Toothpaste, Custard and Chocolate: Maths gets messy

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LONDON MATHEMATICAL SOCIETY GOOD PRACTICE SCHEME

Athena SWAN Silver Award

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Toothpaste, Custard and Chocolate: Maths gets messy

or, How to get this:

through this:

Key component of the chocolate fountain project:

Mathematical Modelling

What can we model?

The chocolate. . .

What can we model?

The chocolate. . .

. . . and the fountain

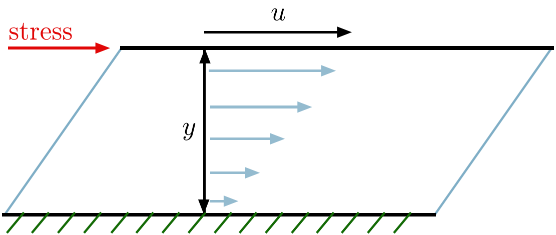
Modelling chocolate

Molten chocolate is a complex material — a highly dense suspension of sugar and cocoa solids in a cocoa butter liquid phase — complicated by the variation in composition of cocoa butter with source and harvest.[1]

How can we model its fluid properties or *rheology*?

[1] Ian Wilson, *Report on Chocolate Congress 2010*, BSR Bulletin, 2010.

Stress and rate of strain

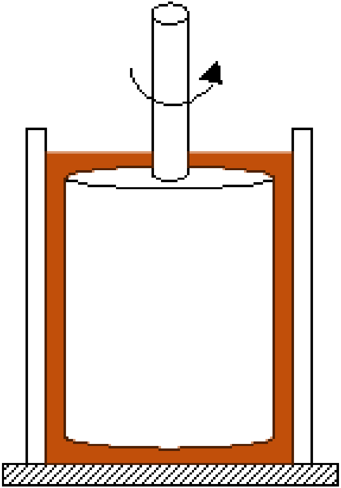


$$\text{stress} = \sigma$$

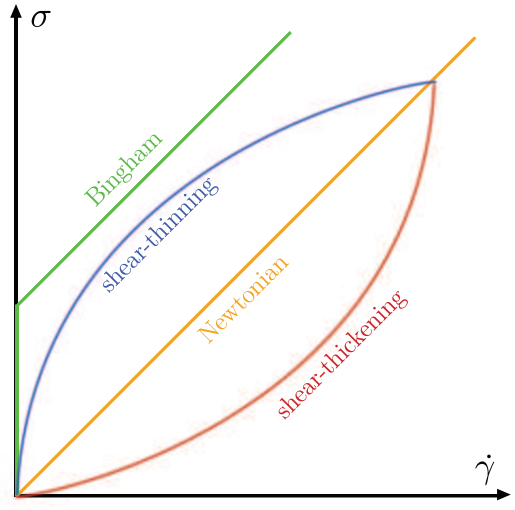
$$\text{rate of strain} = \dot{\gamma} = \frac{\partial u}{\partial y}$$

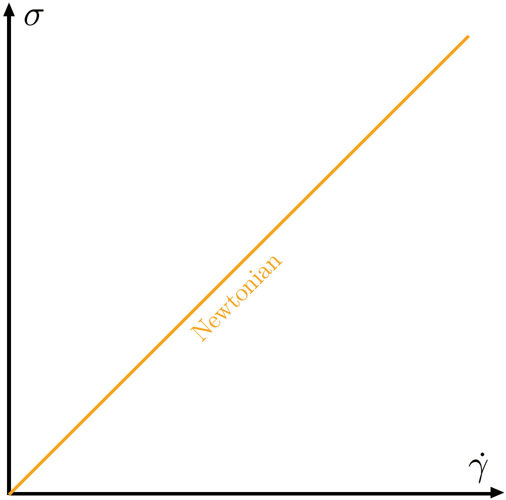
$$\text{viscosity} = \mu = \frac{\sigma}{\dot{\gamma}}$$

Measuring stress and rate of strain

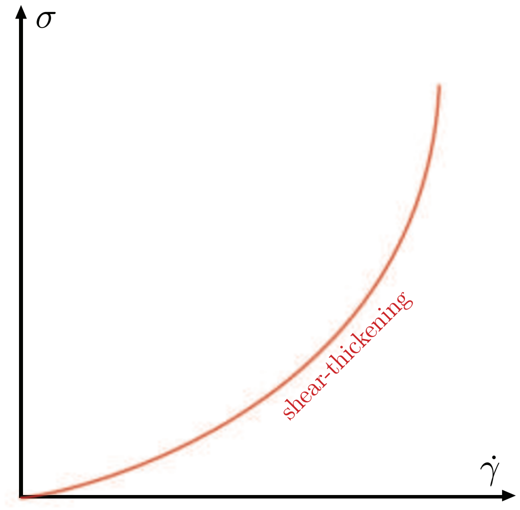


Stress and rate of strain





σ
Newtonian
$\dot{\gamma}$

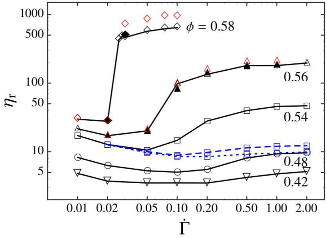


σ
shear-thickening
$\dot{\gamma}$


ARGO

Custard

- Custard/cornflour suspension shear-thickens
- Continuous Shear-Thickening (CST)
  - Moderate suspension concentration
  - Smooth, mild viscosity increase
- Discontinuous Shear-Thickening (DST)
  - High suspension concentration
  - Viscosity increases order of magnitude



Custard powder: how does it thicken?

Early ideas

- Repulsion? (Doesn't happen in attractive suspensions)
- Cluster formation? (but this fails to get enough viscosity change)
- Granular expansion? (incorrectly predicts DST for smooth, hard particles)
- Contact between particles?

