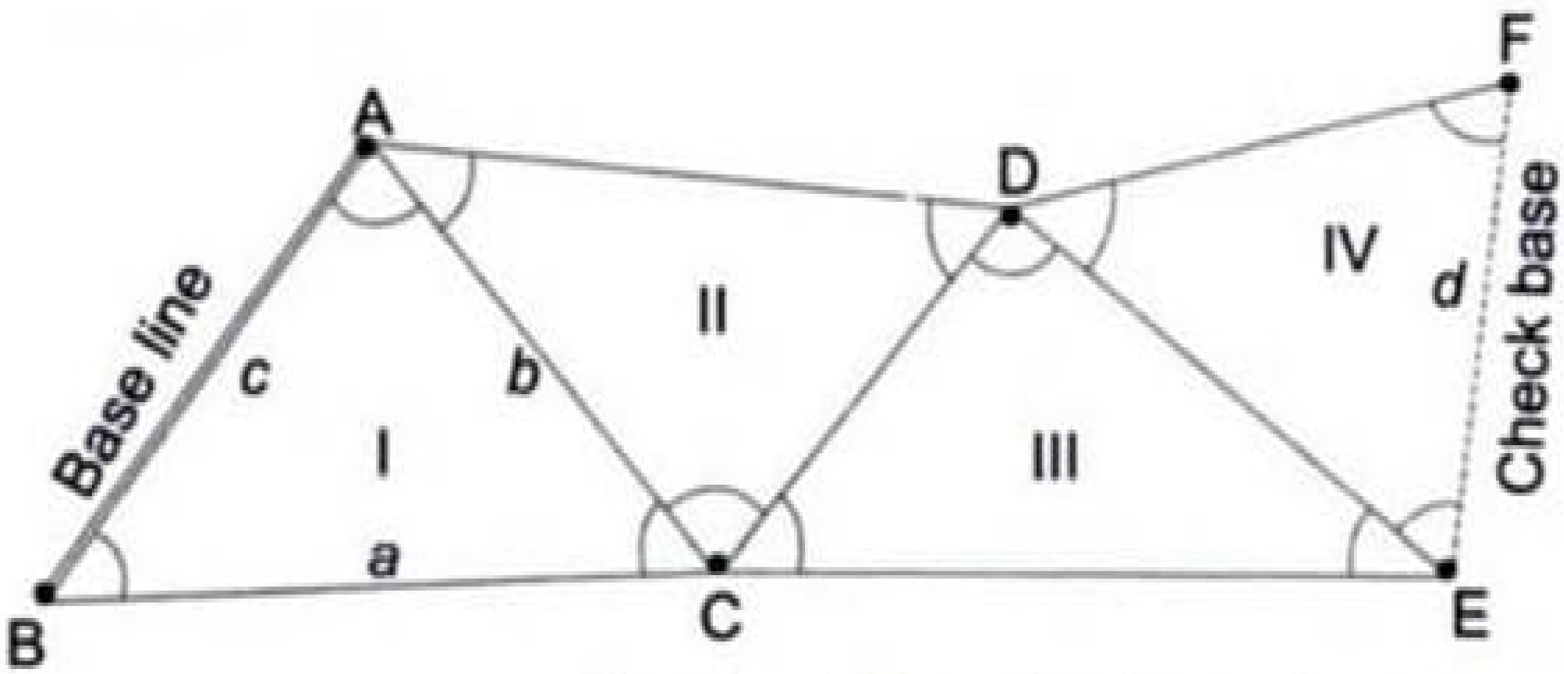
Ptolemy

Mappa Mundi

Ordnance Survey Maps

OS Founded in 1747 by Lieutenant-Colonel David Watson

Maps constructed by triangulation from well defined points

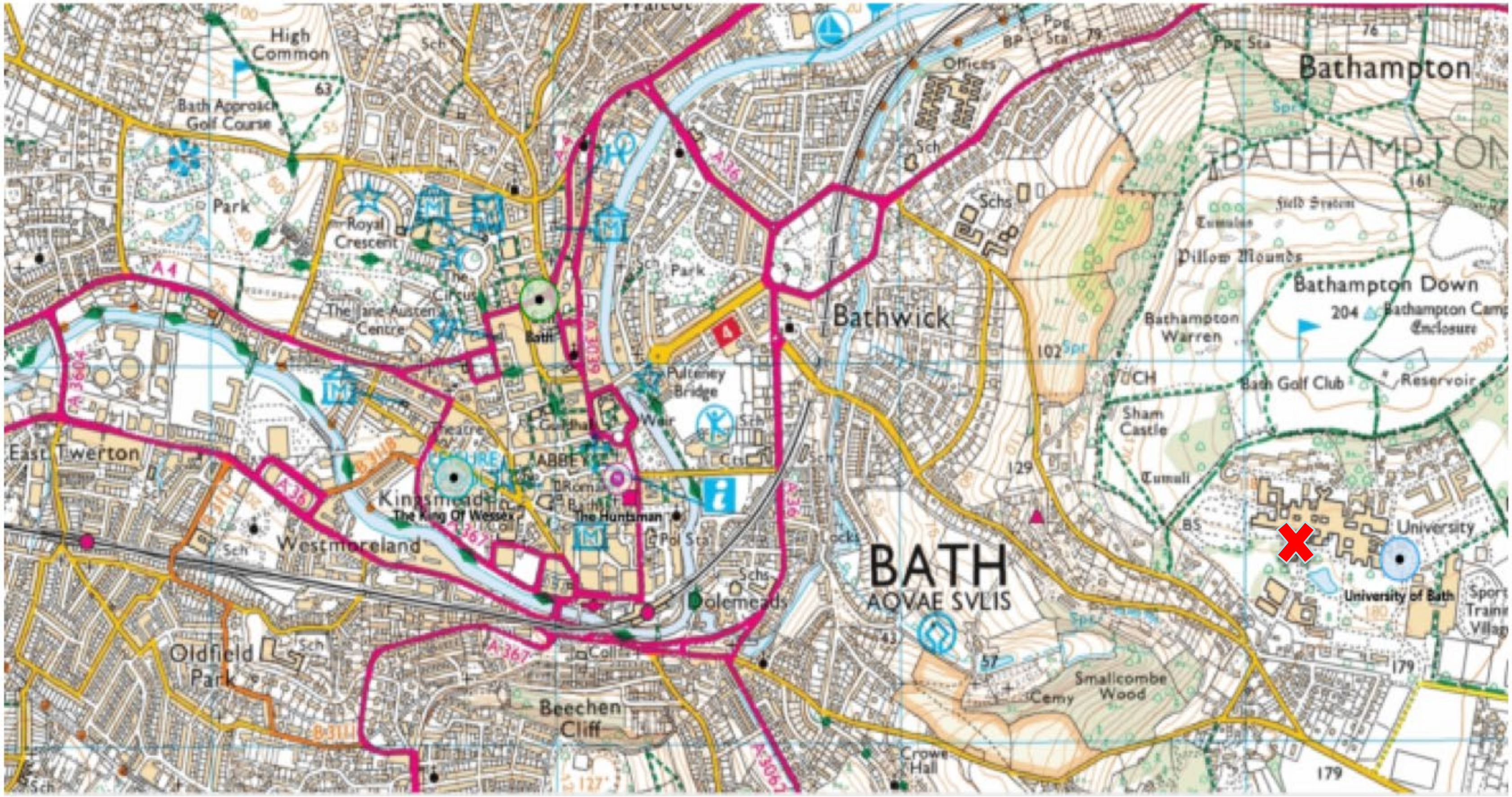


Chain of simple triangles

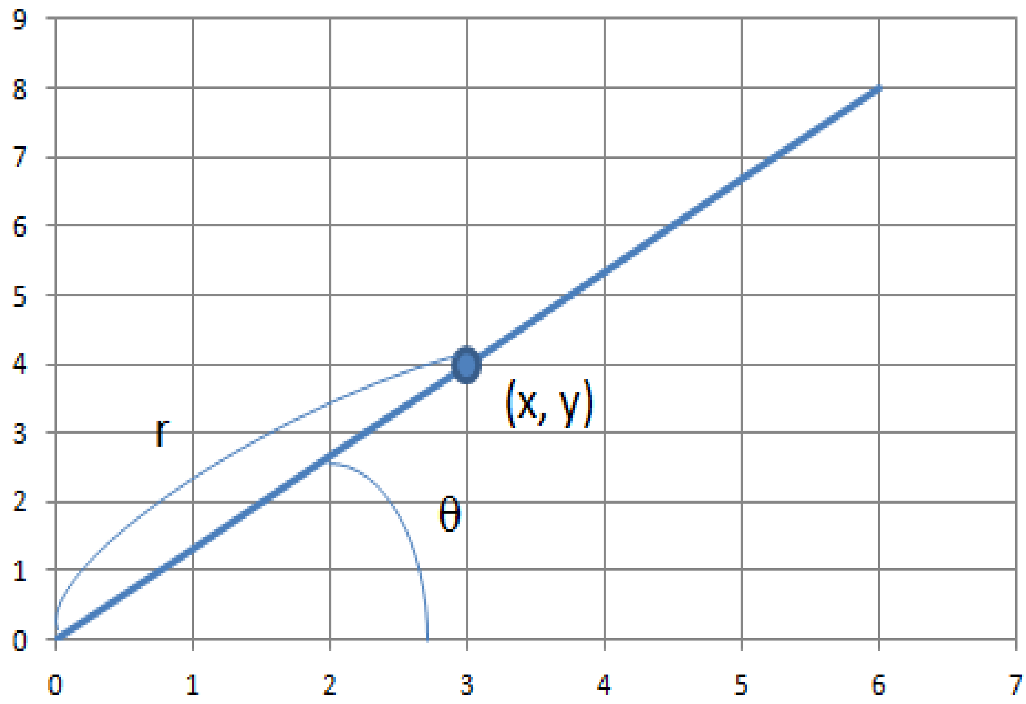
First Trig Point, Cold Ashby 1936

Modern surveying using GPS

1:25 000 scale map of Bath



