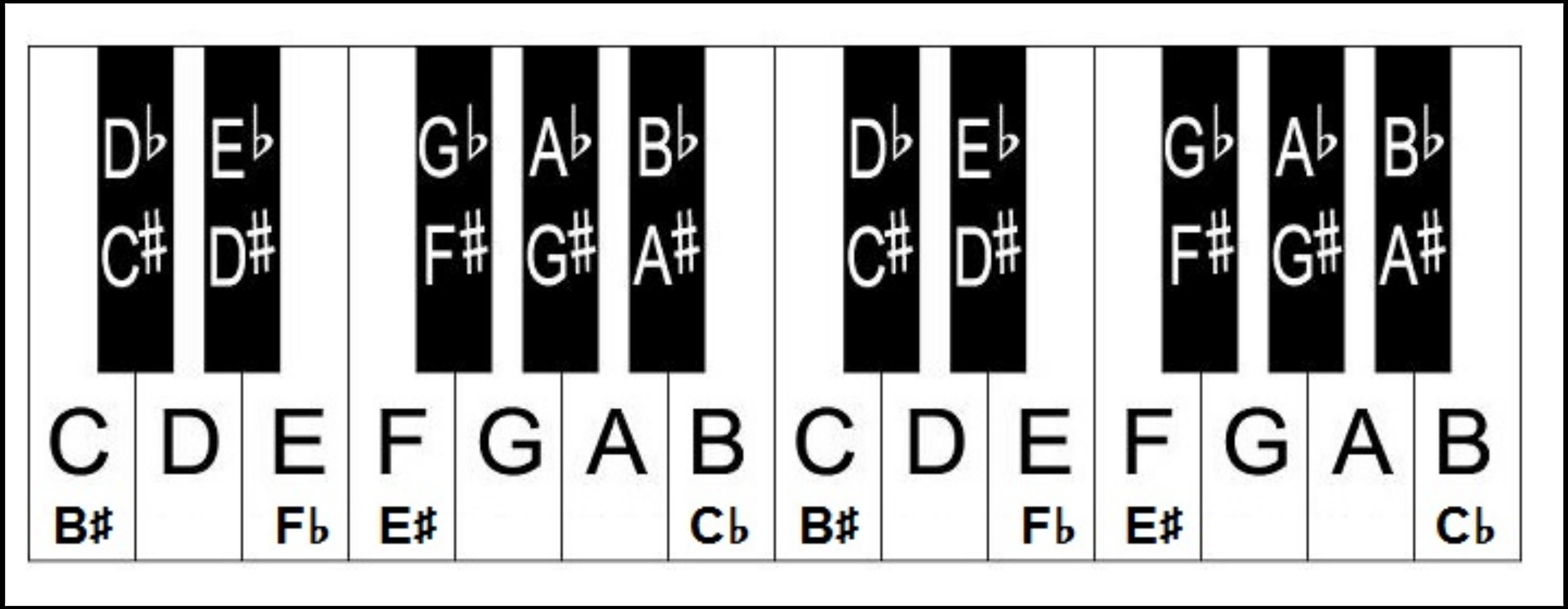
The Mathematics of Musical Composition

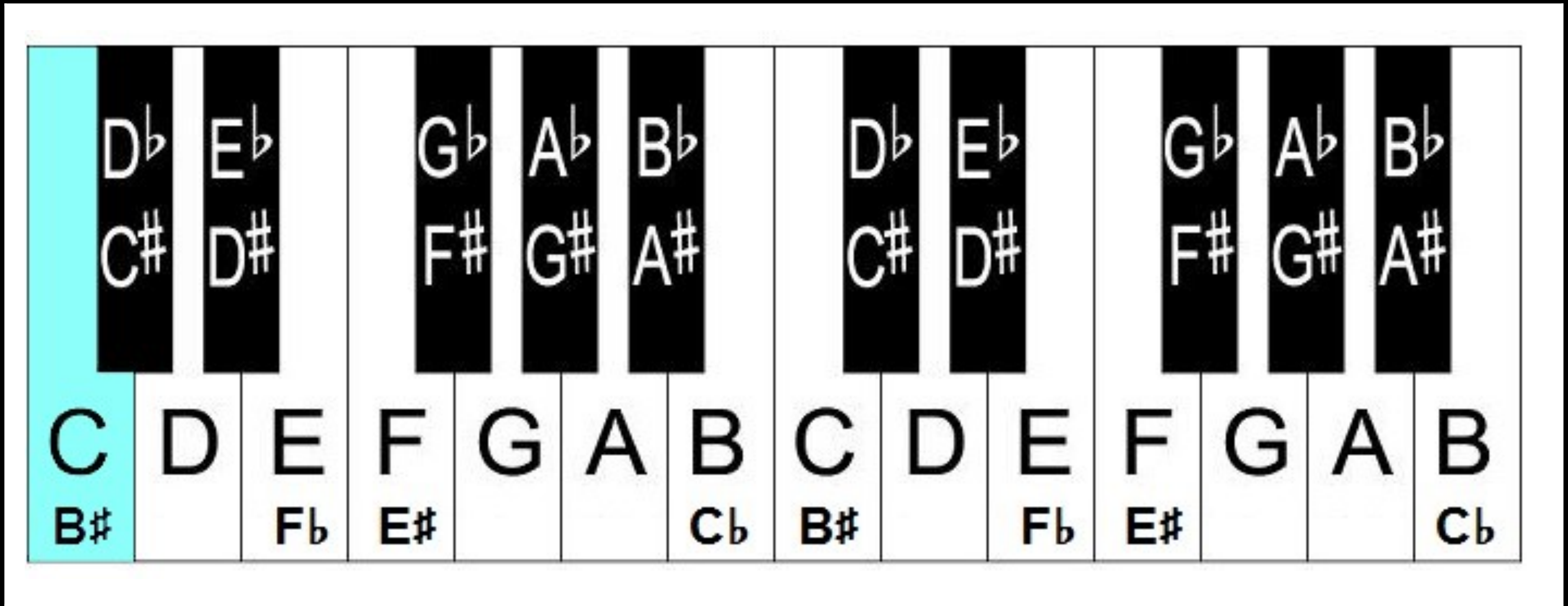
Professor Sarah Hart

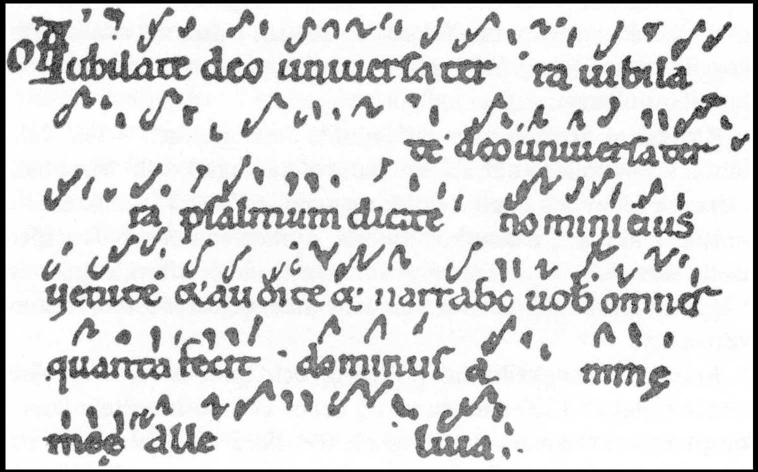
Musical Notation

There are twelve different 'pitch classes'.



- Smallest gap between notes is a semitone. Two semitones is a tone.
- Notes an octave apart have the same name – their ‘pitch class’.
- Diatonic scale has 7 notes; any starting point but same gaps between notes (tone, tone, semitone, tone, tone, tone, semitone)



