GRESHAM COLLEGE

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MATHS IN THE CITY: FUTURE CITIES

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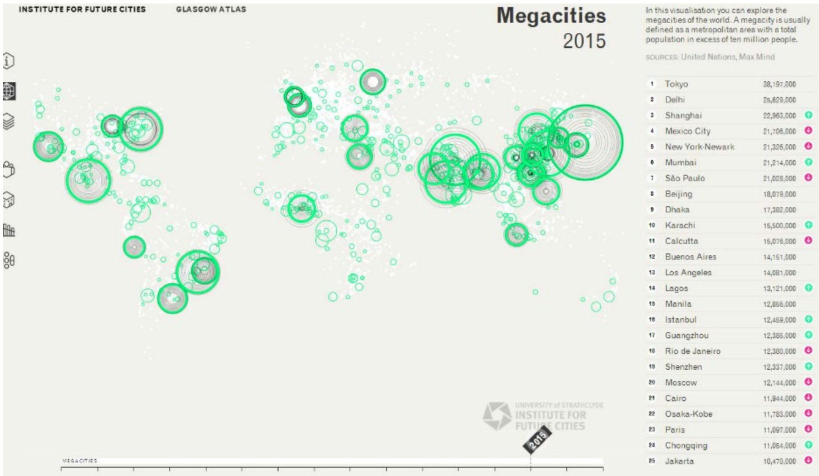
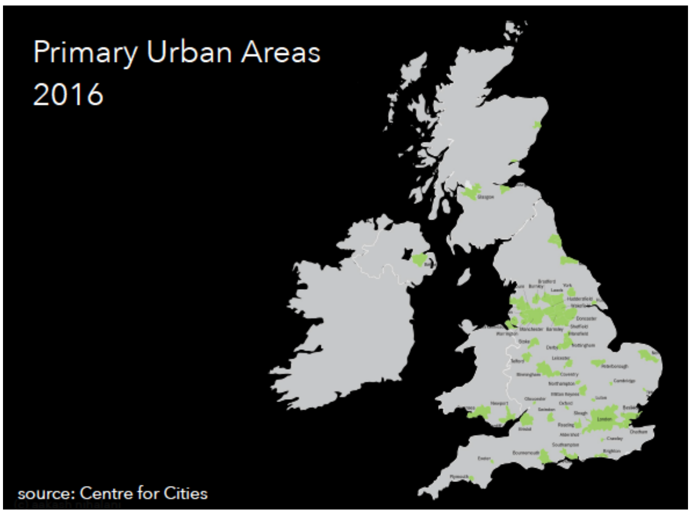
1. Introduction

Today more than half of the population of the world lives in a city. It is estimated that by 2030 this will increase to 60%, and by 2050 to 70%. Cities have been around (certainly) for possibly as long as 6000-8000 years, with the ancient cities of Uruk, Eridu, Jericho and Damascus in the Middle East, and maybe for similar lengths of time in China and in India. Now, huge cities are everywhere, and the figure below shows some of the worlds Mega Cities.



For comparison the main urban centres of the UK in 2016 are shown below.



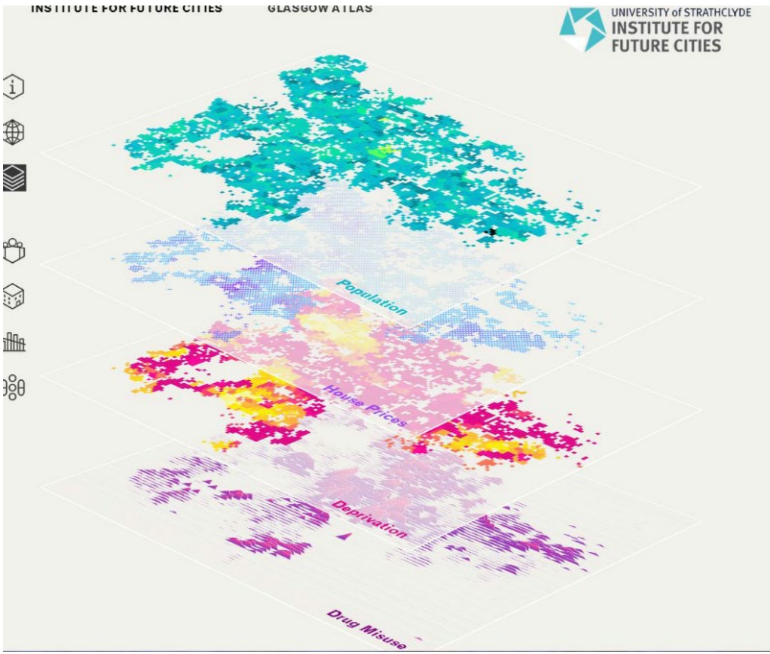
But what is a city? On one level the answer is obvious, it is a place where a lot of people live, and the houses that they inhabit. For example 12 Million people live in Greater London and over 24 Million people live in

Shanghai (illustrated), which, alongside Tokyo is one of the world's largest cities. However, for a city to function it has to comprise much, much more than just the houses for the people who live in it. For example even early cities had to have effective transport, government and communication systems, had to have means of distributing food, and (sadly) for dealing with crime and (significantly) for defending themselves against the outside world (usually through walls or similar). In a modern city, apart, in general for the need for defence, the same needs arise (although they are faster and more complicated), and have to be augmented with infrastructure for health, education, sewage, retail, entertainment, energy, cultural activities, and eating. All of these interact with each other and with the people who use them. In the modern and future *smart city,* we see these traditional infrastructures coordinated and integrated through the use of digital technology (such as smart phones and Aps), augmented with social media.