Navigation by dead reckoning

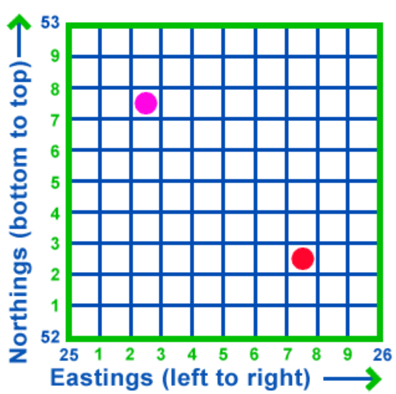


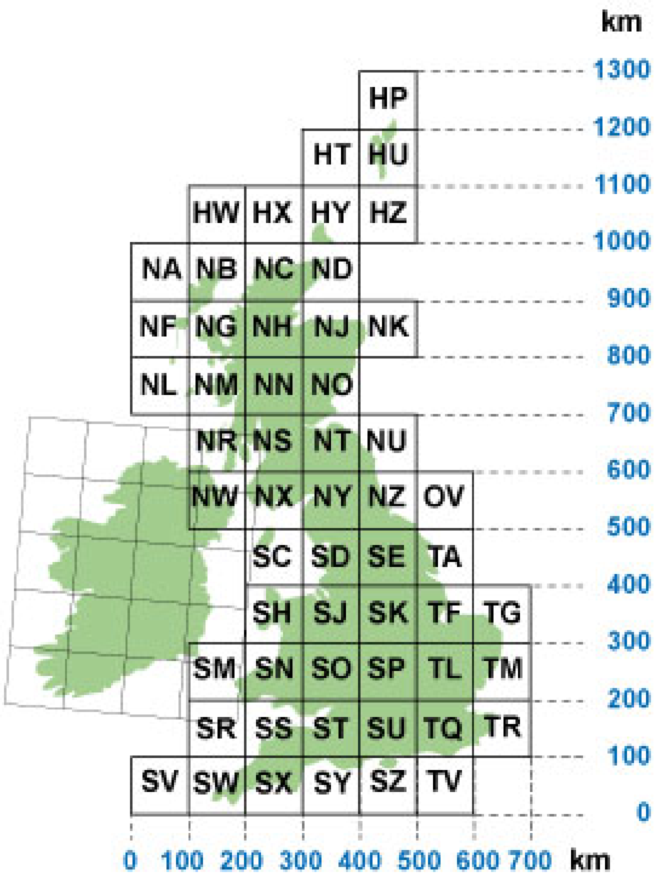
Cartesian coordinates

(x,y) Map reference

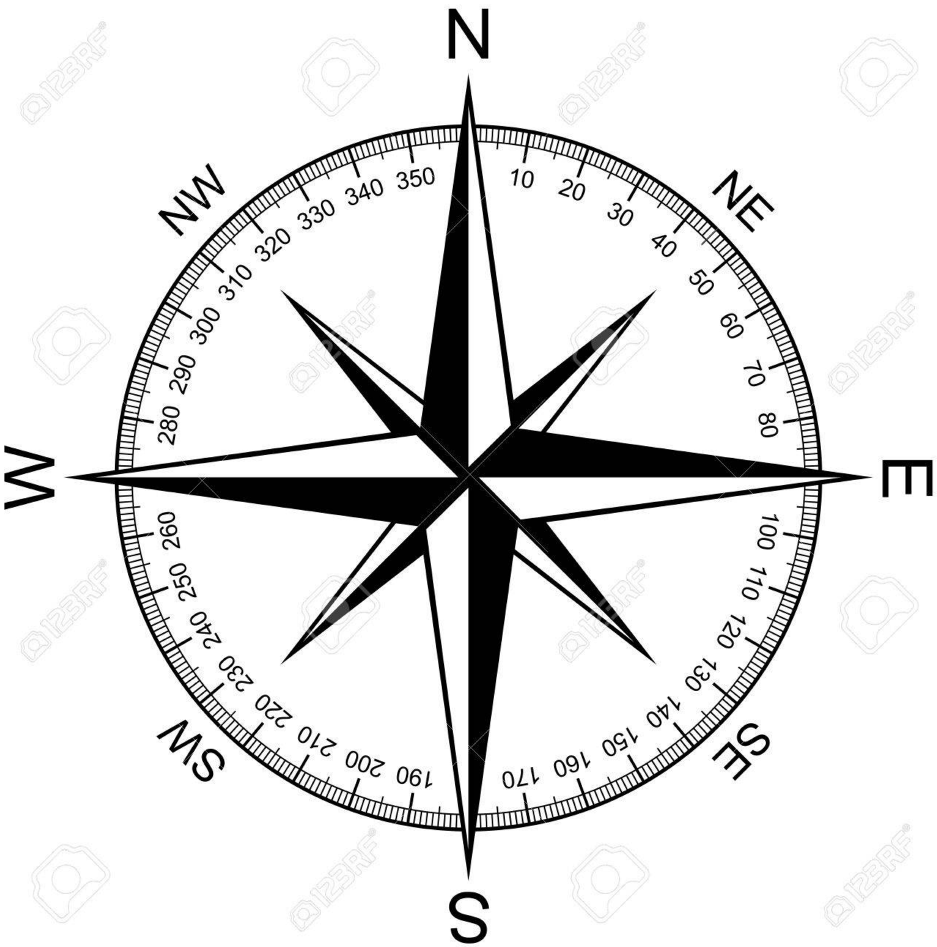
eg. Bath Centre

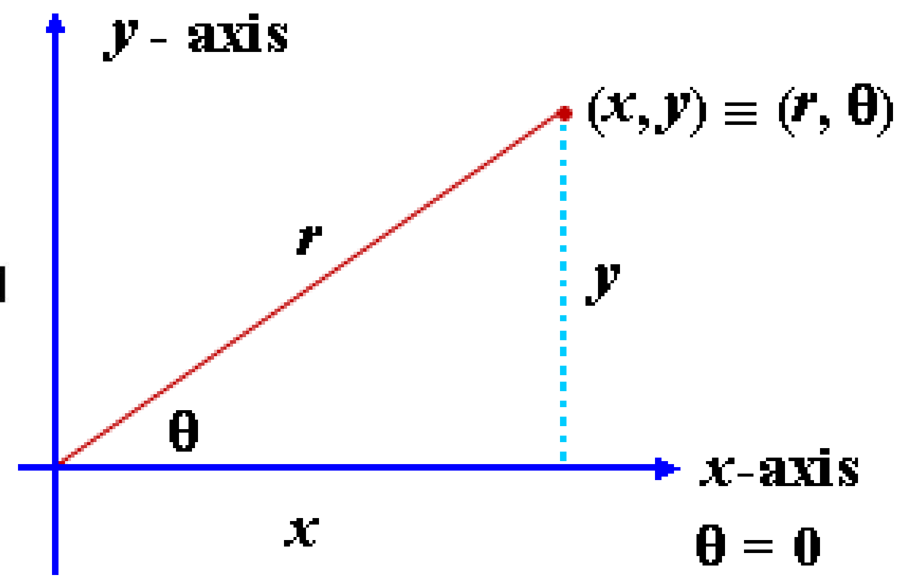
ST 7509 6492





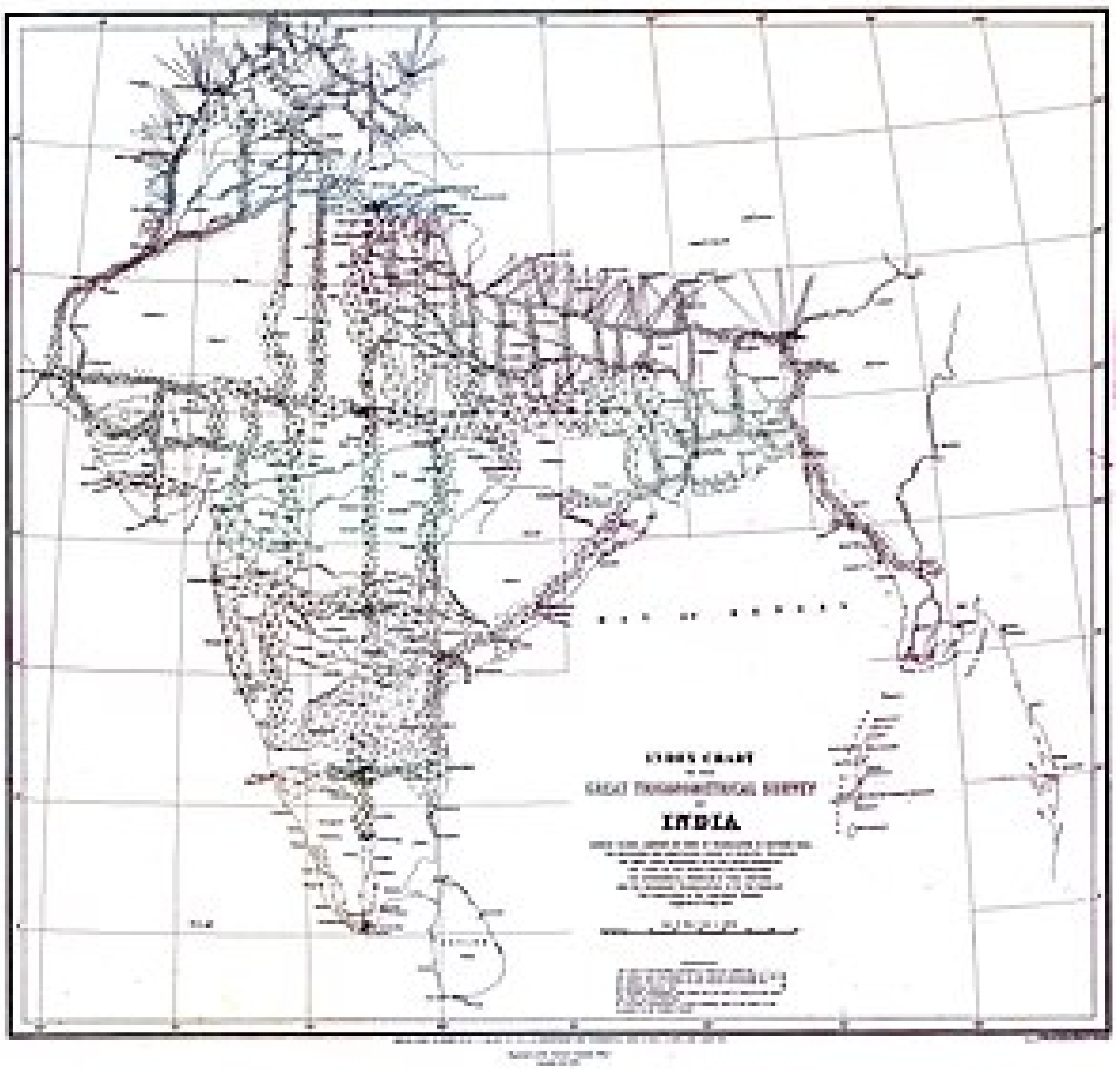
Polar coordinates: Compass bearings and distance estimates