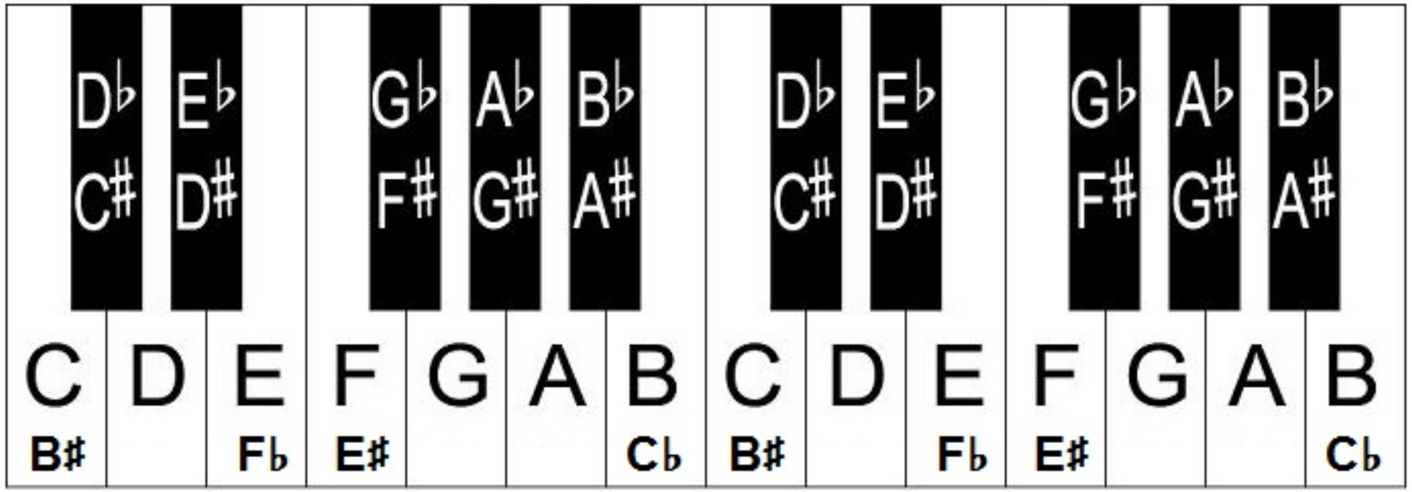


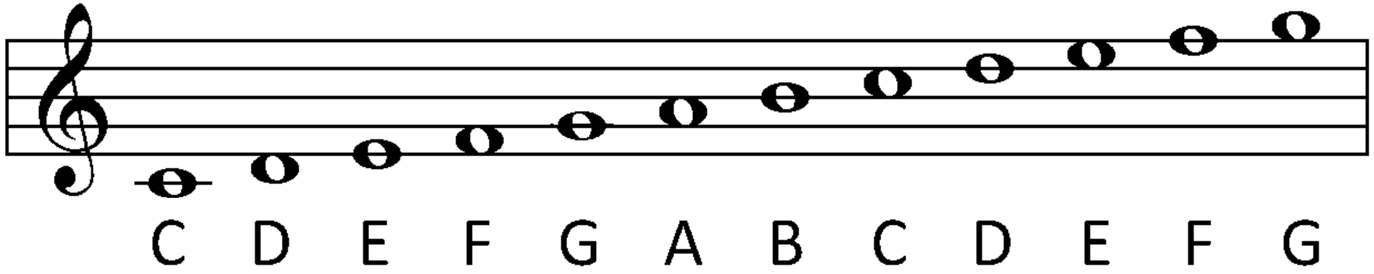
Writing a Score

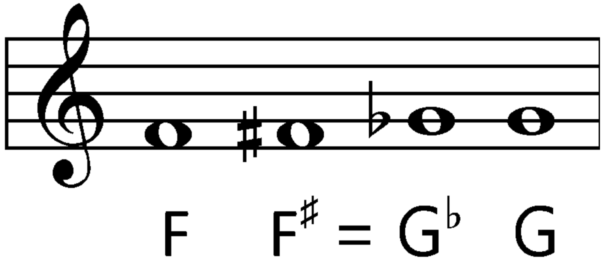
Modern Notation

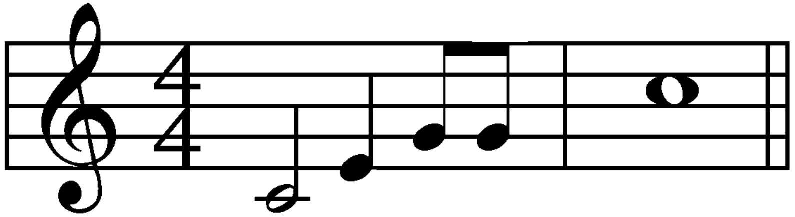
𝅝	(semibreve, whole note)	=	4 ♩
𝅗𝅥	(minim, half note)	=	2 ♩
♩	(crotchet, quarter note)	=	1 ♩
♪	(quaver, eighth note)	=	½ ♩
𝅘𝅥𝅯	(semiquaver, sixteenth note)	=	¼ ♩