In fact we can basically define a city to be a combination of:

A local government + a dense built environment + a suitable infrastructure

It is how these all interact which give a city its character and lead to interesting modelling challenges for mathematicians. Cities themselves are not homogeneous, but often divide up into different communities, and they can show structure at many different scales. The figure below, produces by the Institute for Future Cities at the University of Strathclyde [1] shows four levels of structure in the city of Glasgow, with illustrations of (from the top) population, house prices, deprivation and drug misuse.



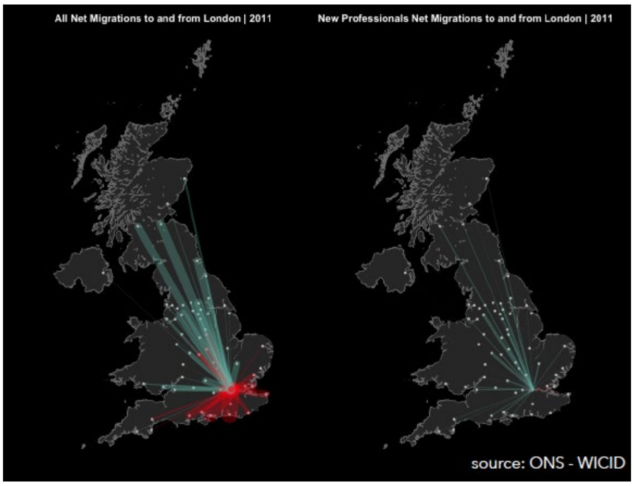
As a result of all of these interactions at many scales, cities are nonlinear complex systems, which provide interesting modelling challenges for mathematicians.

There has been an explosion of growth in interest amongst mathematicians and other quantitative scientists in studying cities. An excellent survey of this is given by Batty in [2]. Whole institutes have been set up to do this such as the Strathclyde Institute For Future Cities. In particular in applying quantitative methods, with testable predictions, to study the way that people and cities, behave. This is using mathematical concepts such as multi-scale analysis, network theory, uncertainty quantification, dynamical systems informed by large, disparate and multi-resolution data sets. This can be done both theoretically, and also using a city as a 'living laboratory' in which we can test different theories. The insights from these allow us to look ahead and to plan the cities of the future and has led to the government report [3] (see also [4]).

The term Future Cities in the title of this talk is often used as a reference to this type of 'urban analytics' which allow us to both understand their behaviour now, and predict their behaviour in the future. In this talk we will see how mathematics can give us insights into a number of aspects of modern urban life. We look at how cities grow, how the people in them communicate and form communities, how they travel around, how crime can be modelled, the important business of retail, and the future smart city. As a conclusion, we will have a short mathematical tour of the City of London.

2. How Do Cities Get Bigger?

Cities grow both by migration into them and also from the expansion of their own population. For example, the figure below shows in blue the migration of people (left) and professionals (right) to London (mainly from the North) and in red where people are leaving to (mainly in the South).



Clearly, the more people there are in a city, the more living spaces that are needed, with the number of such spaces proportional to the number of people. However, this only gives a partial insight in the questions of how cities grow, and whether there is there a limit to how large they can get. Larger cities have more professionals and are denser and more efficient and productive. However, they are also more unequal. Mathematics can give us insights into all of these questions. To help answer this question we need to understand how the infrastructure of a city scales with the size of the city itself.

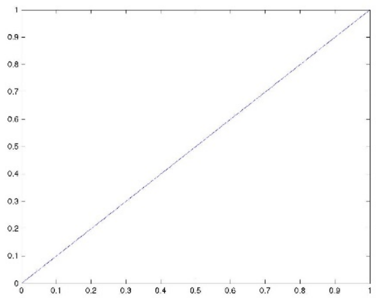
One of the questions I like asking my (first year maths) undergraduates, usually attributed to Enrico Fermi, is 'How many piano tuners are there in Bath?' (The original city referred to was New York.) Whilst this might not look like a maths question, it can be tackled using mathematics. The reasoning is that a certain proportion X of any city population will own a piano, such a piano will need to be tuned once a year. Assuming that it takes about half a day to tune a piano. If the city has N inhabitants, then N*X pianos need to be tuned in one year, so

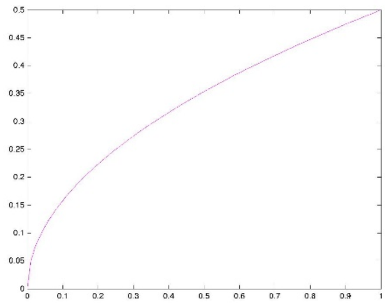
$$Z = N*X/(2*365)$$

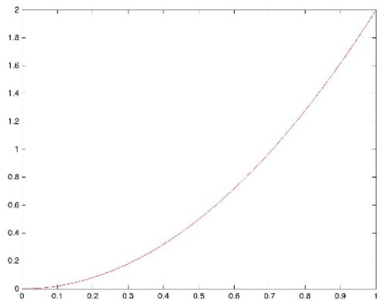
pianos are tuned in each half day. The value of Z then gives us a first estimate for the number of piano tuners.

The purpose of this calculation is that it shows us how many piano tuners need to move into a city if the piano playing population is to survive. The number of piano tuners is something that grows *linearly* with the size of the city. So that if the city doubles in size then we need twice as many piano tuners. Other features of a city grow linearly, such as the number of schools and also the number of jobs. But is this true of everything, for example airports, power stations, shops, train stations, and roads. Similarly, if a city doubles in size, do we need twice and many police officers? These questions of scale are important as we plan for the growth of cities. If we glue two cities together, do we simply get a city double the size, or do we need more or less facilities.