Recent progress

- Can get DST from frictional contact plus fluid forces

Custard: My research

Impose contact at fixed separation and friction

Dilute suspensions (years ago):

- Contact reduces viscosity
- Friction increases viscosity weakly

Moderate concentrations (fairly recent work):

- Friction can increase viscosity but not strongly enough
- Hard contacts (which only we can do) destroy DST

Very dense suspensions (current work):

- Creating new models incorporating contact and microstructure
- Aiming to capture shear thickening and correct reversal response

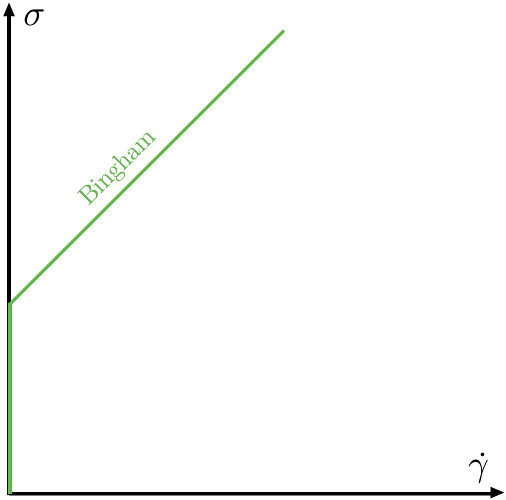
Custard: Potential applications

Ballistic protection

- Kevlar alone does not stop a bullet at point blank range
- Kevlar treated with shear-thickening fluid can!

Cryopreservation

- If DST transition caused by structural changes, could potentially inhibit formation of ice crystals in solvent
- Industrial partners Asymptote have recently shown that some approved *cryoprotectants* (essentially starch in a glycerol solution) do show shear-thickening.
- Behaviour in freezing still to investigate



σ
Bingham
$\dot{\gamma}$


HEINZ
57
VARIETIES
HEINZ
1869
TOMATO
KETCHUP
57 VARIETIES
NET WT 14 OZ (397g)
Heinz
WWW.HEINZ.COM

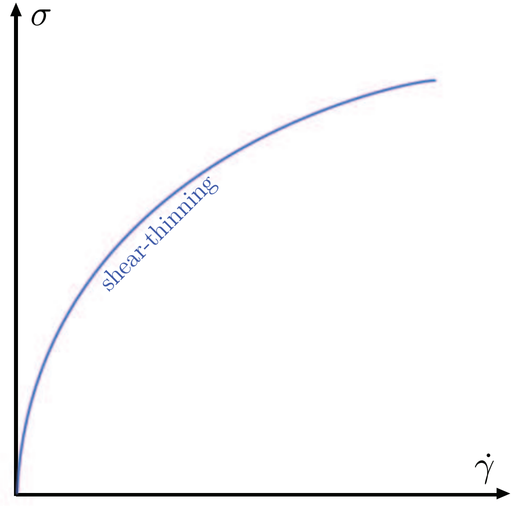
Toothpaste

Truly complex fluid:

- Polymeric fluid matrix
- Silica particles for abrasion
- Different silica particles for rheology
- Active ingredients, flavours, etc.

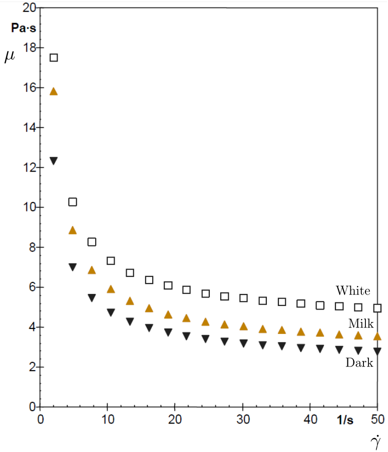
Working with UCL engineers & GSK to model processing

- Experiments on rheology and interparticle forces
- Modelling to predict effect of particles
- CFD to scale up to processing flows


σ
shear-thinning
$\dot{\gamma}$


SCENTED
WHEN DRY

Viscosity against rate of strain: chocolate at 40°C



Clearly shear-thinning.

Modelling chocolate

Newtonian fluid

$$\sigma = \mu\dot{\gamma}$$

Power-law fluid

$$\sigma = k\dot{\gamma}^n$$

Casson's model

$$\sqrt{\sigma} = \begin{cases} \sqrt{\mu_c\dot{\gamma}} + \sqrt{\sigma_y} & \text{if } \sigma \geq \sigma_y \\ \sqrt{\sigma_y} & \text{if } \sigma \leq \sigma_y \end{cases}$$

Modelling chocolate

Newtonian fluid

$$\sigma = \mu\dot{\gamma}$$

$$\mu \approx 14\,\text{Pa s}$$

For water, $\mu = 9 \times 10^{-4}$ Pa s.

Power-law fluid

$$\sigma = k\dot{\gamma}^n$$

Casson's model

$$\sqrt{\sigma} = \begin{cases} \sqrt{\mu_c\dot{\gamma}} + \sqrt{\sigma_y} & \text{if } \sigma \geq \sigma_y \\ \sqrt{\sigma_y} & \text{if } \sigma \leq \sigma_y \end{cases}$$

Modelling chocolate

Newtonian fluid

$$\sigma = \mu\dot{\gamma}$$

$$\mu \approx 14\,\mathrm{Pa\,s}$$

Power-law fluid

$$\sigma = k\dot{\gamma}^n$$

Milk choc, 40^oC: $k \approx 65\,\mathrm{Pa\,s}^n$, $n \approx 1/3$

(Actually 64.728; 0.3409). μ matches at $\dot{\gamma} = 10\,\mathrm{s}^{-1}$.

Casson's model

$$\sqrt{\sigma} = \begin{cases} \sqrt{\mu_c\dot{\gamma}} + \sqrt{\sigma_y} & \text{if } \sigma \geq \sigma_y \\ \sqrt{\sigma_y} & \text{if } \sigma \leq \sigma_y \end{cases}$$

Modelling chocolate

Newtonian fluid

$$\sigma = \mu\dot{\gamma}$$

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Power-law fluid

$$\sigma = k\dot{\gamma}^n$$

$$k \approx 65\,\mathrm{Pa\,s}^n, \quad n \approx 1/3$$

Casson's model

$$\sqrt{\sigma} = \begin{cases} \sqrt{\mu_c\dot{\gamma}} + \sqrt{\sigma_y} & \text{if } \sigma \geq \sigma_y \\ \sqrt{\sigma_y} & \text{if } \sigma \leq \sigma_y \end{cases}$$

$$\mu_c \approx 3.2\,\mathrm{Pa\,s}, \quad \sigma_y \approx 4.6\,\mathrm{Pa}$$

International Confectionery Association 1973–2000.