𝅗𝅥.	=	3 ♩
𝅗𝅥..	=	3½ ♩
♩♩♩ (3)	=	2 ♩
𝄽	=	rest 1 ♩
	=	$\frac{4}{4}$ bar


D♭ C♯
E♭ D♯
G♭ F♯
A♭ G♯
B♭ A♯
D♭ C♯
E♭ D♯
G♭ F♯
A♭ G♯
B♭ A♯
C D E F G A B C D E F G A B
B♯
F♭
E♯
C♭
B♯
F♭
E♯
C♭


C D E F G A B C D E F G

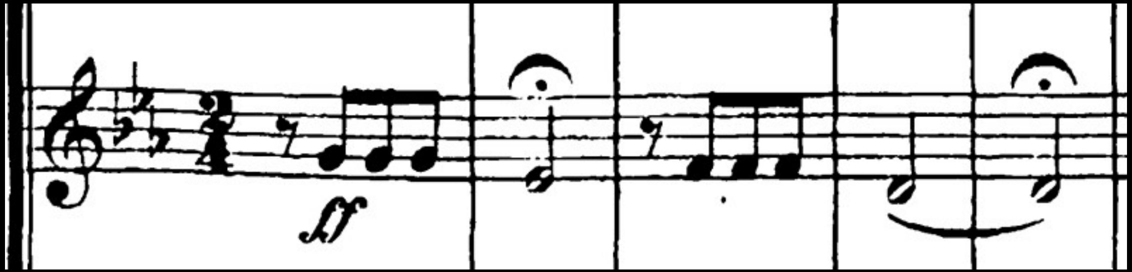

F F♯ = G♭ G



Musical Symmetries – themes and variations

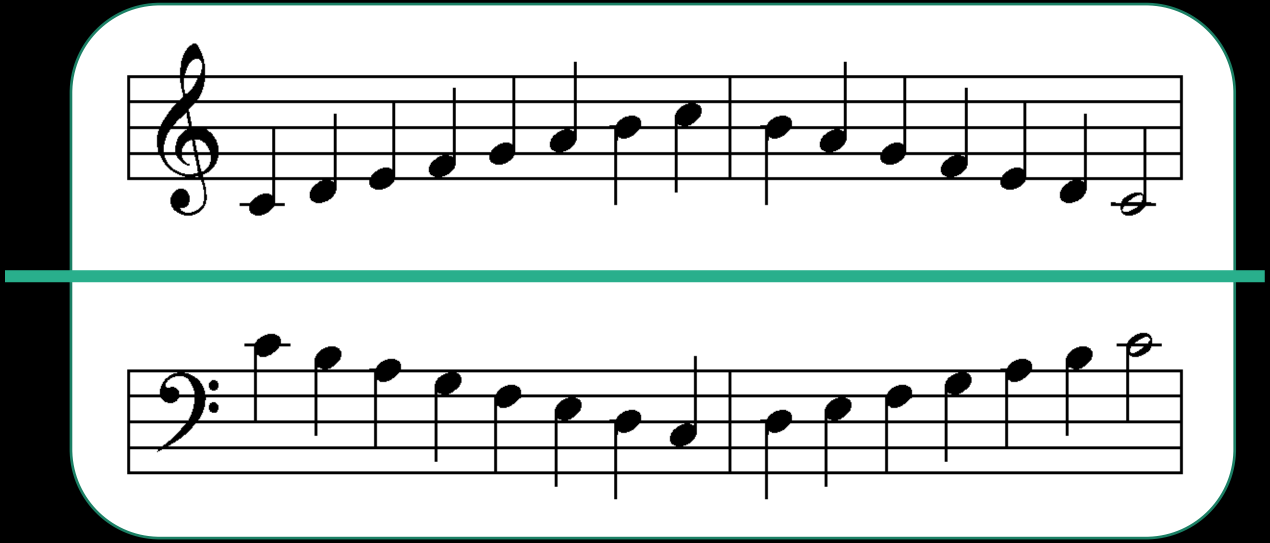
- Repetition

- Translation – “shifting up or down”