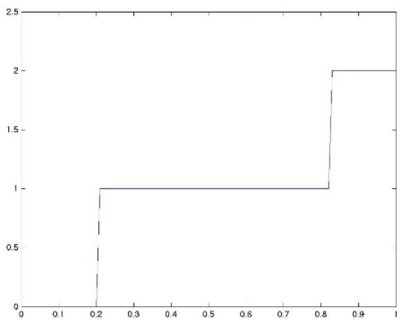
As well as linear growth we can see sub-linear growth (slower than linear), and super linear, and indeed, exponential growth. These are illustrated below







A further type of growth involves thresholds. For example a city has to be of a certain size before it can have an airport or (perhaps) a university. It will then have to grow much larger before another airport (or university) is built.



We get *sub-linear growth* when the size of a city allows economies of scale. For example the city of Bristol has one huge hospital, Southmead (illustrated), rather than lots of small ones.

Alternatively, if we have N people in a city then there are

$$C = N*(N-1)/2$$

possible interactions between them, or to put it another way, this is a measure of the number of parties going on in a city at any one time. The number C grows proportionally to the square of the number of inhabitants and grows at a super linear rate. This is important if we consider the number of journeys that people will be making as part of these interactions, or the number of (mobile) phone calls that they will be making. As a consequence we might expect the number of journeys on roads in a city to increase in a similar manner. This is true, but interestingly the total length of roads in the centre of the city grows at a sub linear rate. This well known effect is due to the increasing densification of city centres. The opposite effect occurs in cities with commuters where the road network does scale much more with the square of the number of commuters. In conclusion, careful research has shown the following different types of growth, all of which we need to plan for as cities grow:

Linear: Educational workers, health workers, piano tuners, sewage

Super linear: Professionals, commuter roads, communication networks, journeys

Sub linear (including threshold): Hospitals, airports, city centre roads.

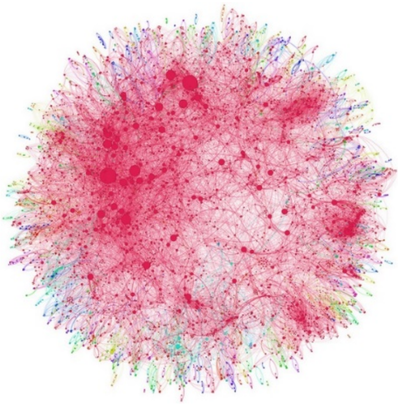
3. Who Knows Who?

3.1 People Networks

It is obvious that cities are made up of people. The character of a city is largely a function of the way that all of these people interact with each other. In particular the nature of their social interactions, and the friendship, and acquaintanceship, groups that they form. Much of the interaction now occurs either through mobile phones or by social media. It is possible to monitor this, and this gives us a way to see how the city is functioning, and its inhabitants are interacting.

A mathematical way of representing this is via a network. Networks form a very powerful mathematical tool for studying many aspects of modern life including roads, services (gas and electricity) and social behaviour and a survey of them is given by Newman in [5]. In such a social network we have nodes who are people, joined by edges that represent the links through friendship and other connections. As we have said above, in a city with N inhabitants the number of nodes is N and the number of edges is proportional to N squared. An example of a small network is shown below

Larger networks are more complicated as the following illustration shows which attempts to show the friendship networks in a large city.

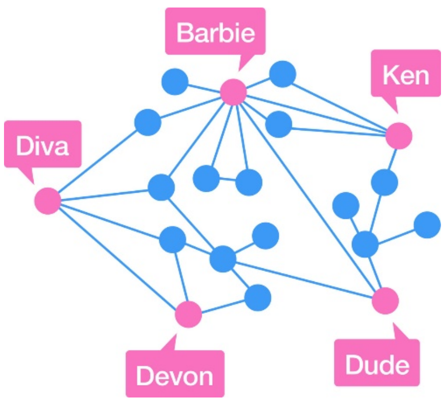


Despite this complexity we can use mathematical techniques to start to make sense of how this network operates. The more links there are to any one person (or node) the more friends that they have. The popularity of a person can thus be judged by this number, called the *degree* of the node.

If a person has many friends, then they will connect to many nodes which on average will have fewer friends than they do. This means that if you are an average person, then you are more likely to connect with people who themselves have lots of friends. Thus, on average your friends have more friends than you have. This surprising, but mathematically rigorous result, is called the *friendship paradox* [6]. It gives us a bit more of an insight into the way that cities grow.

3.2 How cities form communities, and the communities that they form.

Most cities divide into communities, and the rise of social media confirms that such communities are very much present in urban life [7]. Traditionally, for example, these communities may be due to race, class, occupation or religion. In a more modern city, we can also see communities developing through the way that we communicate with each other. It is the way that these communities form and interact which helps give the city its character. Is a city the sum of its communities, or is it more of a whole? Also can one city be made up from the communities of another? We can study these communities through the mathematics of networks and how we can divide them up. For example if a person A is friends with B, B is friends with C and C is friends with A, then we have a friendship group. Typically any network divides up in to groups and if there are few (or no) connections to other groups then these form a community. An example of this is shown below with a group of imaginary (American) friends

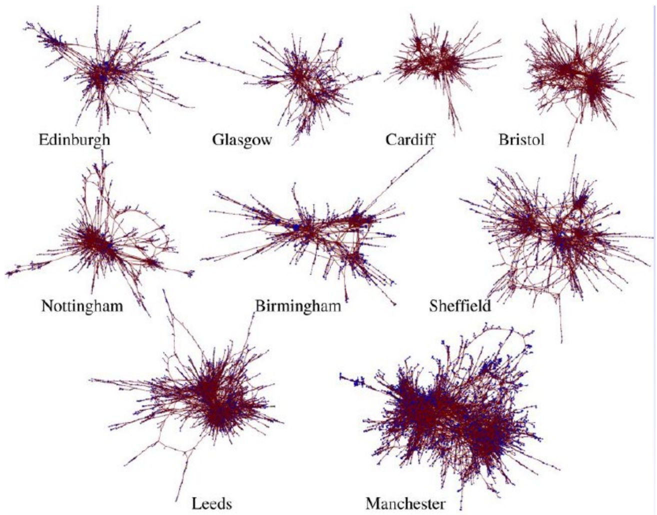


In this network we can see that whilst Barbie has lots of friends, the friends of Ken form a separate community from the friends of Diva.