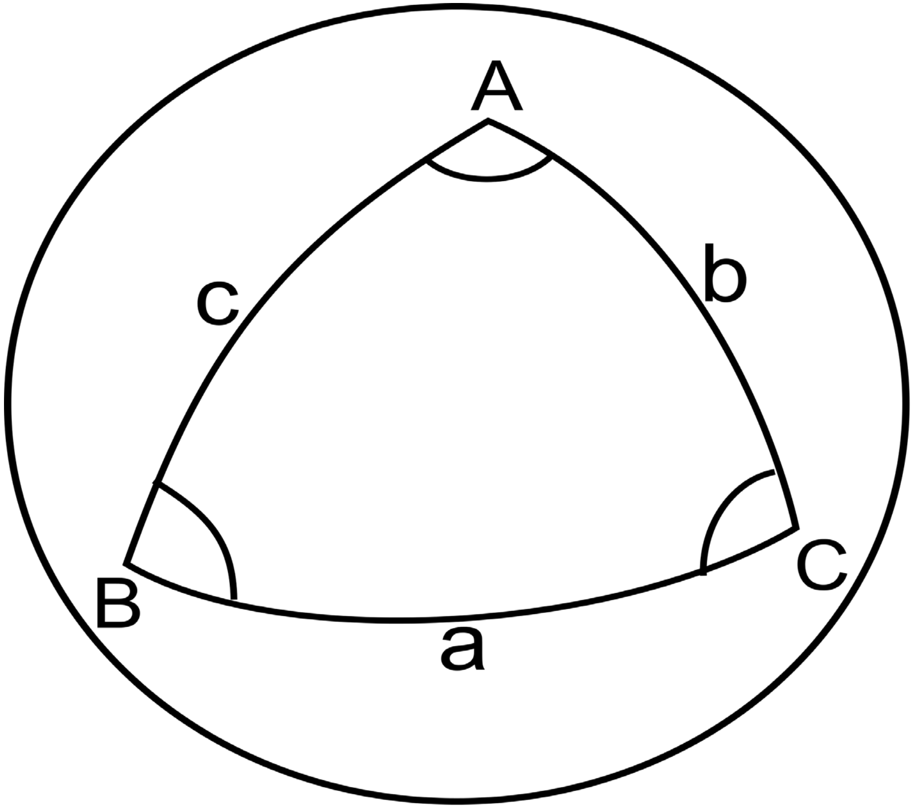


CJB: 660 Double paces = 1 km

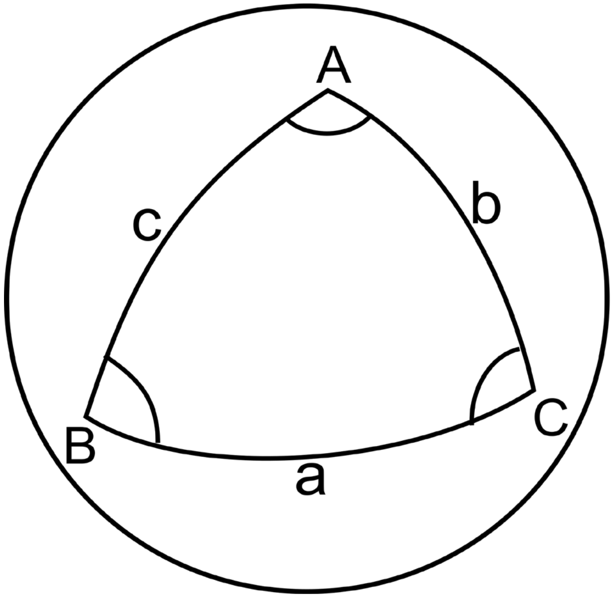
1802-1871 The Great Trigonometrical Survey of India



Triangles were large and the curvature of the Earth became important

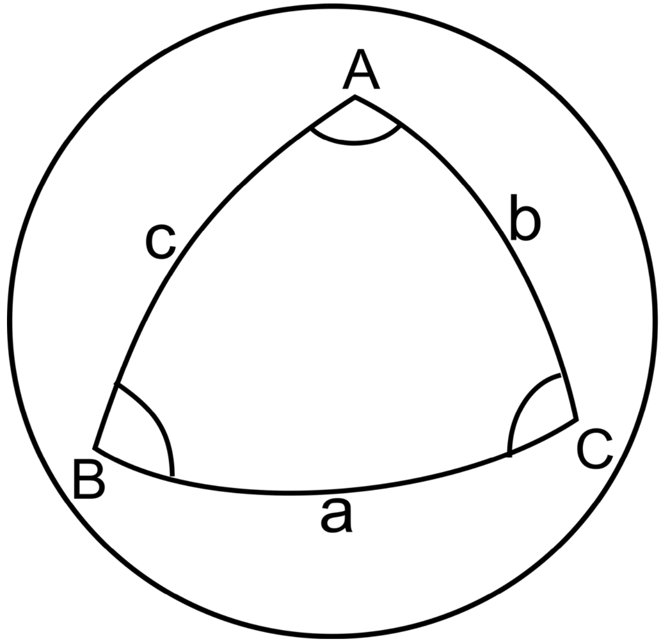


Study using spherical trigonometry



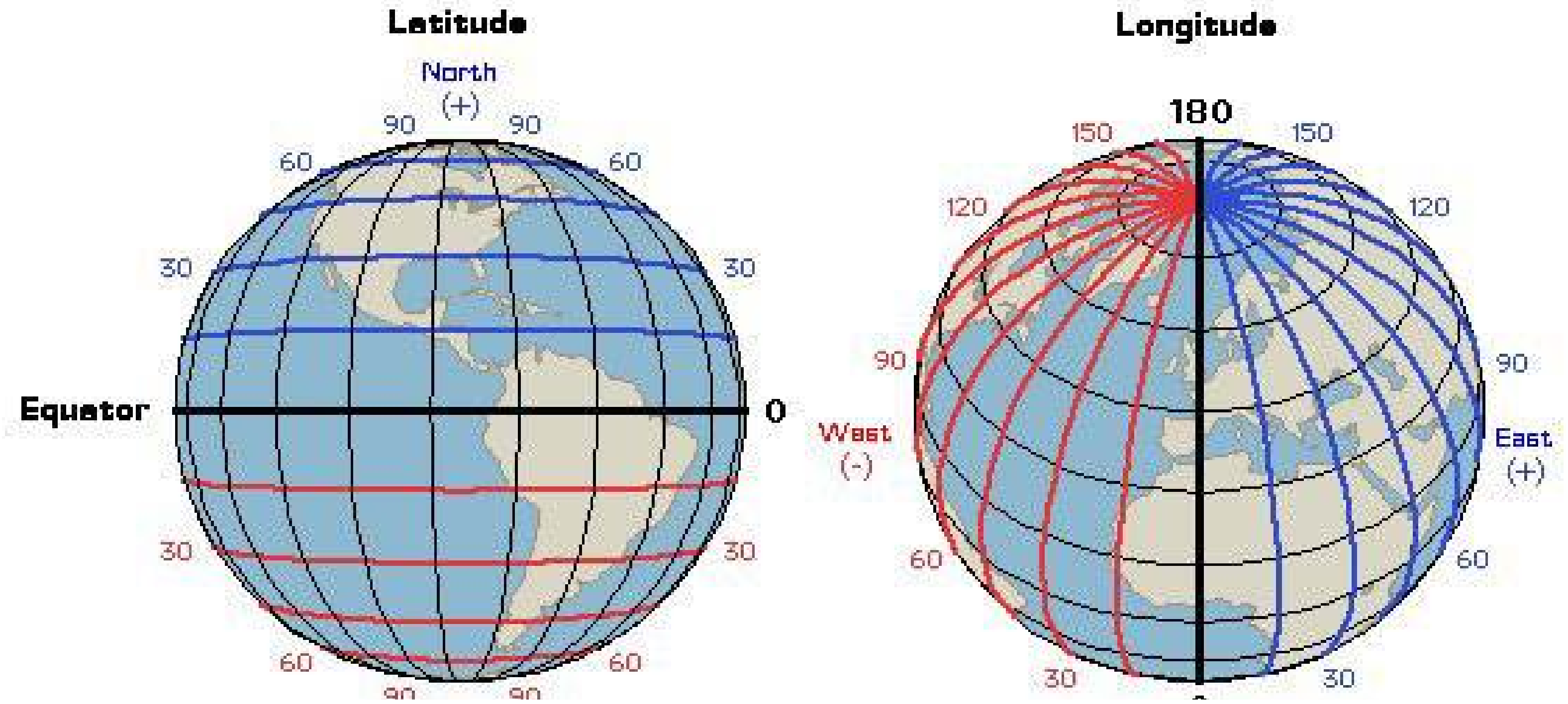
$$\cos(a) = \cos(b)\cos(c) + \sin(b)\sin(c)\cos(A),$$
$$\cos(b) = \cos(c)\cos(a) + \sin(c)\sin(a)\cos(B),$$
$$\cos(c) = \cos(a)\cos(b) + \sin(a)\sin(b)\cos(C).$$

$$\frac{\sin(A)}{\sin(a)} = \frac{\sin(B)}{\sin(b)} = \frac{\sin(C)}{\sin(c)}.$$



$$A + B + C = \pi + 3 \times \text{Area of the triangle}$$

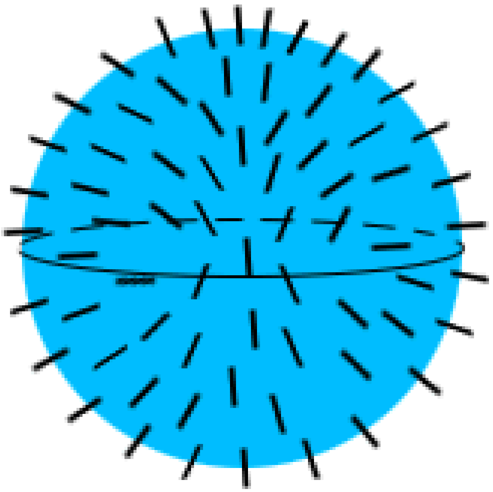
Mapping the world



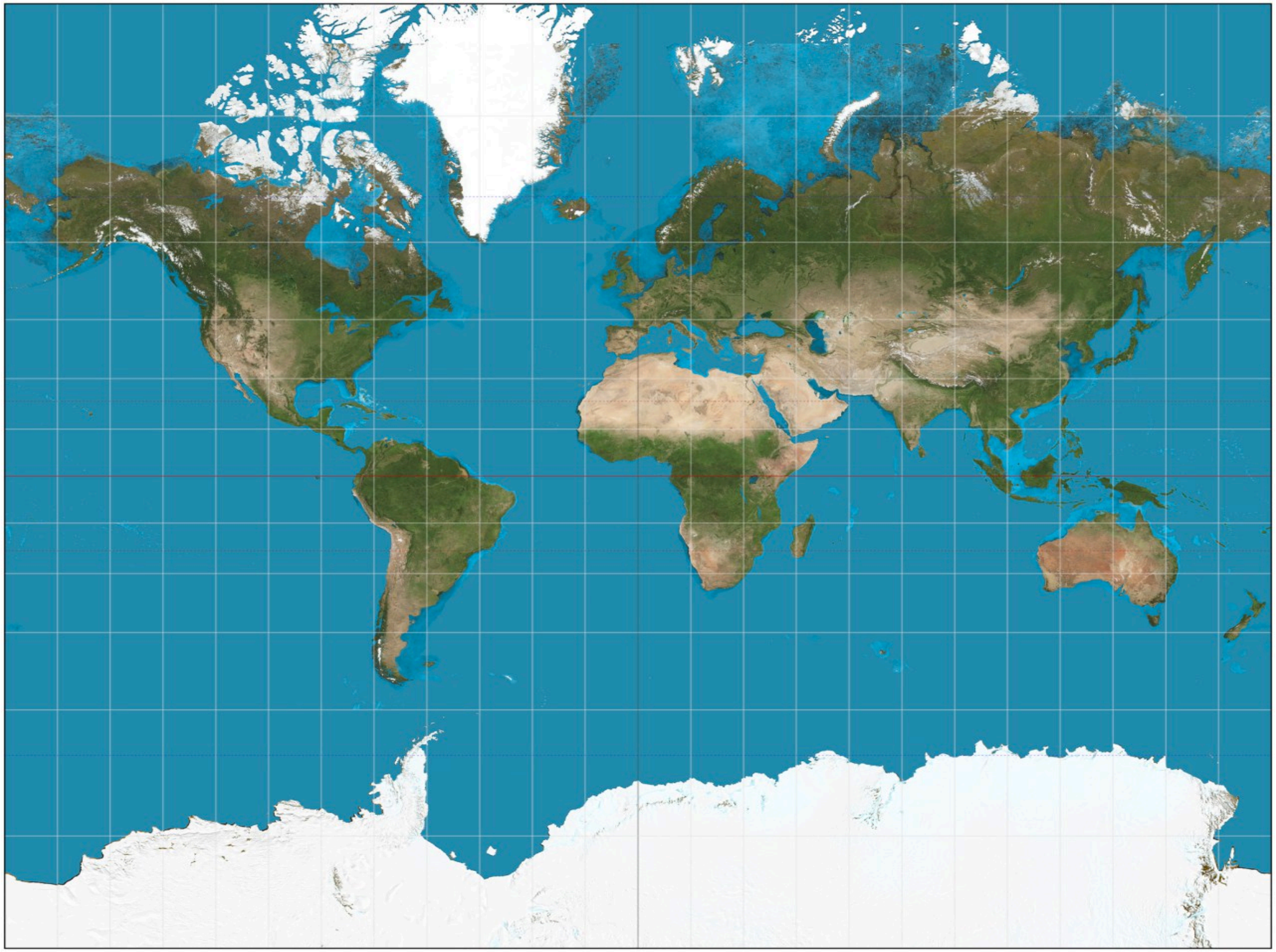
$$
\begin{aligned}
x &= R\cos(\theta)\cos(\phi),\\
y &= R\cos(\theta)\sin(\phi),\\
z &= R\sin(\theta).
\end{aligned}
$$