Musical Symmetries

- Inversion
  "reflect in horizontal mirror"
- Contrary motion scales
- Bach's Fugue in G major
  (Well Tempered Clavier Book 1)



Bars 1-3 →
(right hand)

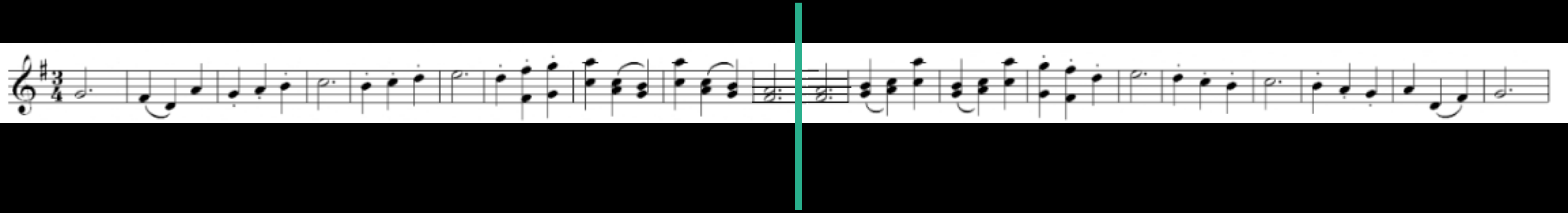
Bars 28-30 →
(left hand)

Musical Symmetries

Retrogression – "reflecting in a vertical mirror"

- *Ma fin est mon commencement* (palindromic rondeau by Guillaume de Machaut – 14thC)
- Two voice canon (Bach's *Musical Offering*)
- *Minuet al roverso* (Haydn Symphony No. 47) ↓

Can define translation, inversion and retrogression so that they don't depend on choice of notation.



Rotation?

- *Der Spiegel* (the mirror), attributed to Mozart
- See also *Quaerendo Invenietis* (seek and ye shall find) - a two-voice canon from Bach’s Musical offering

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Paul Hindemith

Ludus Tonalis (1942)

25 movements symmetrically arranged, with large and small scale symmetries throughout.

Musical Groups

"Now, music is steeped in the problem of symmetry, and symmetry is made accessible by the theory of groups. [..] What is music, if not very often a set of structures made from the permutations of notes, of sounds?"

Iannis Xenakis

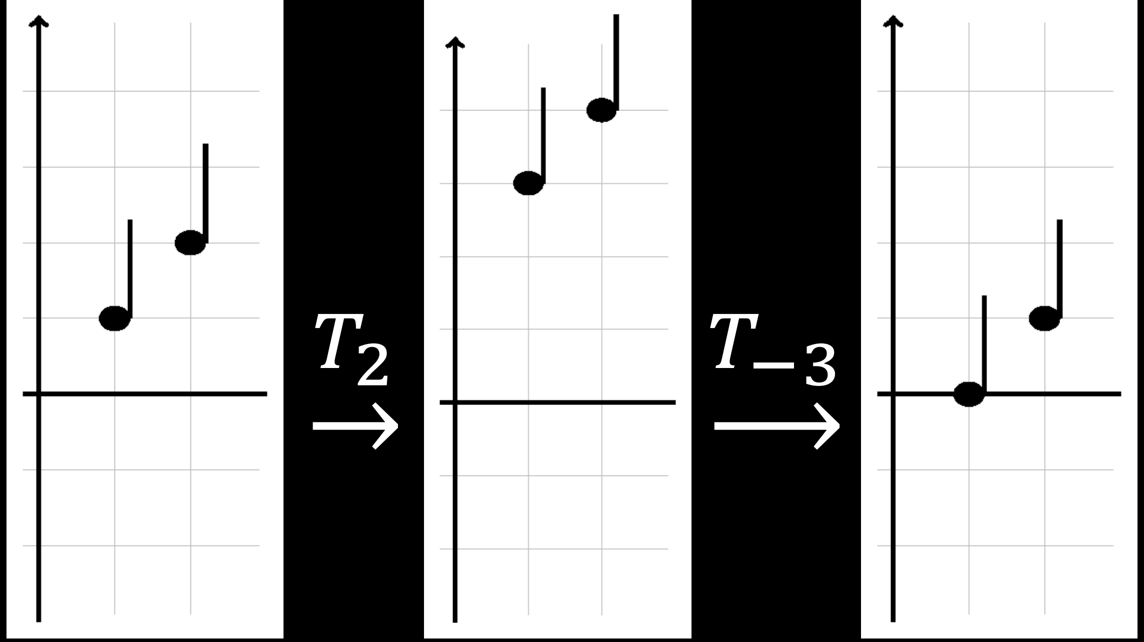
(Gresham Professor of Music 1975-8)