Modelling chocolate

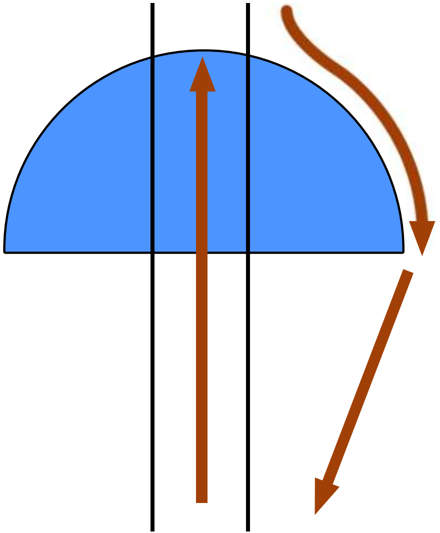
Newtonian fluid

$$\sigma = \mu\dot{\gamma}$$

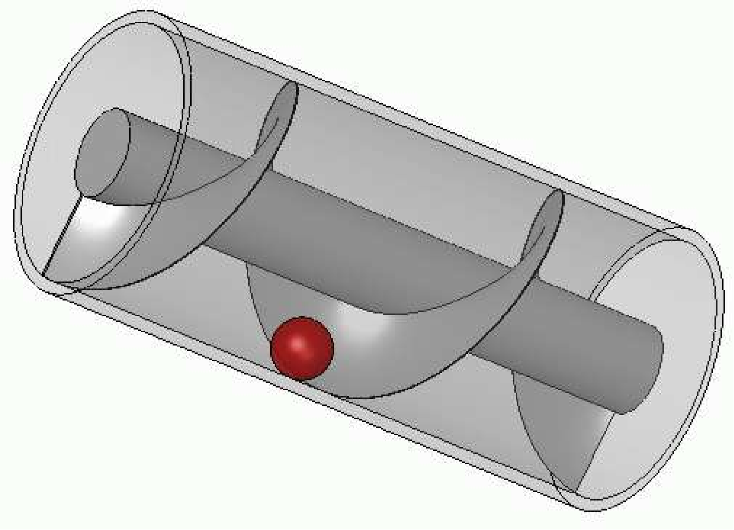
Power-law fluid

$$\sigma = k\dot{\gamma}^n$$

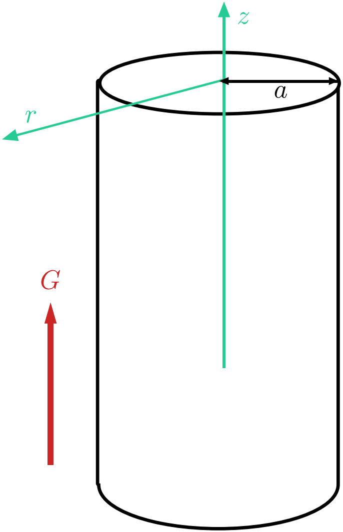
Modelling the Fountain



Pipe flow



Pipe flow



Navier–Stokes equation

Euler equations:

$$\rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\nabla p + \mathbf{F}$$

Newtonian Navier–Stokes equation:

$$\rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\nabla p + \mu \nabla^2 \mathbf{u} + \mathbf{F}$$

General Navier–Stokes equation:

$$\rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\nabla p + \nabla \cdot \boldsymbol{\sigma} + \mathbf{F}$$

Pipe flow

$$\rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\nabla p + \nabla \cdot \boldsymbol{\sigma} + \mathbf{F}$$

$$\nabla \cdot \mathbf{u} = 0$$

In coordinates

$$\rho \left(\frac{\partial u_r}{\partial t} + u_r \frac{\partial u_r}{\partial r} + u_z \frac{\partial u_r}{\partial z} \right) = -\frac{\partial p}{\partial r} - \left[\frac{1}{r} \frac{\partial}{\partial r}(r \sigma_{rr}) - \frac{\sigma_{\theta\theta}}{r} + \frac{\partial \sigma_{rz}}{\partial z} \right] + F_r$$

$$\rho \left(\frac{\partial u_z}{\partial t} + u_r \frac{\partial u_z}{\partial r} + u_z \frac{\partial u_z}{\partial z} \right) = -\frac{\partial p}{\partial z} - \left[\frac{1}{r} \frac{\partial}{\partial r}(r \sigma_{zr}) + \frac{\partial \sigma_{zz}}{\partial z} \right] + F_z$$

$$0 = \frac{1}{r} \frac{\partial}{\partial r}(\rho r u_r) + \frac{\partial}{\partial z}(\rho u_z).$$

Pipe flow

Assume: **steady flow**

$$\rho\left(\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + u_z\frac{\partial u_r}{\partial z}\right) = -\frac{\partial p}{\partial r} - \left[\frac{1}{r}\frac{\partial}{\partial r}(r\sigma_{rr}) - \frac{\sigma_{\theta\theta}}{r} + \frac{\partial \sigma_{rz}}{\partial z}\right] + F_r$$

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Pipe flow

Assume: **no radial flow** $(u_r = 0)$

$$\rho\left(u_r\frac{\partial u_r}{\partial r} + u_z\frac{\partial u_r}{\partial z}\right) = -\frac{\partial p}{\partial r} - \left[\frac{1}{r}\frac{\partial}{\partial r}(r\sigma_{rr}) - \frac{\sigma_{\theta\theta}}{r} + \frac{\partial \sigma_{rz}}{\partial z}\right] + F_r$$