Map projections are needed to represent the curved surface of the Earth on a flat surface

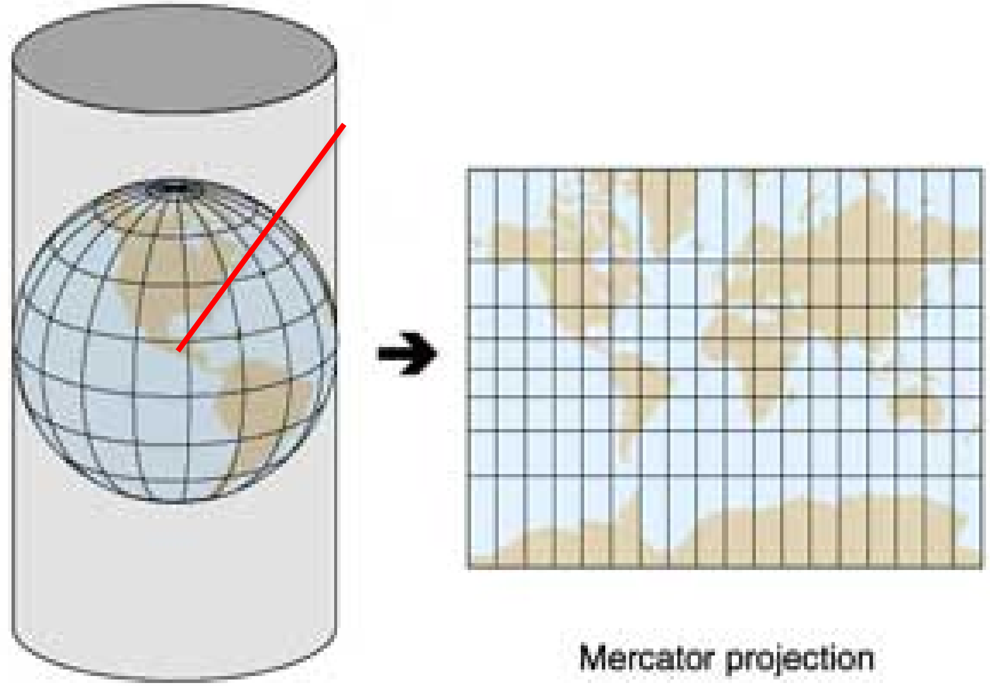
Problem: Hairy Ball Theorem of geometry says that this will always lead to some distortion at some point

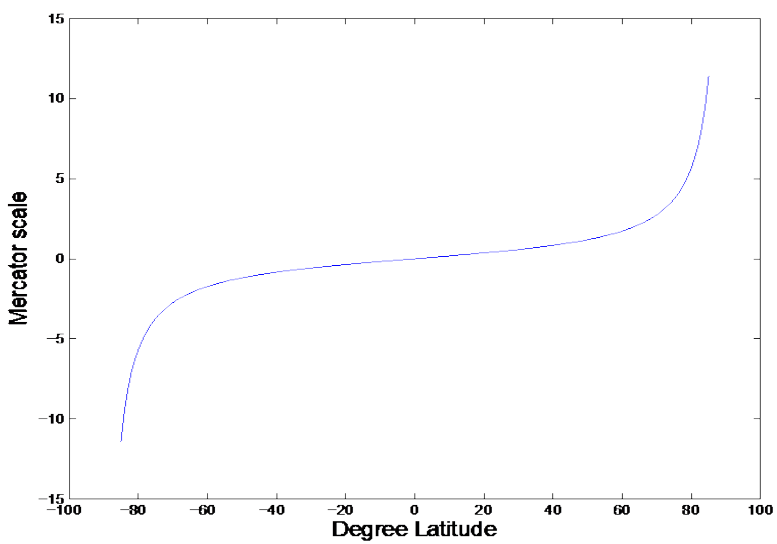


All projections are a compromise between accuracy, convenience and politics!



Mercator Projection: 1569


Mercator projection


Mercator scale
Degree Latitude
15
10
5
0
−5
−10
−15
−100
−80
−60
−40
−20
0
20
40
60
80
100

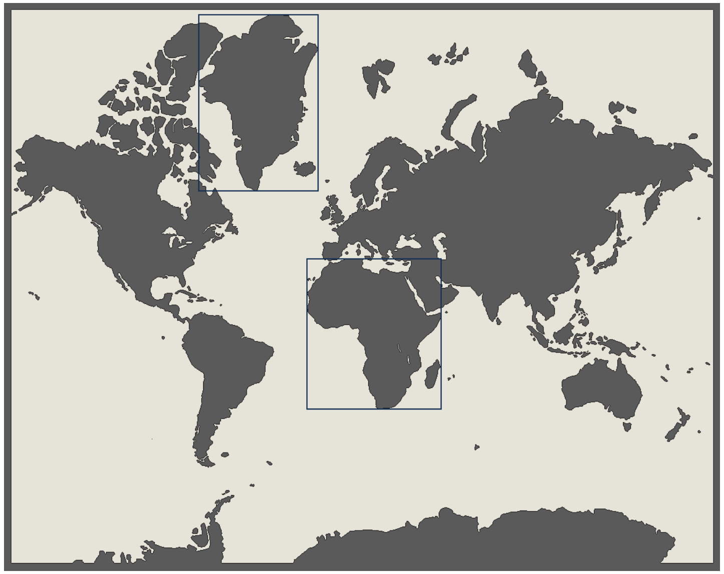
$$y = R\tan(\theta)$$

Advantages of the Mercator projection

Easy to construct

Lines of Longitude and Latitude are straight

Rhumb lines are straight



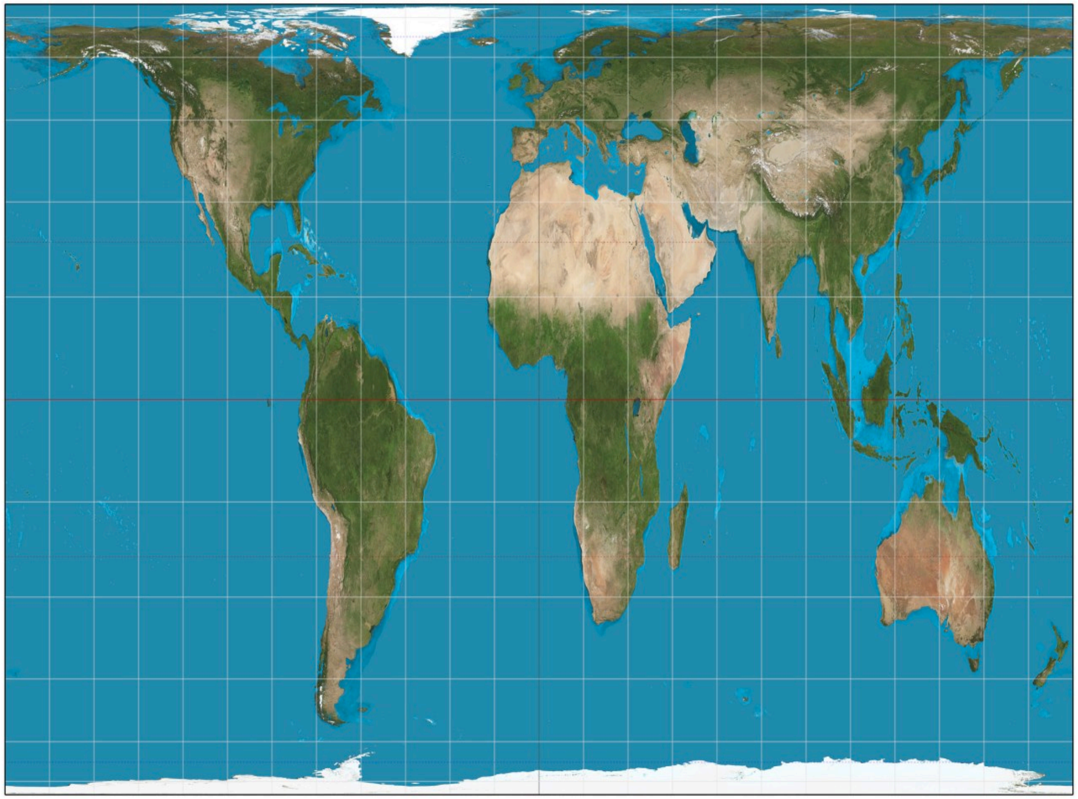
Disadvantages:

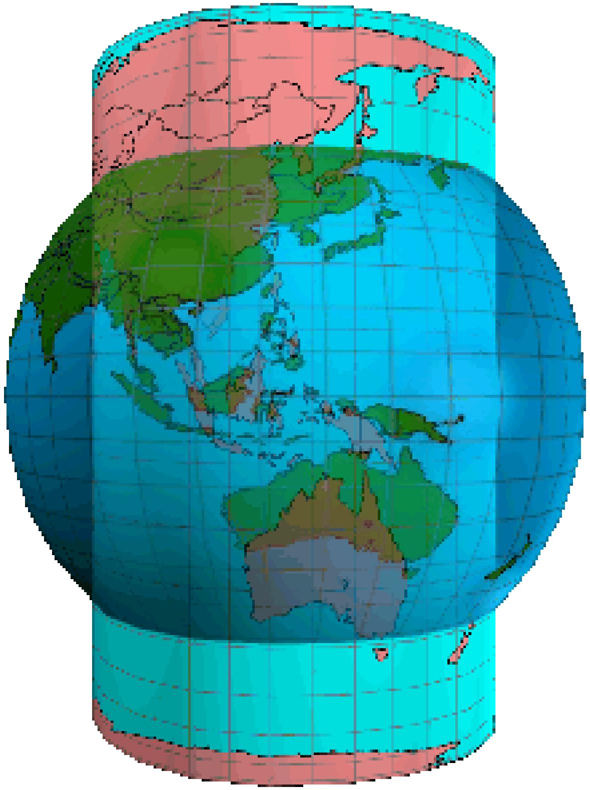
Very significant distortion at the poles