To be precise we can think of any network as having communities which have lots of connections within them, but few with others. Any network will have R essentially self-contained communities and the *modularity* Q of a network is a measure of how easily communities form. If Q = 1 the network has a clear community structure, and of Q = 0 it is a single community. A convenient way of representing, and then studying, social interactions mathematically, which allows us to calculate Q, is to use an NxN adjacency matrix A. This is an array of numbers with N rows and columns. Here N is the number of people in the city. We think of lining up all of the people in the city and then number them from 1 to N. The adjacency matrix has entries either 0 or 1. If the i th person knows (or is 'friends' with) person j, then $A_{i,j} = 1$, otherwise $A_{i,j} = 0$. Using the adjacency matrix, Newman [5] gives the following formula for Q, where the sum is taken over all pairs of nodes.

$$Q = \frac{1}{2m} \sum_{i,j} \left(A_{ij} - \frac{k_i k_j}{2m} \right) \delta\left(C_i, C_j\right)$$

Here, m is the total number of edges in the network, k_i is the degree (the number of friends) of the ith node (person), and $\delta(C_i, C_j)$ is one if I and j are in the same community, and is zero otherwise. We can now look at a practical way of applying this formula to a city. The figure below, taken from [7] shows the social network of a number of cities in the UK, determined by looking at Twitter accounts. In this network each node is a Twitter account, and the edges show which Twitter accounts are connected to each other.



Next we see if we can divide the city into communities. The way to do this is to take an existing community and divide it into two. If the value of Q increases, then this process continues until Q can increase no further. We see below from [7] how my former home town of Bristol is progressively divided into 74 such communities with a final value of Q=0.803.

If we compare different cities we get the following results, in which we can see that the city of Birmingham is much less divided into communities than Manchester and London.

city	#accts	av. degree	degree var.	clust. coeff.	*R*	*Q*	diff.
Birmingham	1321	3.017	17.24	0.111	45	0.795	0.047
Edinburgh	1645	2.609	7.95	0.070	38	0.841	0.027
Glasgow	1802	2.535	8.49	0.065	39	0.865	0.044
Nottingham	2066	3.054	18.71	0.119	55	0.827	0.068
Cardiff	2685	3.310	21.24	0.196	44	0.859	0.083
Sheffield	2845	3.092	16.19	0.128	52	0.855	0.088
Bristol	2892	3.138	18.30	0.107	74	0.803	0.019
Leeds	5263	3.541	53.38	0.101	133	0.735	0.015
Manchester	7646	3.182	29.14	0.072	145	0.820	0.037
London	16 171	3.001	20.64	0.097	156	0.869	0.159

Finally we compare communities across different cities and ask which cities can be made from the communities of the other. Leeds, for example, has a wide variety of social communities but the same cannot be said of Bristol. Therefore Bristol can be made up of parts of Leeds. A social campaign that works in Bristol would not necessarily work in Leeds. Similarly, Edinburgh can be made up of parts of Glasgow, but not vice versa.

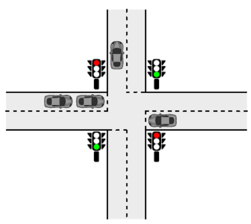
communities from	Bi	Ed.	Gl.	No.	Ca.	Sh.	Br.	Le.	Ma.
Birmingham							✓	✓	
Edinburgh	✓			✓			✓		✓
Glasgow	✓	✓		✓					✓
Nottingham	✓	✓					✓	✓	
Cardiff	✓	✓		✓			✓	✓	✓
Sheffield	✓	✓	✓	✓	✓		✓	✓	✓
Bristol	✓	✓		✓					✓
Leeds	✓	✓	✓	✓	✓	✓	✓		✓
Manchester	✓	✓	✓	✓	✓	✓	✓	✓	

This analysis has lots of implications for the future as it gives us both a way of studying how a city will grow, and also a means of surveying the people in that city to ask what sort of future they wish to have.