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Pipe flow

Assume: **no radial flow** ($u_r = 0$)

$$
\begin{aligned}
0 &= -\frac{\partial p}{\partial r} - \left[\frac{1}{r}\frac{\partial}{\partial r}(r\sigma_{rr}) - \frac{\sigma_{\theta\theta}}{r} + \frac{\partial \sigma_{rz}}{\partial z}\right] + F_r \\
\rho\left(u_z \frac{\partial u_z}{\partial z}\right) &= -\frac{\partial p}{\partial z} - \left[\frac{1}{r}\frac{\partial}{\partial r}(r\sigma_{zr}) + \frac{\partial \sigma_{zz}}{\partial z}\right] + F_z \\
0 &= \frac{\partial}{\partial z}(\rho u_z).
\end{aligned}
$$

Pipe flow

Continuity equation tells us: $u_z = u_z(r)$

$$0 = -\frac{\partial p}{\partial r} - \left[\frac{1}{r}\frac{\partial}{\partial r}(r\sigma_{rr}) - \frac{\sigma_{\theta\theta}}{r} + \frac{\partial \sigma_{rz}}{\partial z}\right] + F_r$$

$$\rho\left(u_z\frac{\partial u_z}{\partial z}\right) = -\frac{\partial p}{\partial z} - \left[\frac{1}{r}\frac{\partial}{\partial r}(r\sigma_{zr}) + \frac{\partial \sigma_{zz}}{\partial z}\right] + F_z$$

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Pipe flow

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Pipe flow

Stresses are a function of velocity so $\sigma_{ij} = \sigma_{ij}(r)$

$$0 = -\frac{\partial p}{\partial r} - \left[\frac{1}{r}\frac{\partial}{\partial r}(r\sigma_{rr}) - \frac{\sigma_{\theta\theta}}{r} + \frac{\partial \sigma_{rz}}{\partial z}\right] + F_r$$

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$$0 = -\frac{\partial p}{\partial z} - \frac{1}{r}\frac{\partial}{\partial r}(r\sigma_{zr}) + F_z$$

Pipe flow

Gravity acts downwards

$$0 = -\frac{\partial p}{\partial r} - \left[\frac{1}{r}\frac{\partial}{\partial r}(r\sigma_{rr}) - \frac{\sigma_{\theta\theta}}{r}\right] + F_r$$

$$0 = -\frac{\partial p}{\partial z} - \frac{1}{r}\frac{\partial}{\partial r}(r\sigma_{zr}) + F_z$$

Pipe flow

Gravity acts downwards

$$0 = -\frac{\partial p}{\partial r} - \left[\frac{1}{r}\frac{\partial}{\partial r}(r\sigma_{rr}) - \frac{\sigma_{\theta\theta}}{r}\right]$$

$$0 = -\frac{\partial p}{\partial z} - \frac{1}{r}\frac{\partial}{\partial r}(r\sigma_{zr}) - \rho g$$

Pipe flow

Assume: **No normal stress differences** so only $\sigma_{rz} = \sigma_{zr}$ nonzero

$$0 = -\frac{\partial p}{\partial r} - \left[\frac{1}{r}\frac{\partial}{\partial r}(r\sigma_{rr}) - \frac{\sigma_{\theta\theta}}{r}\right]$$

$$0 = -\frac{\partial p}{\partial z} - \frac{1}{r}\frac{\partial}{\partial r}(r\sigma_{zr}) - \rho g$$

Pipe flow

Assume: **No normal stress differences** so only $\sigma_{rz} = \sigma_{zr}$ nonzero

$$0 = -\frac{\partial p}{\partial r}$$

$$0 = -\frac{\partial p}{\partial z} - \frac{1}{r}\frac{\partial}{\partial r}(r\sigma_{zr}) - \rho g$$

Pipe flow

Pressure gradient is constant, $-\partial p/\partial z = G$

$$0 = -\frac{\partial p}{\partial z} - \frac{1}{r}\frac{\partial}{\partial r}(r\sigma_{zr}) - \rho g$$

Pipe flow

Pressure gradient is constant, $-\partial p/\partial z = G$

$$0 = G - \frac{1}{r}\frac{\partial}{\partial r}(r\sigma_{zr}) - \rho g$$

Pipe flow

Combine forces to form total pressure head H

$$0 = G - \left[\frac{1}{r}\frac{\partial}{\partial r}(r\sigma_{zr})\right] - \rho g$$

Pipe flow

Combine forces to form total pressure head H

$$0 = -\left[\frac{1}{r}\frac{\partial}{\partial r}(r\sigma_{zr})\right] + H$$

Pipe flow

Left to solve

$$\frac{\partial}{\partial r}\left(r\sigma_{zr}\right) = Hr$$

which integrates to give

$$\sigma_{zr} = \frac{Hr}{2}$$

where the boundary condition was $\sigma_{zr}(0) = 0$ by symmetry.

Pipe flow

So we have

$$\sigma_{zr} = \sigma = \frac{Hr}{2}$$

But remember for a power-law fluid,

$$\sigma = k\dot{\gamma}^n$$

and recall that in a pipe

$$\dot{\gamma} = \frac{\mathrm{d}u_z}{\mathrm{d}r}$$

so

$$k\left(\frac{\mathrm{d}u_z}{\mathrm{d}r}\right)^n = \frac{Hr}{2}$$

Pipe flow

So we have

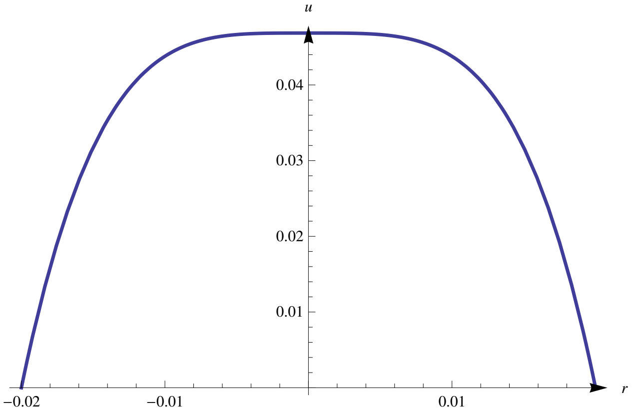
$$\frac{\mathrm{d}u_z}{\mathrm{d}r} = \left(\frac{H}{2k}\right)^{1/n} r^{1/n}$$