Great circle lines are curved and look longer

Gall Peters projection: 1973

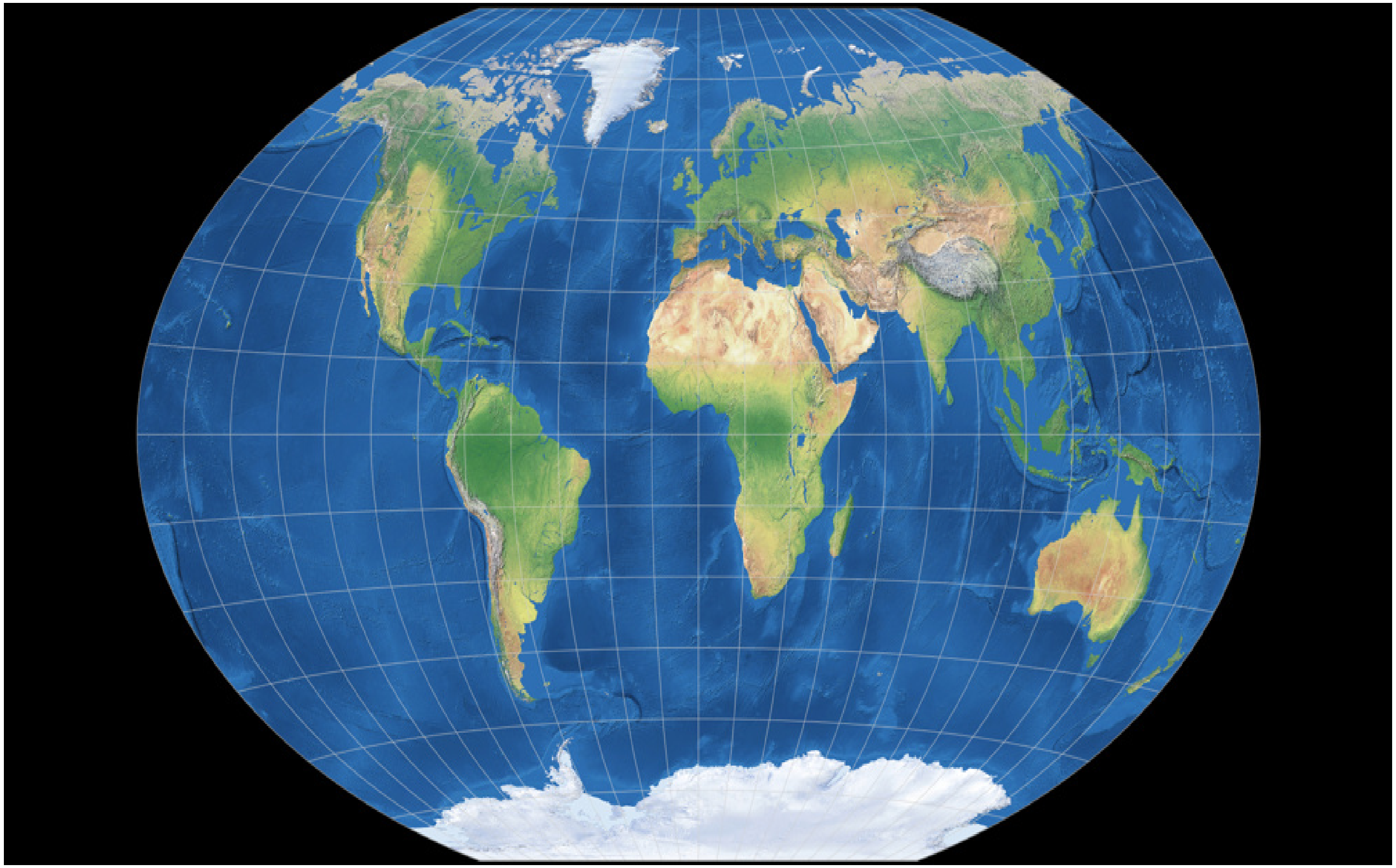
More emphasis of equatorial regions



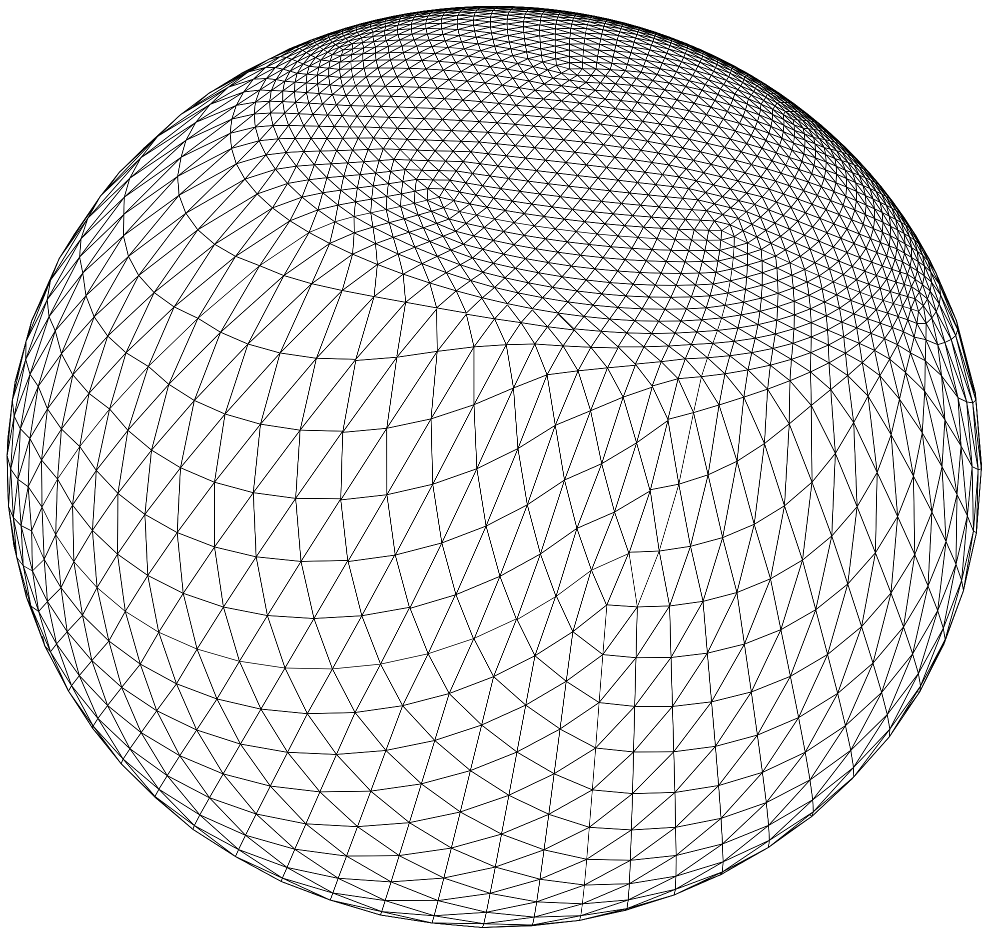


Winkel-Tripel Bartholemew Projection, 1921

Used by the Times. Good compromise, but not very useful for navigation



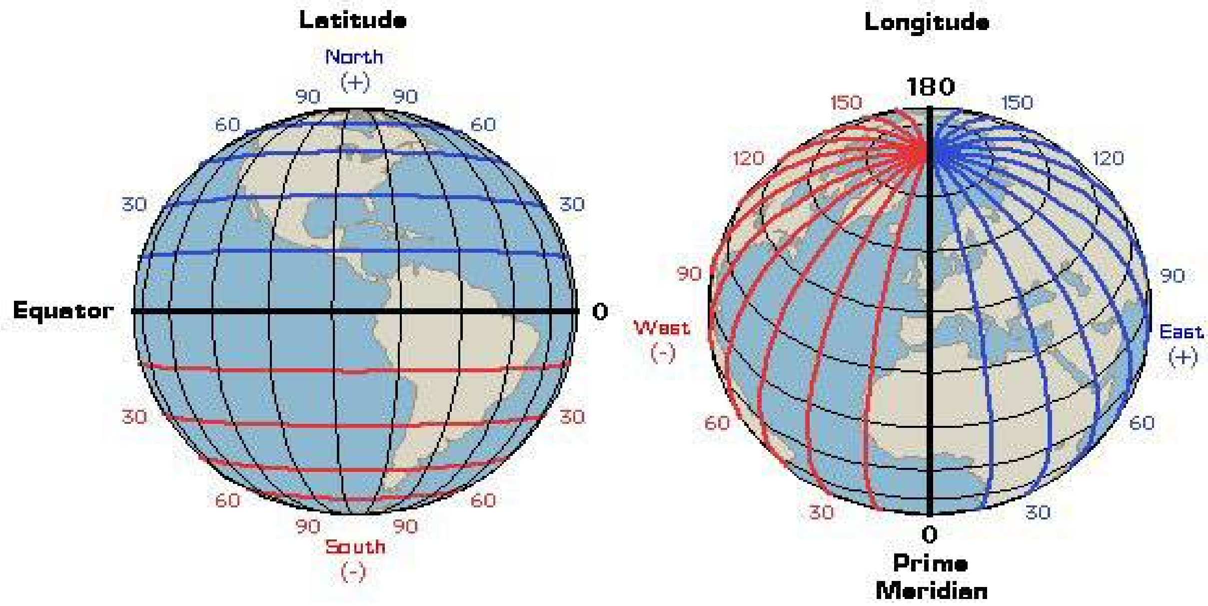
Forecasting the weather: Icosahedral grid [Met Office +CJB]



Celestial Navigation

Huge question for exploration and economics:

Finding location on the Earth



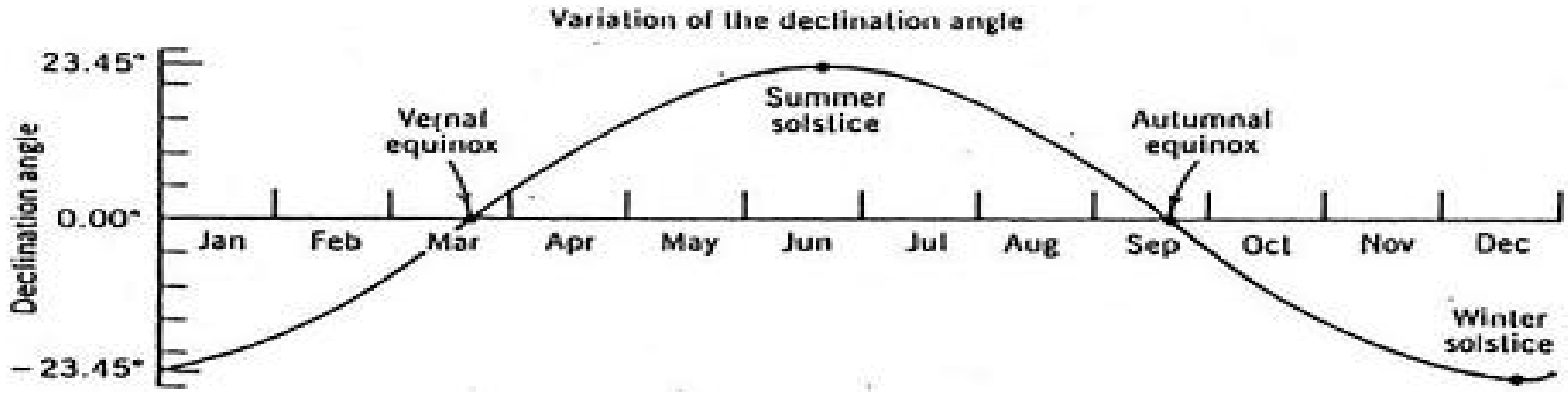
Need to find Latitude and Longitude

Columbus used a combination of dead reckoning and crude celestial navigation

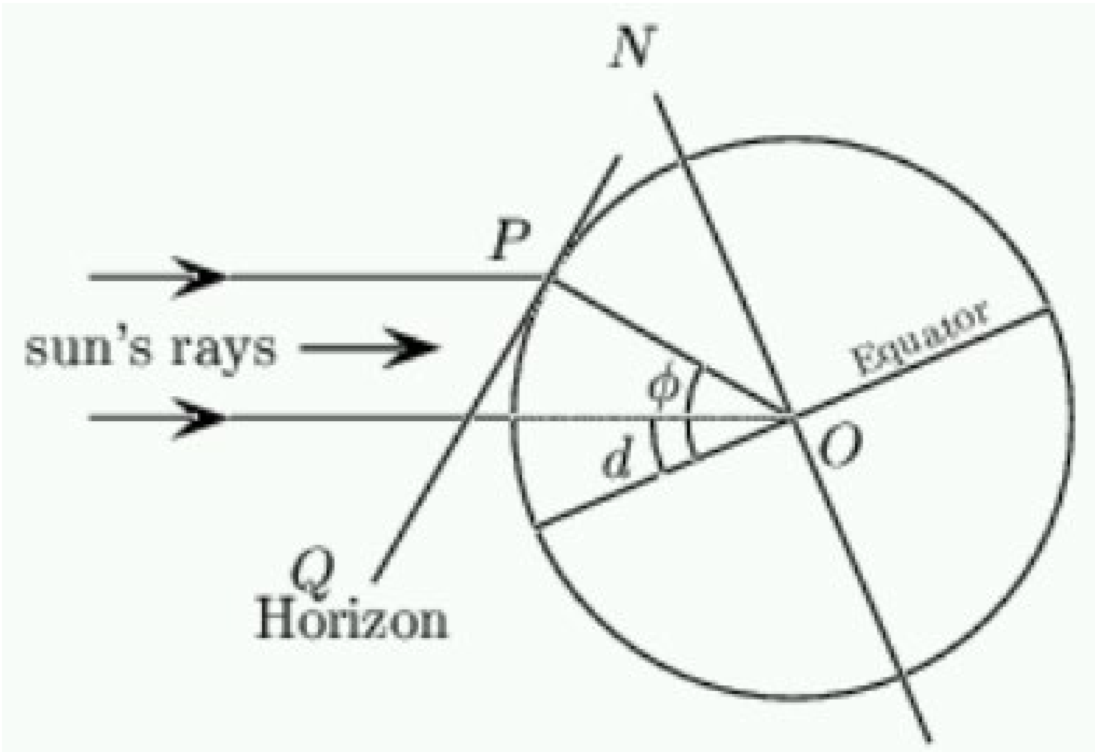
Calculation of Latitude

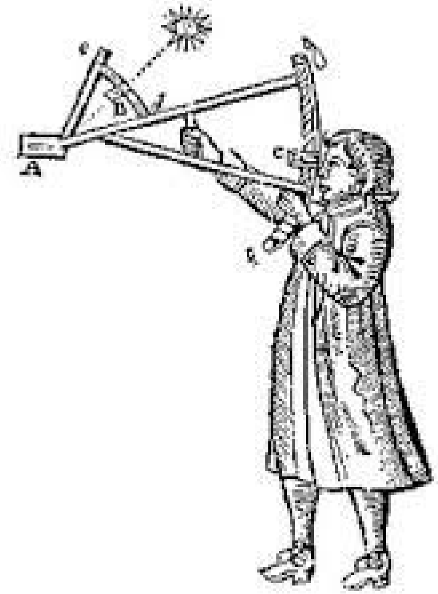
This is relatively easy provided that we can measure the height Z of the Sun above the Horizon at Noon

Declination d: Angle Sun makes relative to the Equator



$$d = \arcsin\left(\sin(-23.44)\cos\left(\frac{360}{365.24}(N+10) + 1.91\sin\left(\frac{360}{365.24}\right)\right)\right).$$





$$Z = 90 + d - \phi$$

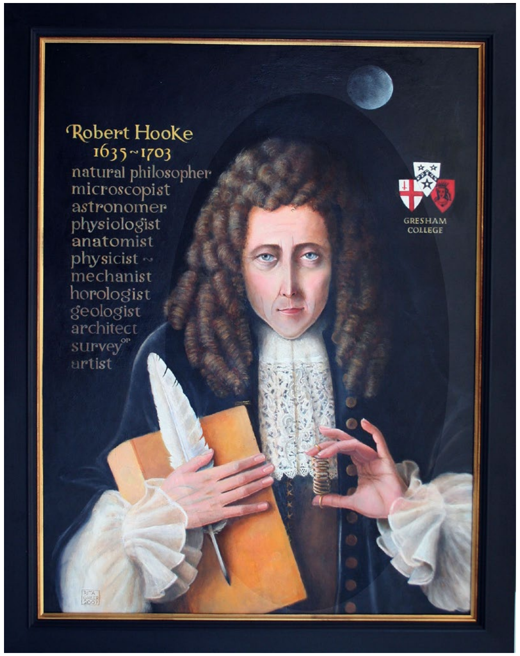
$$\phi = 90 + d - Z$$

Finding Longitude

This is MUCH harder

Obtained the reputation of an impossible problem!

Tackled by Newton, several Astronomers Royal, and Gresham Professor Robert Hooke



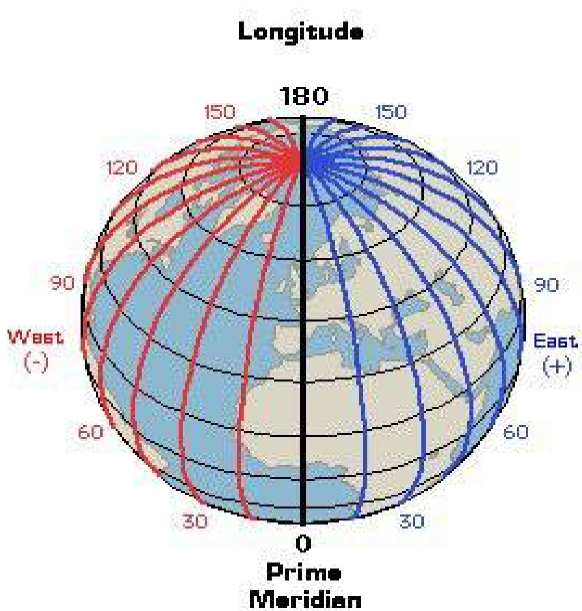
British Government offered a prize of up to £20,000 (equivalent to £2.89 million in 2018) under the 1714 Longitude Act

Key to Longitude is finding time accurately

Earth rotates 360 degrees in 24 Hours

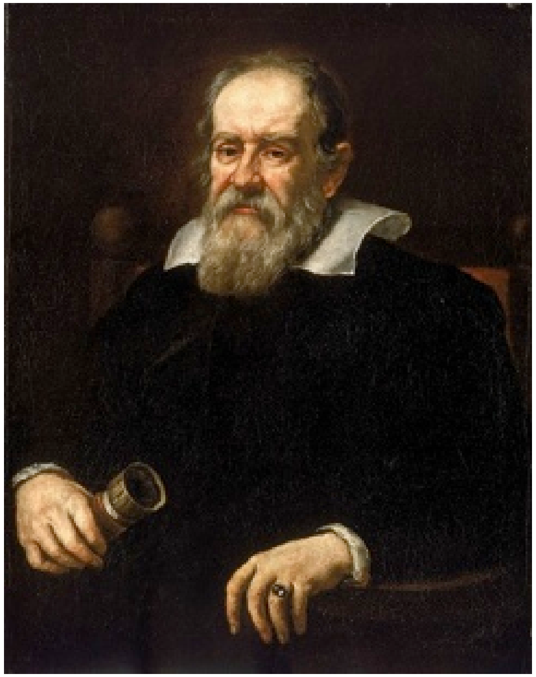
1 hour = 15 degrees of Longitude

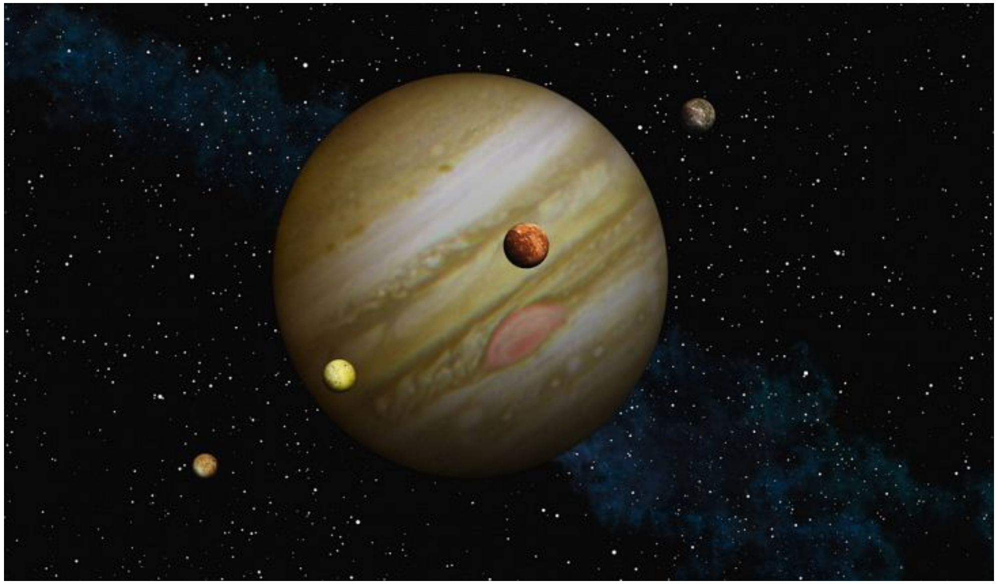
4s = 1 minute of Longitude = 1 Nautical Mile nm



Should time be determined by celestial or mechanical means?

Celestial method 1: Moons of Jupiter





Moons of Jupiter give an accurate celestial clock

Could be used to determine Longitude accurately on land

Louis XIV seeking to make his country the world leader in science commissioned astronomers to use measurements of Io's eclipses to improve the map of France

It was the most accurate map ever produced up to that time, and it revealed distances were actually shorter than had been believed.

France was smaller than thought and Louis complained that he was losing more of his realm to astronomers than to his enemies

Celestial method 2: Lunar Distances

1750s: map-maker Tobias Mayer devised the lunar distance method for finding longitude at sea.

Sailors measured the angle between the moon and a star to establish the time in Greenwich, and then compared it with the local time on board ship.