4. How Do We Travel in A City?

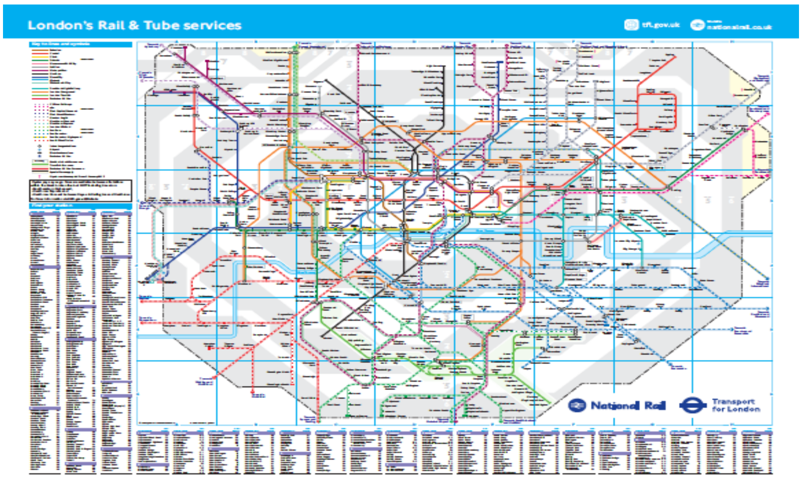
Keeping cities accessible and moving is one of the biggest challenges road authorities face today and into the future. Cities have road networks that have both to be planned from scrat5ch, and also need to evolve from existing road networks. As we shall see, even quite small changes to these networks can make a big change to the resulting flow of the traffic. An alternative, and somewhat cheaper and easier, was to control the traffic is to monitor it using sensors, and then have an intelligent system of traffic lights or similar to control the actual flow. Urban areas contain millions of sensors, ranging from loop detectors embedded in the road to the increasing number of connected vehicles. Many of these sensors only monitor one point in the city, and mathematical models are needed to combine all of this data together, and to then use it for smart and efficient control of the traffic lights. A further use of this data is to inform approaching drivers about scheduled green phases of the lights in advance, offering them the possibility to adjust their driving speed for a more efficient through flow. One of the activities is the optimal management of traffic through intelligent control of traffic lights. The control of the individual traffic lights optimises the traffic through flow based on model predictions of individual arrival times of vehicles. Often this uses a (so called) Poisson process in which the traffic arrives fairly randomly, rather like calls at a telephone exchange. I'll come back to this presently when we look at crime. Optimal traffic management controls them all, in such a way that a complete *network of intersections* can be optimised.

The networks which we have already introduced to look at the way that people interact with each other can therefore also be used to represent, understand, improve, and optimise, the road network in a city. As a simple way of doing this we think of the junctions in the road network (illustrated) as the nodes in the network, and the roads as the edges.



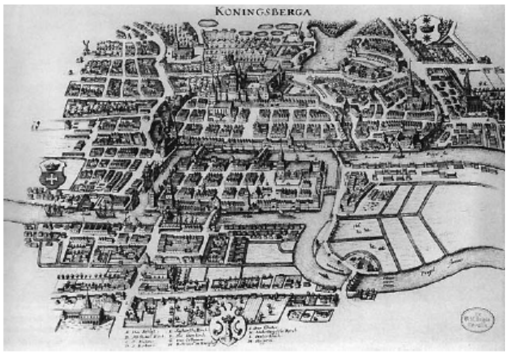
In the case of a road network we introduce some extra information into this. For example, we have directed networks, which indicate whether a road is one way or two way. Secondly, we can attribute a number to show the maximum carrying capacity of the road, in terms of cars per hour.

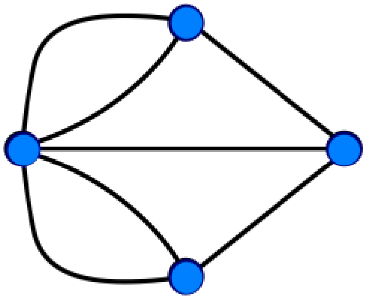
A classic network for a transport system in a city is given by the Tube Map for London, which was introduced in 1933.



http://www.nationalrail.co.uk/passenger_services/maps/

In fact the first use of networks was by Euler and for exactly the question of urban planning (well sort of). Euler was asked the question of whether the inhabitants of the town of Konigsberg could walk around the town, and in particular across to the two islands in the main river, only crossing the bridges to the islands exactly once.





On the left of the two figures above we can see the original town plan, and on the right the fully equivalent network, with the lines representing the paths over the bridges, and the nodes the junctions of these paths. Each node in a general network has connects to either an even number of paths or an odd number, and is called respectively an even or an odd node. A result in network theory states that to go around a network passing over every path once and once only the network must have *at most two odd nodes*. As this network has four odd nodes, such a route is impossible.

Modern urban road networks are a lot more complex, as can be seen from a network given by the street map of the centre of Edinburgh. A SatNav device will make use of this network in order to find the fastest route from one point to another, allowing for the fact that many roads are one-way only.

Using such a network, city planners can both work out how to expand the road network and also the effect of restructuring the network, perhaps to allow for more pedestrianisation by closing roads. Rather surprisingly, this can often improve the traffic circulation. An example of this occurred in New York in 1990 when they closed 42nd Street for Earth Day and the traffic flow actually improved.

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What if They Closed 42d Street and Nobody Noticed?

By GINA KOLATA
Published: December 25, 1990

ON Earth Day this year, New York City's Transportation Commissioner decided to close 42d Street, which as every New Yorker knows is always congested. "Many predicted it would be doomsday," said the Commissioner, Lucius J. Riccio. "You didn't need to be a rocket scientist or have a sophisticated computer queuing model to see that this could have been a major problem."

But to everyone's surprise, Earth Day generated no historic traffic jam. Traffic flow actually improved when 42d Street was closed.

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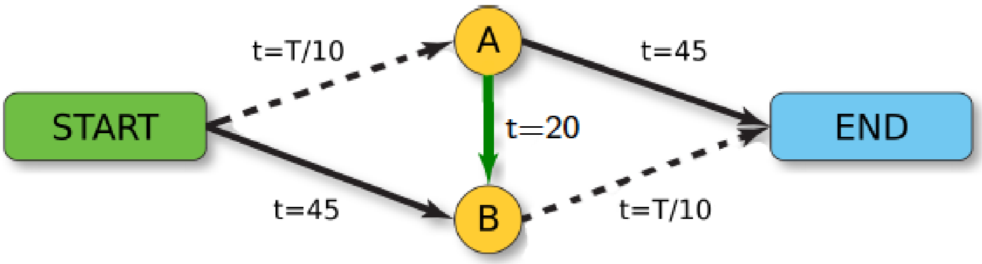
This is an example of Braess's paradox, which was discovered in 1968 by the German mathematician Dietrich Braess who noticed that adding a road to a congested road traffic network could increase overall journey time. His idea was that if each driver is making the decision as to which route for them is the quickest, then a shortcut could be chosen too often leading to an overloading of parts of the road network and thus greater average travel times.