So we solve this with the boundary condition $u(a) = 0$ and get,

$$u_z = \left(\frac{H}{2k}\right)^{1/n} \frac{r^{1+1/n} - a^{1+1/n}}{1 + 1/n}.$$

Pipe flow

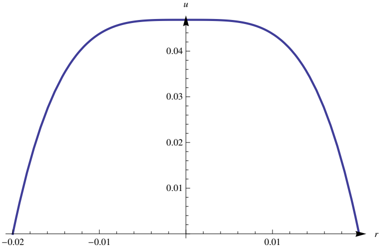
$$u_z = \left(\frac{H}{2k}\right)^{1/n} \frac{r^{1+1/n} - a^{1+1/n}}{1+1/n} \qquad n = \frac{1}{3}$$



Pipe flow

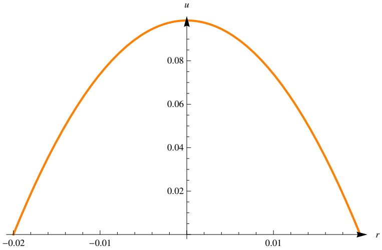
Power law $(n = 1/3)$

$$u_z = \left(\frac{H}{2k}\right)^{1/n} \frac{r^{1+1/n} - a^{1+1/n}}{1+1/n}$$



Newtonian $(n = 1)$

$$u_z = \frac{H}{2k} \frac{r^2 - a^2}{2}$$

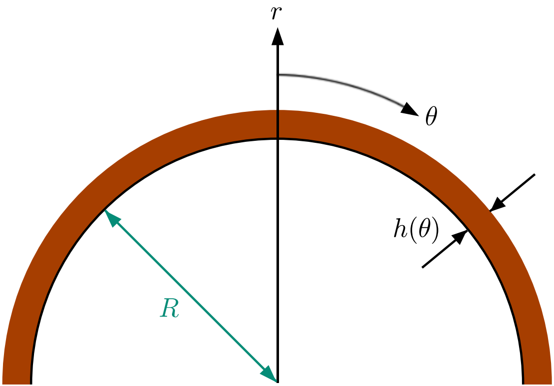


Dome flow

Same procedure as before, we take Navier–Stokes

$$\rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\nabla p + \nabla \cdot \boldsymbol{\sigma} + \mathbf{F}$$

and put it into cylindrical coordinates on this geometry



Simplifications

1. Flow is within the plane ($u_z = 0$)
2. Flow variation is within the plane
   $$(\partial u_i/\partial z = 0,\ \partial \sigma_{ij}/\partial z = 0,\ \partial p/\partial z = 0)$$
3. Flow is steady
4. Gravity acts downwards

Governing equations

$$\rho\left[u_r\frac{\partial u_r}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_r}{\partial\theta}-\frac{u_\theta^2}{r}\right]$$

$$=-\frac{\partial p}{\partial r}-\left[\frac{1}{r}\frac{\partial}{\partial r}(r\sigma_{rr})+\frac{1}{r}\frac{\partial\sigma_{r\theta}}{\partial\theta}-\frac{\sigma_{\theta\theta}}{r}\right]-\rho g\cos\theta$$

$$\rho\left[u_r\frac{\partial u_\theta}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{u_r u_\theta}{r}\right]$$

$$=-\frac{1}{r}\frac{\partial p}{\partial\theta}-\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\sigma_{r\theta}\right)+\frac{1}{r}\frac{\partial\sigma_{\theta\theta}}{\partial\theta}\right]+\rho g\sin\theta$$

$$0=-\left[\frac{1}{r}\frac{\partial}{\partial r}(r\sigma_{rz})+\frac{1}{r}\frac{\partial\sigma_{\theta z}}{\partial\theta}\right]$$

$$0=\frac{\partial}{\partial r}(ru_r)+\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}.$$

Rates of strain

Stress tensor for generalised Newtonian fluid is $\sigma_{ij} = \eta(\dot{\gamma})\dot{\gamma}_{ij}$.

$$\begin{aligned}
\dot{\gamma}_{rr} &= -2\frac{\partial u_r}{\partial r} \\
\dot{\gamma}_{\theta\theta} &= -\frac{2}{r}\frac{\partial u_\theta}{\partial \theta} - \frac{2u_r}{r} \\
\dot{\gamma}_{zz} &= 0 \\
\dot{\gamma}_{r\theta} = \dot{\gamma}_{\theta r} &= -\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{r}\frac{\partial u_r}{\partial \theta} \\
\dot{\gamma}_{rz} = \dot{\gamma}_{zr} &= 0 \\
\dot{\gamma}_{\theta z} = \dot{\gamma}_{z\theta} &= 0
\end{aligned}$$

Rates of strain

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\dot{\gamma}_{zz} &= 0 \\
\dot{\gamma}_{r\theta} = \dot{\gamma}_{\theta r} &= -\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{r}\frac{\partial u_r}{\partial \theta} \\
\dot{\gamma}_{rz} = \dot{\gamma}_{zr} &= 0 \\
\dot{\gamma}_{\theta z} = \dot{\gamma}_{z\theta} &= 0
\end{aligned}
$$

So $\sigma_{zz} = \sigma_{rz} = \sigma_{zr} = \sigma_{\theta z} = \sigma_{z\theta} = 0$.