This required very precise observations

English Astronomer Royal Nevil Maskelyne was a keen advocate of the lunar method

Big problem: Calculating the exact location of the moon in advance

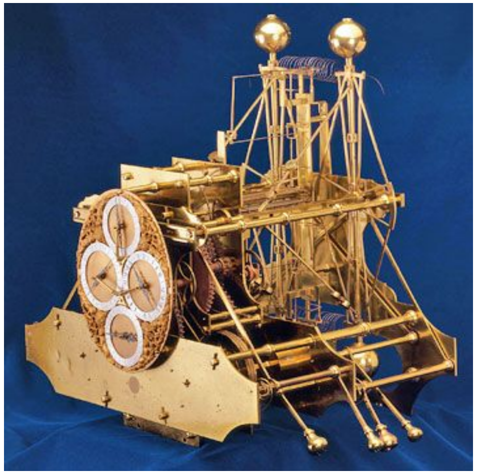
Solution: Determine and solve the differential equations describing the motion of the moon relative to the stars.

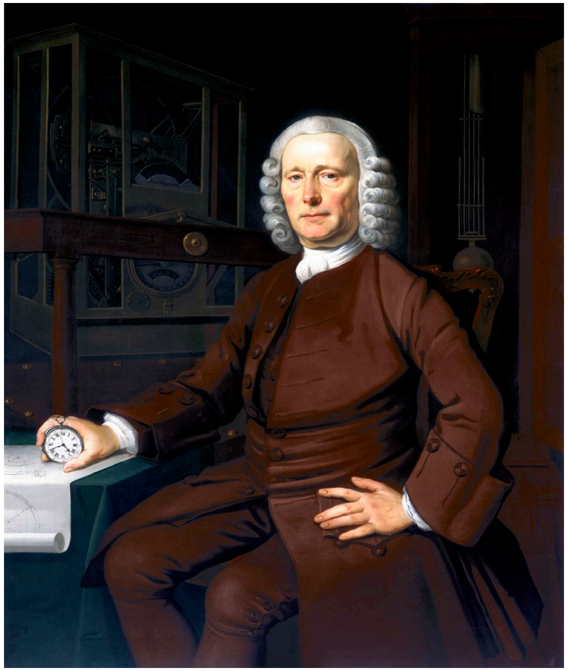
Solved by the great mathematician Leonhard Euler who was awarded £800 by the Longitude Board

£3000 to Mayer's widow

Mechanical method: John Harrison

Frictionless and compensated clocks and watches



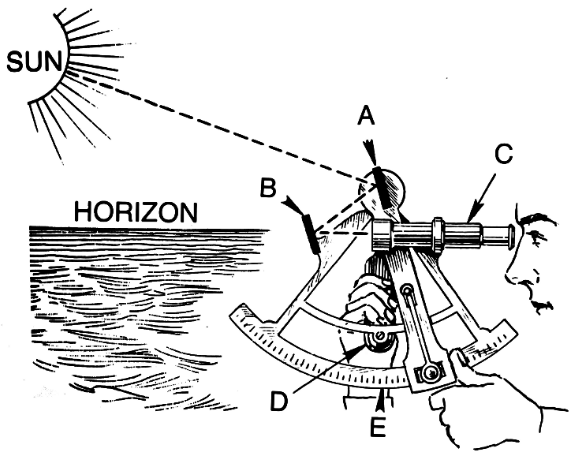


H1: 1736

H4: 1761

Modern celestial navigation: The intercept method

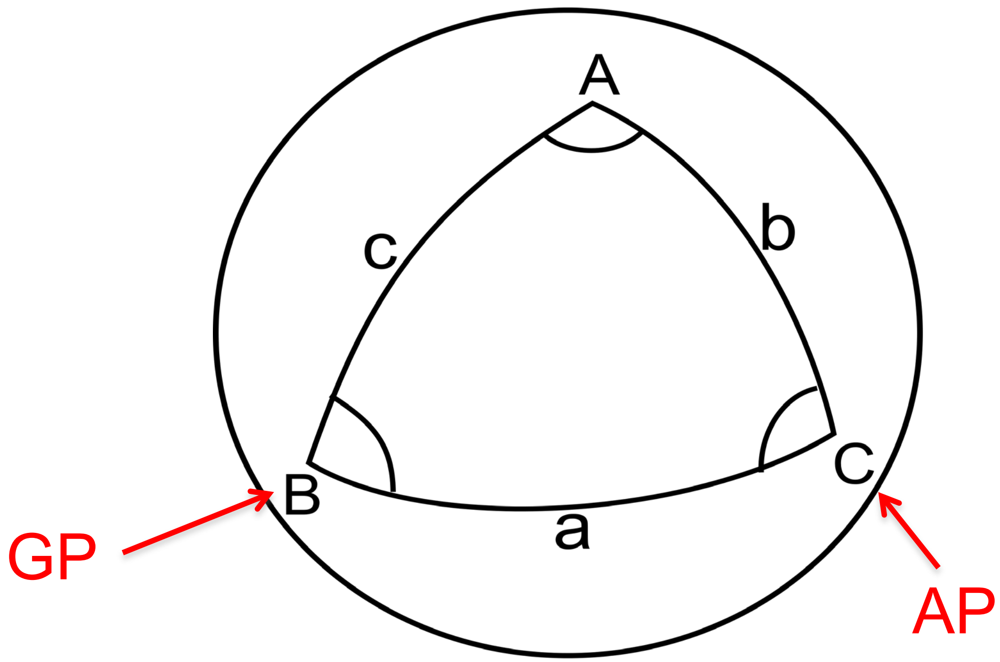
- Have an assumed position AP on the Earths surface
- Measure the height Ho of a celestial object at a precise time from your location



- Determine the geographical position GP of the object from Epheremides table

  (point on the Earths surface directly below the object)

- Use spherical trigonometry to find the height Hc = 90-a and azimuth Z = C of the celestial body relative to the AP



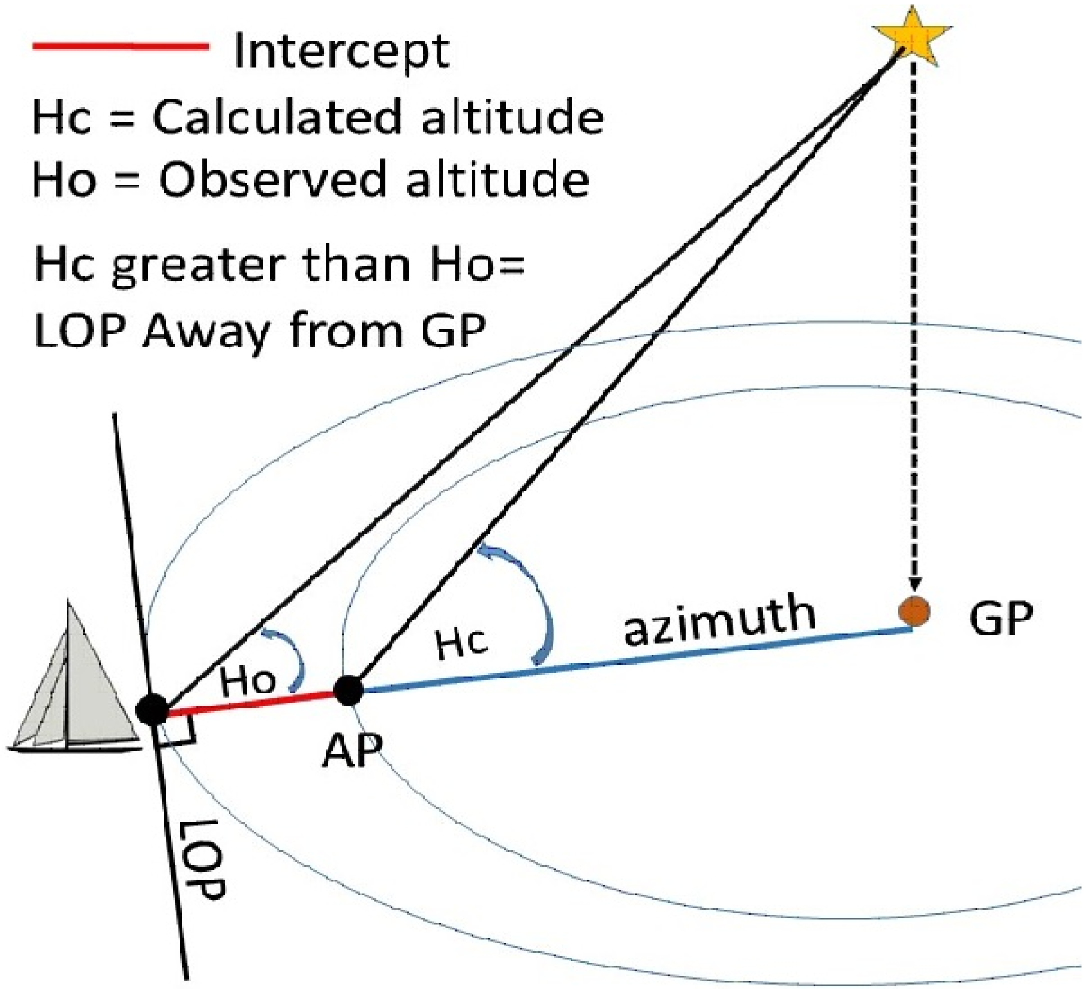
A: Relative angle of AP to GP (known)

b: 90 – Latitude of AP (known)

c: 90 – Latitude of GP (known)

$$\cos(a) = \cos(b)\cos(c) + \sin(b)\sin(c)\cos(A),$$
$$\sin(C) = \sin(c)\sin(A)/\sin(a).$$

Correct AP along azimuth line by comparing Ho and Hc



Fix position by taking several sights

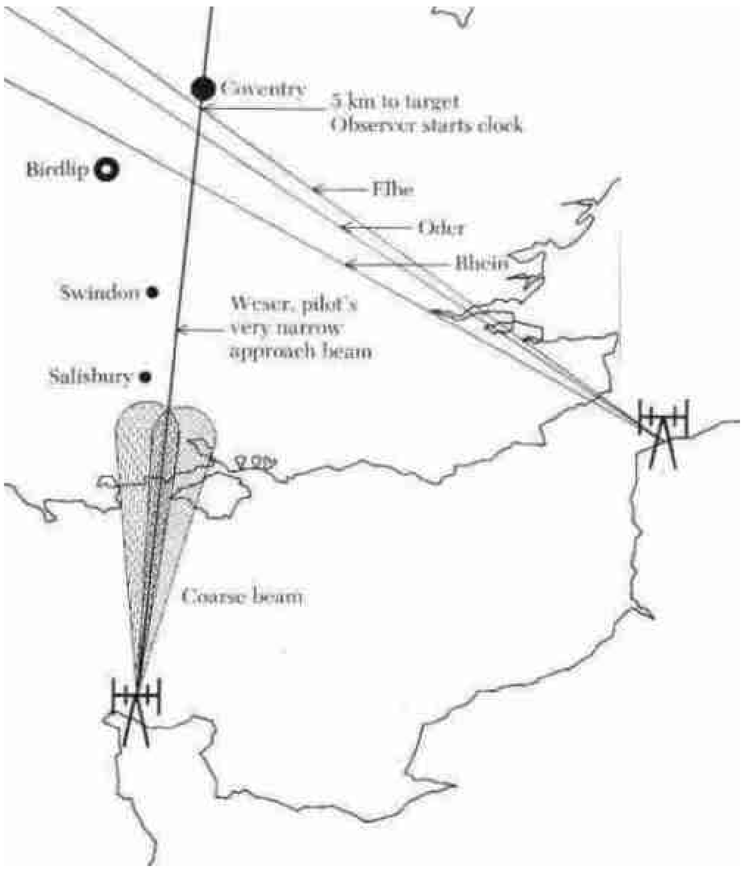
Navigation in WW2

Vital need to navigate bomber aircraft to their targets

Celestial navigation was very hard and not very accurate

German approach: 1940

Knickebein and X-Gerat Beams



Initially very effective

But could only direct a small number of aircraft to one location and was easy to jam

[R V Jones]

Gee: Hyperbolic Navigational System, 1941

Master and two Slave stations transmitted simultaneous radio pulses.