We can illustrate this with the road network shown below.



In this network we have two roads marked by dashes, which have the feature that they rapidly clog up with traffic, which then slows down and takes more time to get through. The time it takes for an individual car to travel down each road in minutes is given by T/10 where T is the total number of cars using that road. The two roads marked in solid are both better and do not get clogged so easily. Instead it takes a car 45 minutes to go down them, regardless of the total amount of traffic. The green road is a short cut from A to B, which takes 2o minutes to go down.

Suppose firstly that the green short cut is not open and we want to pass 300 cars through the network going from the Start to the End. If more cars want to go through A than through B they will find that their journey time will be longer, because they will clog up the road to A. Hence other cars will switch to the route through B. As there is no advantage to any car to go through either A or B, then the system will soon settle down, so on average 150 cars take the route through A and 150 through B. The journey time for them is then

$$150/10 + 45 = 60 \text{ minutes.}$$

Now suppose that we open up the short cut from A to B. If one of the 150 cars going through A takes the short cut, then the time for them to get to the End will be

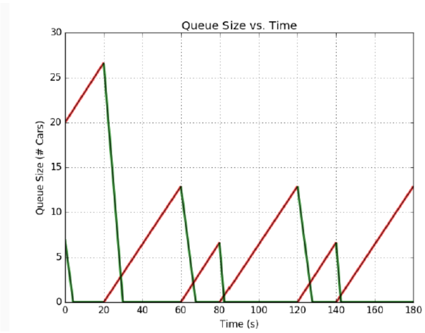
$$150/10 + 20 + 151/10 = 50.1 \quad \text{minutes}$$

Thus it makes sense for them to change to use the short cut. As they do it, so more cars will change, and other cars will be attracted to the route via A. This process will continue until all of the journeys take the same time, and there is not incentive to change. A little mathematics shows that this arises when: 250 cars travel to A of which 200 change and 50 go on. In contrast 50 cars travel to B. The total travel time for each car is then

$$250/10 + 45 = 250/10 + 20 + 250/10 = 45 + 250/10 = 70 \text{ minutes.}$$

Adding the short cut combined with each car driver acting in what they see as their own best interest, **has increased the journey time by 10 minutes!**

Such considerations of the behaviour networks have to be considered with great care in the design, for example, of traffic light systems. Current systems which use monitors, aim to reduce the waiting time at a particular network. Typical queue times, taken from a survey I conducted, are shown below.



However, in a future city, it is clear that local queue time optimisation is not good enough. Instead the traffic lights should all be linked together so that situations involving Braess' paradox can be avoided.

5. What about crime?

The New York Times recently asked for a poll of its readers to describe crime that they had encountered in London. I was greatly encouraged when they received replies of such heinous crimes as 'people smiling on the underground', 'standing on the left side of the escalator', 'not bringing their bins in', 'speaking to them on the bus' and, worst of all, 'putting the tea in before the milk'. It made me feel proud to be Londoner.

Los Angeles Times

latimes.com

COLUMN ONE

Stopping crime before it starts

Computer analysis of data can tell police where criminals are headed, theorists say. The LAPD is pursuing the technology.

LAUSD PRESSES UNION ON TEST SCORES

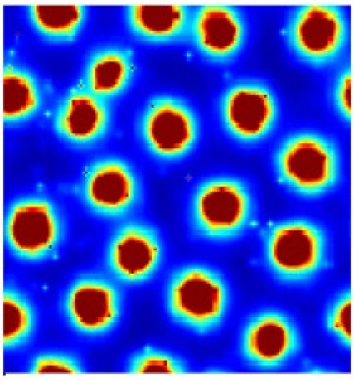
The district wants new labor contracts to include 'value-added' data as part of teacher evaluations.

The Los Angeles Unified School District will ask labor unions to adopt a new approach to teacher evalu-

FRIDAY PRAYER: Worshipers greet each other after services at the Islamic Center of Southern California in Los Angeles. Leaders of several faiths came out to support the proposed Muslim center in New York.

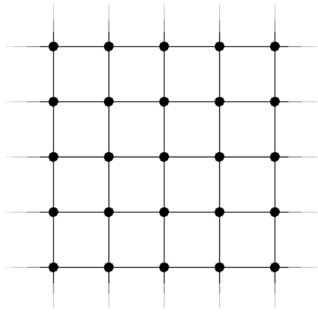
However, even in London, serious crime is a problem of major importance in a modern city. We have already looked at the way that cities divide into communities. In a similar way, crime has a way of dividing into geographical communities in a city. Urban crime is ubiquitous and heterogeneous: while it can certainly affect a whole city, clusters of crimes are often localized in time and space, forming "hot spots" of increased criminal activity. Simple spatial heterogeneity in the environment is insufficient to explain the temporal variations in crime recurrence. Rather, the emergence of hot spots is linked to repeat victimisation in that a successful offender likely to re-offend in the same or nearby location probably not long after a previous incident. The study of such patterns in crime has attracted the attention of many mathematicians. In particular the group of

Prof. Andrea Bertozzi at UCLA has developed sufficiently sophisticated models of criminal behaviour to act as major advisors to the Los Angeles Police Force. Interestingly, these models are based on mathematics used to study the movement of slime mould! A response by law enforcement forces to a hot spot of crime is to then deploy additional resources to the hot spot to deter further crime. This is called the *cops on the dots* strategy. It has sometimes successfully dispersed the hot spot, but in other times this strategy has merely displaced the crime to surrounding areas [8]. An example of a crime hot spot scenario is given below in which areas of high crime attractiveness are shown in red.