Governing equations

$$\rho\left[u_r\frac{\partial u_r}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta}-\frac{u_\theta^2}{r}\right]$$
$$=-\frac{\partial p}{\partial r}-\left[\frac{1}{r}\frac{\partial}{\partial r}(r\sigma_{rr})+\frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta}-\frac{\sigma_{\theta\theta}}{r}\right]-\rho g\cos\theta$$

$$\rho\left[u_r\frac{\partial u_\theta}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{u_r u_\theta}{r}\right]$$
$$=-\frac{1}{r}\frac{\partial p}{\partial \theta}-\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\sigma_{r\theta}\right)+\frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta}\right]+\rho g\sin\theta$$

$$0=-\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\sigma_{rz}\right)+\frac{1}{r}\frac{\partial \sigma_{\theta z}}{\partial \theta}+\frac{\partial \sigma_{zz}}{\partial z}\right]$$

$$0=\frac{\partial}{\partial r}\left(ru_r\right)+\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}.$$

Governing equations

$$\rho\left[u_r\frac{\partial u_r}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta}-\frac{u_\theta^2}{r}\right]$$

$$=-\frac{\partial p}{\partial r}-\left[\frac{\partial \sigma_{rr}}{\partial r}+\frac{1}{r}\sigma_{rr}+\frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta}-\frac{\sigma_{\theta\theta}}{r}\right]-\rho g\cos\theta$$

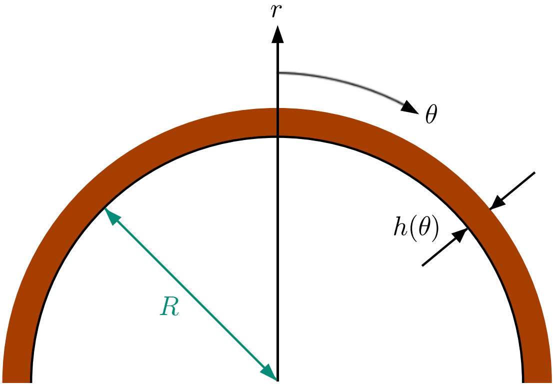
$$\rho\left[u_r\frac{\partial u_\theta}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{u_r u_\theta}{r}\right]$$

$$=-\frac{1}{r}\frac{\partial p}{\partial \theta}-\left[\frac{\partial \sigma_{r\theta}}{\partial r}+\frac{2\sigma_{r\theta}}{r}+\frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta}\right]+\rho g\sin\theta$$

$$0=\frac{\partial}{\partial r}(ru_r)+\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}.$$

Scaling considerations

Remember the geometry:



Scaling considerations

Scale in the following way

$$\widehat{u}_\theta = \frac{u_\theta}{U}, \qquad \widehat{u}_r = \frac{u_r}{V}, \qquad \widehat{h} = \frac{h}{H}, \qquad \widehat{r} = \frac{r-R}{H}.$$

$$u_\theta \sim U, \qquad u_r \sim V, \qquad h \sim H, \qquad r \sim R, \qquad \partial/\partial r \sim 1/H.$$

Continuity equation gives size of V

$$0 = \frac{\partial}{\partial r}(r u_r) + \frac{\partial u_\theta}{\partial \theta}$$

$$\frac{RV}{H} \sim U \qquad H \ll R \qquad V \ll U$$

Governing equations

$$\rho\left[u_r\frac{\partial u_r}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta}-\frac{u_\theta^2}{r}\right]$$

$$=-\frac{\partial p}{\partial r}-\left[\frac{\partial \sigma_{rr}}{\partial r}+\frac{1}{r}\sigma_{rr}+\frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta}-\frac{\sigma_{\theta\theta}}{r}\right]-\rho g\cos\theta$$

$$\rho\left[u_r\frac{\partial u_\theta}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{u_r u_\theta}{r}\right]$$

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$$0=\frac{\partial}{\partial r}(r u_r)+\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}.$$

Governing equations

Left to solve

$$0 = -\frac{\partial p}{\partial r} - \frac{\partial \sigma_{rr}}{\partial r} - \rho g \cos\theta$$
$$0 = -\frac{1}{r}\frac{\partial p}{\partial \theta} - \frac{\partial \sigma_{r\theta}}{\partial r} + \rho g \sin\theta$$

Balance of gravity and fluid stresses: thin film flow.

Governing equations

Left to solve

$$0 = -\frac{\partial p}{\partial r} - \frac{\partial \sigma_{rr}}{\partial r} - \rho g \cos\theta$$
$$0 = -\frac{1}{r}\frac{\partial p}{\partial \theta} - \frac{\partial \sigma_{r\theta}}{\partial r} + \rho g \sin\theta$$

Balance of gravity and fluid stresses: thin film flow.

Geometry is reduced to just the slope!
Lava flow; industrial coating flows; tear films in the eye ...

Solution

For our two fluid models:

$$\text{Newtonian: } \sigma = \mu\dot{\gamma},\ \mu = 14$$

$$\text{Power-law: } \sigma = k\dot{\gamma}^n,\ k = 65,\ n = 1/3$$

given no-slip on the dome and no-traction at the surface, we obtain velocity profiles ($Y = 0$ is the dome, $Y = h$ is the surface):

$$u_N = \frac{1}{2}Y(2h - Y)\frac{\rho g \sin\theta}{\mu}.$$

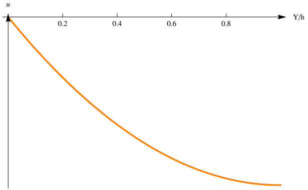
$$u_P = 1230\sin^{3/2}(\theta)\left[h^{5/2} - (h - Y)^{5/2}\right].$$

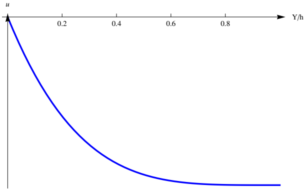
Velocity profiles

At fixed $\theta = \pi/2$,

	Newtonian	Chocolatey power-law
$u(Y) \propto$	$Y(2h - Y)$	$h^{5/2} - (h - Y)^{5/2}$