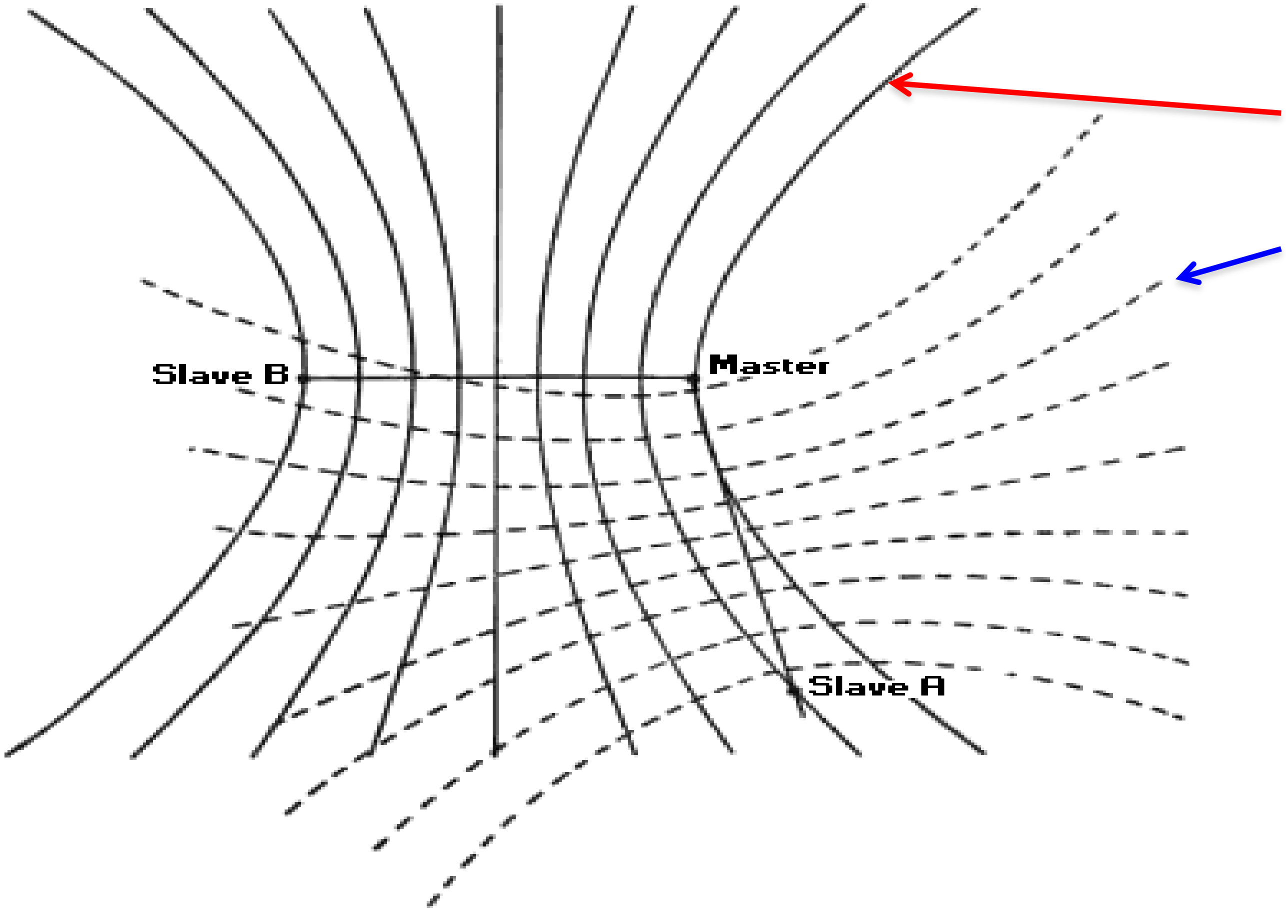
Aircraft navigator measured the time differences of the arrival of the pulses

TA: Master and Slave A

TB: Master and Slave B

DA = c TA = difference in distance of aircraft from Master to Slave A

Curves of constant difference in distance are hyperbolae



$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Or a rotation of this

Aircraft lies at the intersection of the two hyperbolae

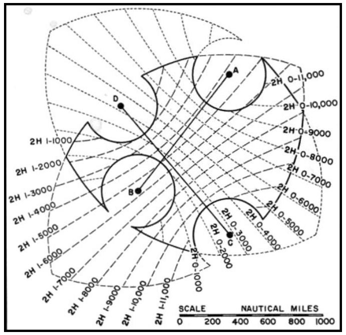
Accuracy was about 1 mile at a distance of 300 miles

Used from 1941 onwards.   Effective at the start, then less so due to jamming

Supplemented with the Oboe system for more precise navigation and bomb aiming

Hyperbolic systems were first introduced in WW1 to detect guns by sound ranging

Still in use today in the Loran system



21^{st} Century: Global Positioning System GPS

Very precise navigational system also based on time difference measurements.

Uses (typically 5) satellites to find a position

Each satellite transmits two signals at 1227 MHz and 1575MHz

Signal gives precise time and location of satellite at transmission

At X = (x,y,z) on Earth's surface the time of reception of signals from 5 visible satellites is recorded

Times for the transmission are: T1,T2,T3,T4,T5

Satellite positions are: X1,X2,X3,X4,X5

Measured time of reception is: TR

True time: T = TR + TO

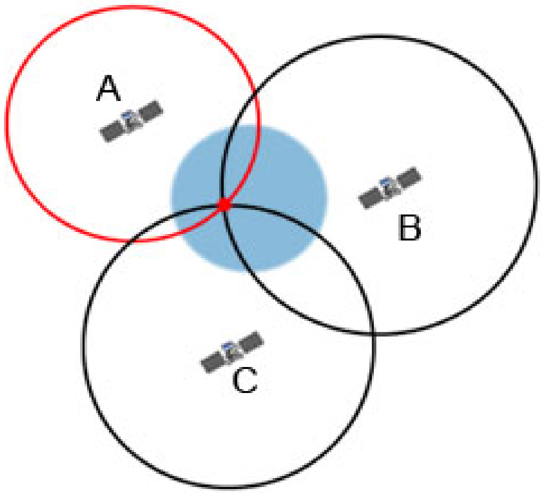
TO the receiver time offset.

Distance Di from receiver to satellite Si is

$$Di = c\,(T - Ti)$$

Four unknowns: Position (x,y,z) and time offset TO

Hence given Di for four satellites we can locate the receiver



Fifth satellite allows for the assessment and correction of any positioning errors.

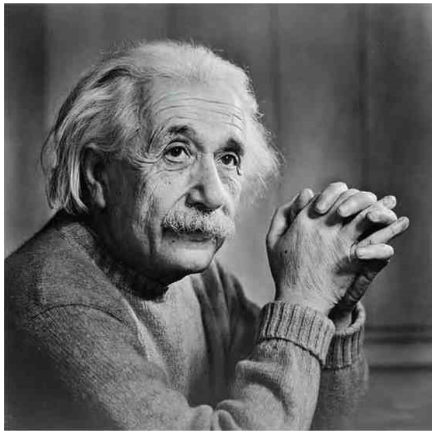
The GPS positioning accuracy is dependent on a number of interacting factors.

Some of these are similar to those in Celestial Navigation and include errors in calculating the position of the satellite and in the timing of the clocks on both the satellite and the receiver.

As all errors get multiplied by the (very large speed of light), a high precision is required.

Whereas Harrison's chronometer had to be accurate to seconds per day, the clock in a GPS satellite has to be accurate to micro seconds per day

Errors due to Einstein's Special and General Theories of Relativity



Satellite is moving fast: Clocks slow down

Satellite is in a reduced gravitational field: Clocks speed up

Combined effect = 38 micro seconds per day (11km)

Corrected in advance

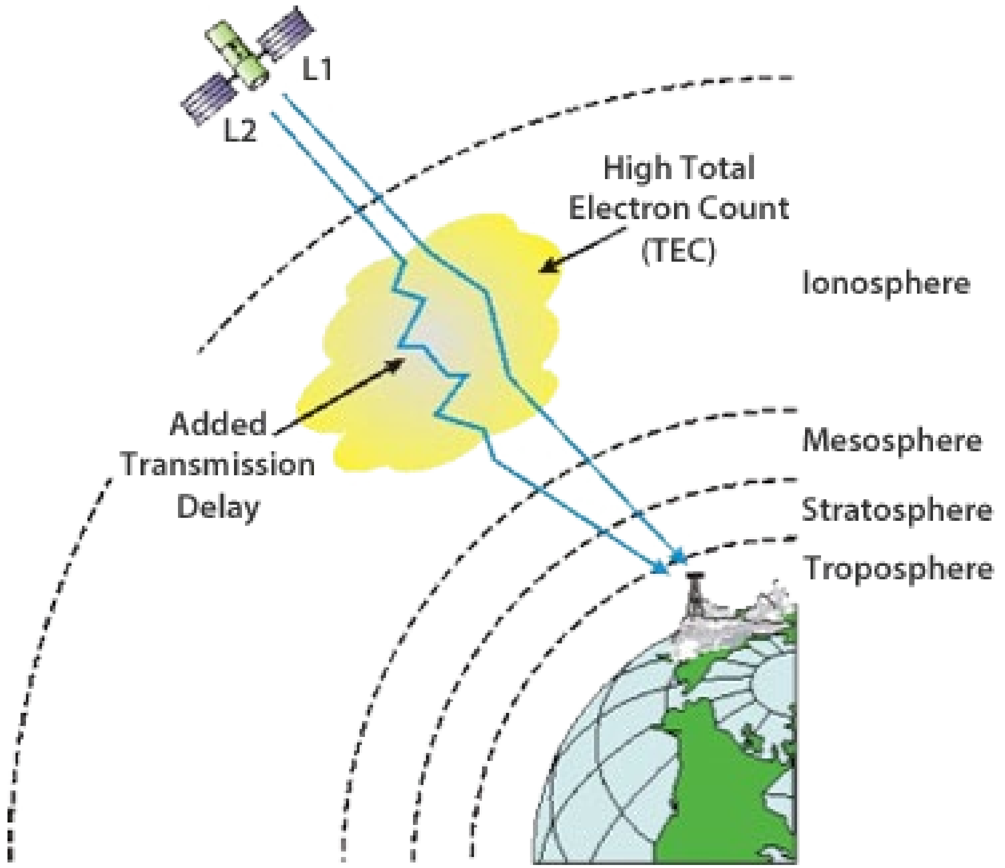
Other errors:

1. Propagation paths are not always straight

2. The speed of light isn't always constant due to electrons in the Ionosphere

Corrected by sending two signals

[Cathryn Mitchell, Bath]



Space Weather: An end to electronic navigation?