These pictures are constructed using a form of the agent-based models which I described in a previous lecture (Can you Do Maths in a Crowd?). In these we have individuals (agents) who are programmed to obey certain rules. Simple agent-based models can give us some insights into criminal behaviour. In one such model, developed at UCLA, the criminal's decisions are influenced by an attractiveness field A(x,t) which measures how likely they are to commit the crime in any particular location x at the time t. The dynamics of this attractiveness field are then coupled to the dynamics of the criminal themselves.

In one such model, the criminals move randomly around a square lattice (think of the road system in Los Angeles) where at each junction in the lattice (a node) they chose a road to go down. See below for a simplified picture of the Los Angeles road network



The criminals move from each node to another one in a time dt. At each node they either commit a crime where they are, or they move to the next node with a probability that depends upon A. The probability that they commit a crime is given by

$$p(x,t) = 1 - e^{-A(x,t)dt}$$

In this expression, the larger A is the more likely they then are to commit the crime. Once a criminal has committed a crime they exit the system. However, more criminals may be generated at each node due to this activity. This is (again) modelled by a Poisson process, in which criminals arrive at random intervals, in much the same random way that the cars arrive at a road junction (or buses at a bus stop!).

To model the growth, and spread, of crime, the value of A not only increase as more crimes are committed in the neighbourhood, but also diffuses to adjacent nodes (in much the same way that a gas will diffuse). This models the process of victimisation.

If the lattice is very dense in the whole city (so that the distances between the nodes are proportionately small) then this system simplifies to a partial differential equation model given by

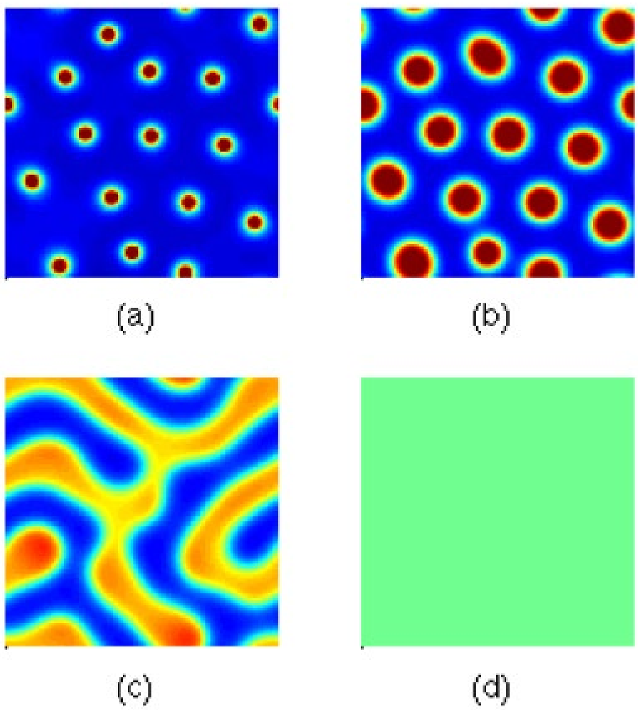
$$\frac{\partial A}{\partial t} = \eta \triangle A + \rho A - A + A_0$$
$$\frac{\partial \rho}{\partial t} = \nabla \cdot (\nabla \rho - 2\rho \nabla \log A) - \rho A + \overline{B}.$$

Here rho is the density of the criminals, and A0 and B are constants. This model can be studied using the techniques of mathematical analysis combined with careful computer calculations. (It is in fact very similar to the equations for the movement of slime mould in which rho is the density of the mould and A is a chemical attractor that the mould secretes). These models give the hot spots shown above.

It is possible to improve these models by introducing additional agents who are the police forces themselves and lead to a deterrence d to the criminal activity. It is assumed that d will be concentrated on the hot-spots. This gives

$$\frac{\partial A}{\partial t} = \eta \Delta A + d\rho A - A + A_0,$$
$$\frac{\partial \rho}{\partial t} = \nabla . \left(\nabla \rho - 2\rho \nabla \log A \right) - d\rho A + b\bar{B},$$
$$d(x) = \frac{1}{2} \left(1 - \tanh[\mu(A(x, t_-)) - A_c] \right).$$

The patterns of crime that we then see depend upon the various parameters in the model. The following four figures taken from [8] show the effects of varying policing strategies with low level policing top left, to targeted high level policing bottom right. The figure on the bottom left is interesting as it shows the effect of a certain level of policing to spread the crime into broader regions around the city.



(a) (b)

(c) (d)

6. Retail

On a less dangerous note, one of the primary activities in a city is that of retail. Indeed, one of the main reasons that tourists want to visit London (or indeed Bath) is to go shopping. The most popular retail centre in London is the Westfield Centre, with 50 million visitors per year, Here is an illustration of another popular shopping centre at Covent Garden.

The subject of modelling retail has gained attention from mathematicians since the 1950s. and is now a mature area of research. Of particular note is the group at UCL headed by Sir Alan Wilson, who headed up the report in [3] and looked at retail in London.

As an example of the research of this group reported in [4] which looks at retail in 200 centres distributed across the 60 wards in London. There is a flow of (money from) people from the wards to the centres which can be represented by a 600x200 matrix, S_{ij}. In [2] this model is matrix is given by

$$S_{ij} = A_i \, e_i \, P_i \, W_j \, e^{-\beta c_{ij}}, \quad i = 1..600, j = 1..200$$

Here e_i is the per capita expenditure and P_i the population in ward i. W_j is a measure of the pulling power of the jth retail centre, and c_{ij} is a measure of the cost of going from i to j. The number A_i is a calibration that is measured by experiment.