Dome





Velocity profiles

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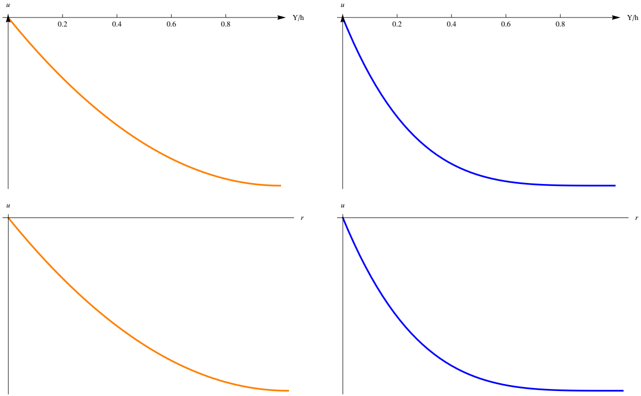


Dome

Pipe

$u(Y) \propto$	$Y^2 - h^2$	$h^{-3}(Y^4 - h^4)$

Film thickness

Fixing flux across the film, we have
for **Newtonian fluid**,

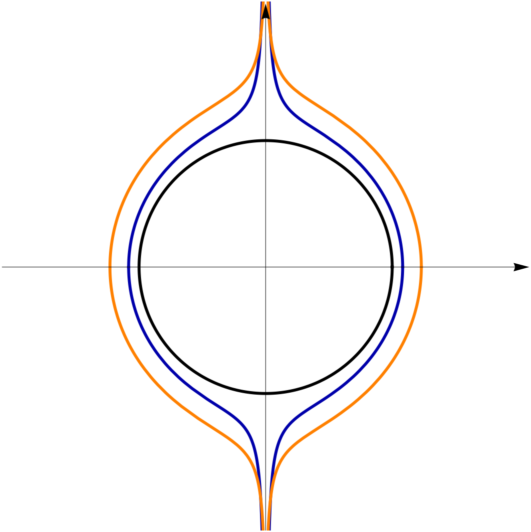
$$h(\theta) \propto \sin^{-1/3}\theta,$$

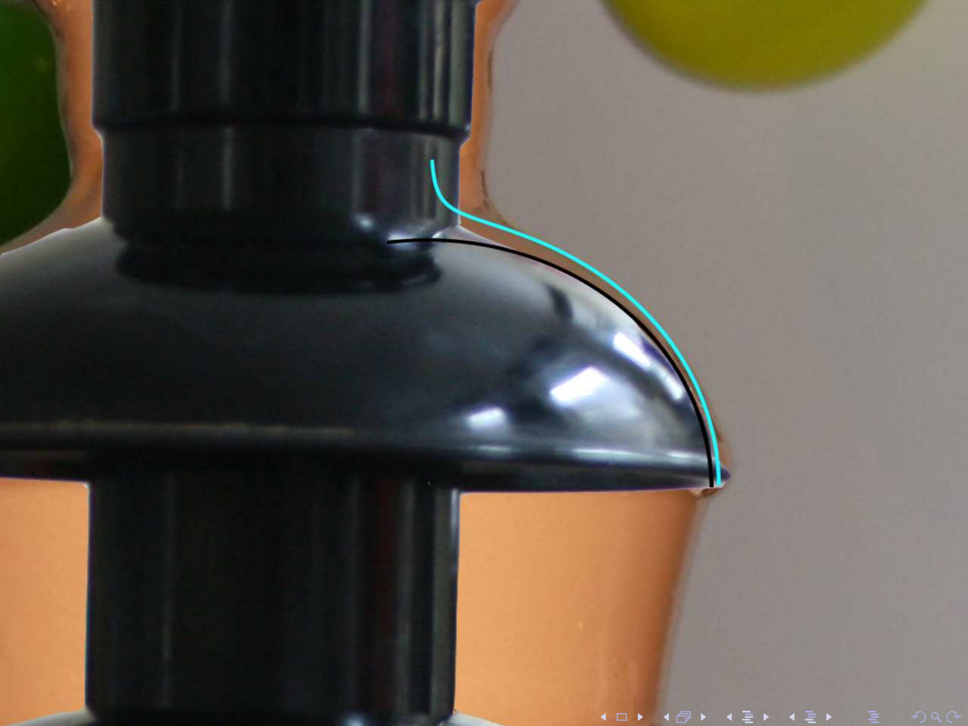
for **chocolatey power-law fluid**,

$$h(\theta) \propto \sin^{-3/7}\theta.$$

Film thickness

$h(\theta) \propto \sin^{-1/3}\theta$ $h(\theta) \propto \sin^{-3/7}\theta$





Falling sheet

Falling sheet

Difficult problem:

- Two free surfaces
- How does sheet thickness and distance along the sheet relate?
- What is the position of the sheet in space?

Falling sheet

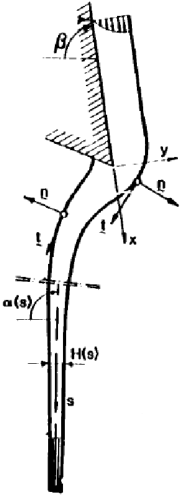
Difficult problem:

- Two free surfaces
- How does sheet thickness and distance along the sheet relate?
- What is the position of the sheet in space?

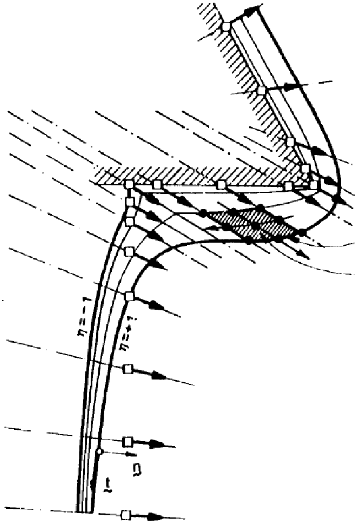
Harder problem:

- What happens at the top of the sheet?