This model has been well tested, and it allows a retailer to assess to cost/impact of opening a new store in a certain location.

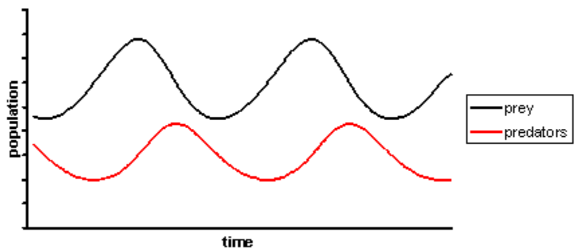
As a city grows into the future, this model can be extended to see how the shopping centres themselves will grow. The growth of the centre W_j itself is governed by the differential equation

$$dW_j/dt = \epsilon \, [D_j - kW_j]$$

where D_j is the revenue attracted to the centre, and kW_j is the cost of running it. In this model W_j will grow or shrink until an equilibrium is reached when

$$W_j = \frac{D_j}{k}.$$

The challenge of applying this model is that the value of D_j depends upon all of the other shopping centres, leading to a very hard system to solve!



The dynamics of the differential equation model above allows us to model how the various shopping centres will compete with each other for consumers as they struggle to grow, or simply to survive. This model is very similar to Lotka-Volterra models of mathematical biology [9] which model how different animal species compete. Various different types of dynamics can be observed. On is that one retailer grows to the expense of all others and becomes the single (huge) retailer for the whole city. This would be an example of the sub linear growth I described above. Another is that we might see cycles, where one retailer grows and then declines, to be replaced by another, which then declines and so on. This is illustrated below

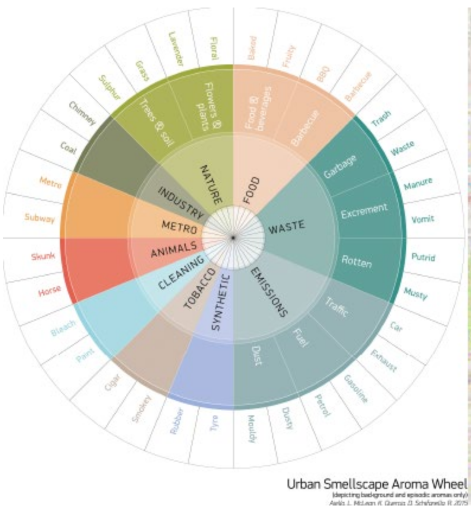
7. The Smart City

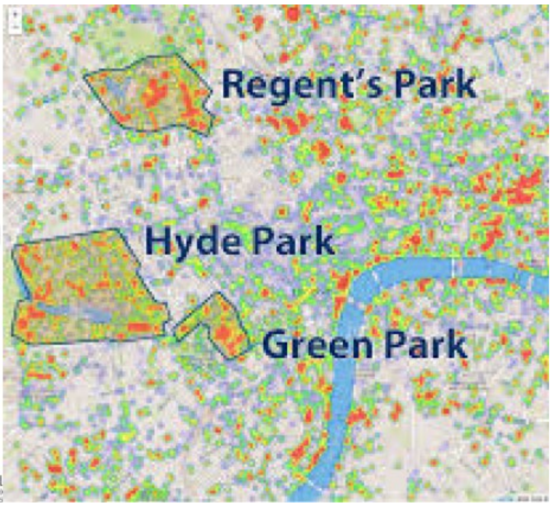
A smart city is an urban area that uses different types of electronic data collection sensors, such as mobile phone apps, to supply information. This information is then used to manage assets (such as health and transport) and resources efficiently. This includes data collected from people, devices such as computers and even WiFi stations on lamp posts, and assets, such as smart electricity meters. This data is then processed and analysed to monitor and manage traffic and transportation systems, power plants, water supply networks, waste management, law enforcement, information systems, government, sewage, schools, libraries, retail centres, hospitals, and other community services. All this is principle leads to a more efficient use of resources. However, we have to be very wary of the effects of Braes' Paradox leading to an overload of the network as a result and thus a slower response overall. In fact I am rather concerned that this does not seem to have been considered yet in the planning of the smart city. The smart city concept ultimately integrates information and communication technology, and various physical devices connected to the network (the Internet of things) to optimize the efficiency of city operations and services and connect to the people in it. Smart city technology allows city officials to interact directly with both community and city infrastructure and to monitor what is happening in the city and how the city is evolving. Unfortunately, it also gives criminals the same access and we need to be very wary of this. The smart city is, with great likelihood, the city of the future. Perhaps fittingly for mathematicians, the organisation of intelligence in a smart city has been inspired in part, from the functioning of the code-breaking centre of Bletchley Park [10].

8. Maths in the City of London

As we are in the City of London (which is where I was born and grew up) I thought that it would be nice to end this lecture with a guide to the mathematical sites of London, which you can explore as you leave. A wonderful mathematical tour of London is described in the guide in [11]. However places which you really should not miss are: the whispering gallery and dome of St Pauls, the tombs of Newton, Dirac and Hawking in Westminster Abbey, the Gherkin, the Rhind Papyrus and other mathematical historical texts in the British museum, the fractal paintings by Pollock in the Tate Gallery and (in my opinion best of all, but them I'm biased) the many labyrinths on the walls of the stations of the London Underground (and of course the London Underground itself).

And very finally, should you wish to investigate it, here is a carefully constructed map of smelly London!





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