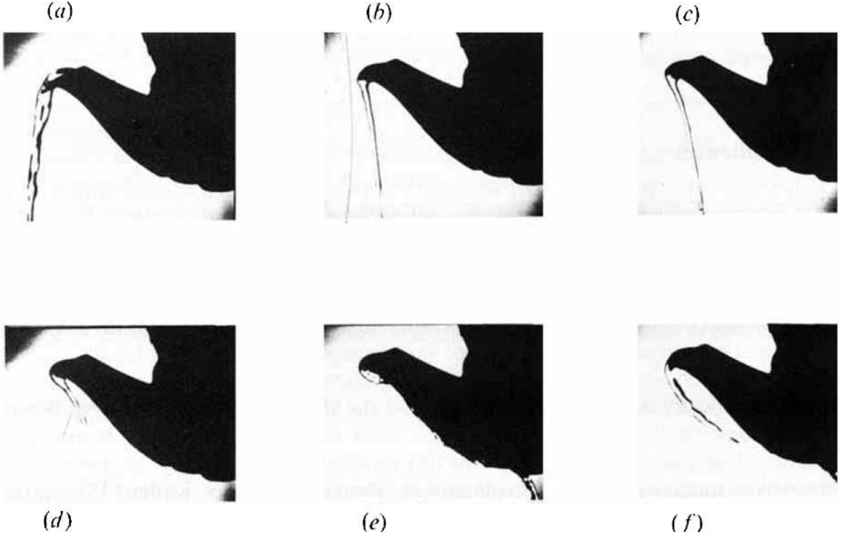
Falling sheet



Falling sheet

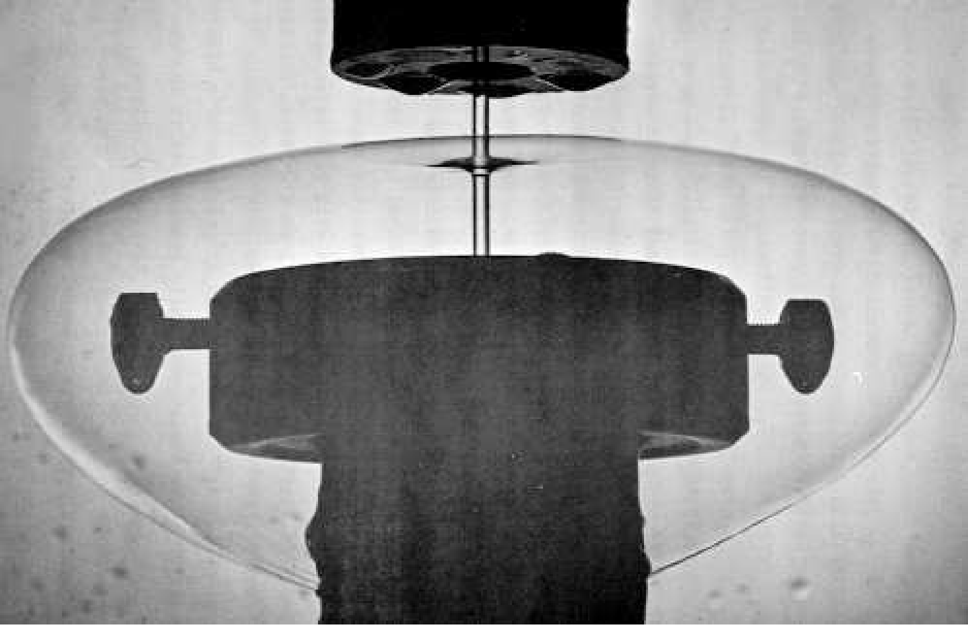


Teapot Effect



What causes the Teapot Effect?

- Surface tension?
- Hydrodynamics?
- Air pressure?
- Wetness of teapot?



Water bells

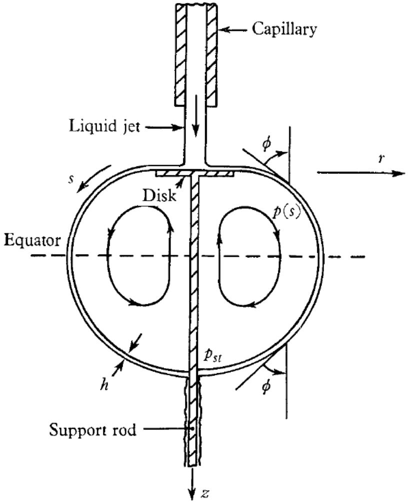
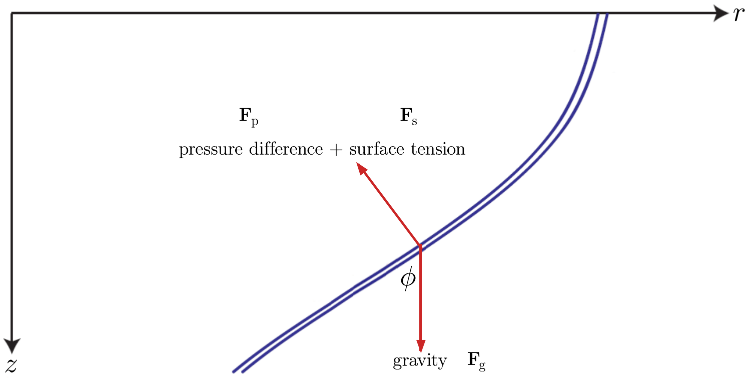


FIGURE 1. Sketch of water-bell and nomenclature.

Falling sheet



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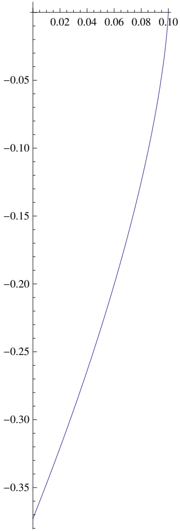
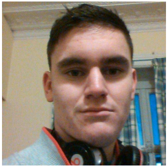


Falling sheet

Support

Adam Townsend



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The fluid dynamics of the chocolate fountain

Adam K Townsend[1] and Helen J Wilson

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Abstract

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- **Liquid flows inwards because of effects of gravity and surface tension**

By RUSS SWAN FOR MAILONLINE

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By Rachel Feltman November 24, 2015

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Pipe flow region is a good starter flow for non-Newtonian fluids
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Chocolate is a nightmare fluid to model...
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