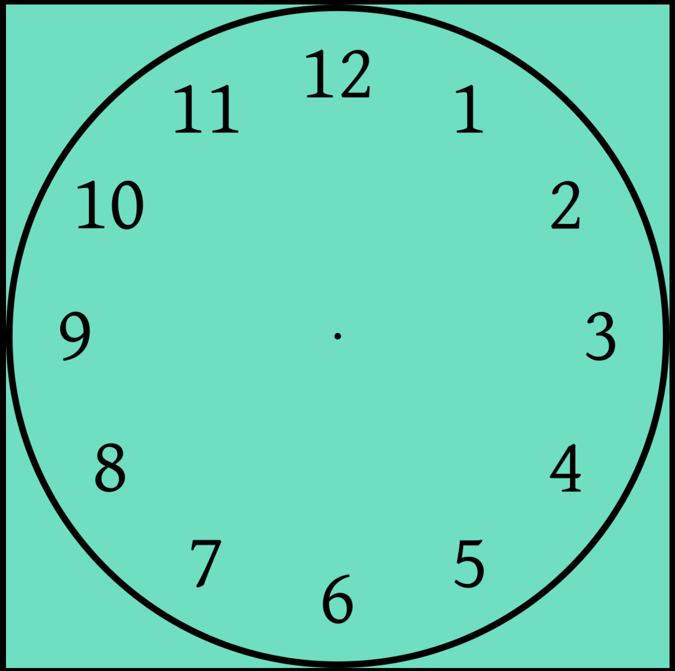
Ground Rules

- Focus on interval-preserving transformations (just as in geometry we preserve distance)
- Avoid dependence on stave notation (C major "bias").
- Measure intervals in semitones (not in notes on the diatonic scale)

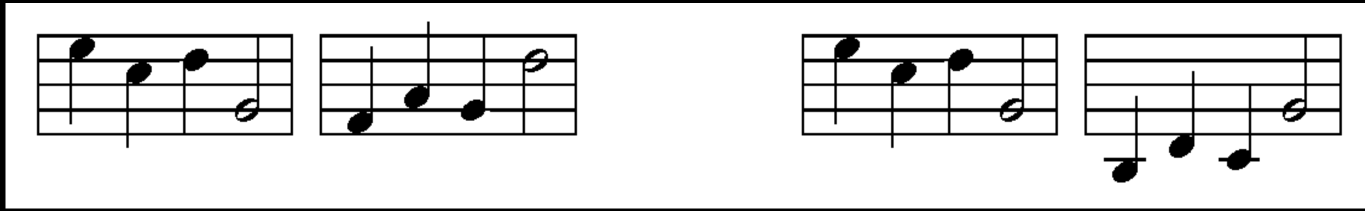
Translation

- T_n is translation up by n semitones.
- $T_n T_m(x) = T_n(T_m(x)) = T_{m+n}(x)$
- $T_0 =$ "up 0"; the **identity** map
- T_{12} maps to same notes one octave higher – same pitch class; often consider $T_{12} = T_0$
- For example $T_7 T_7 = T_{7\oplus 7} = T_2$
- With this convention, $T_n T_m = T_{m \oplus n}$ and there are exactly 12 "different" translations.
- Write $T = T_1$. Any T_n is just T^n.



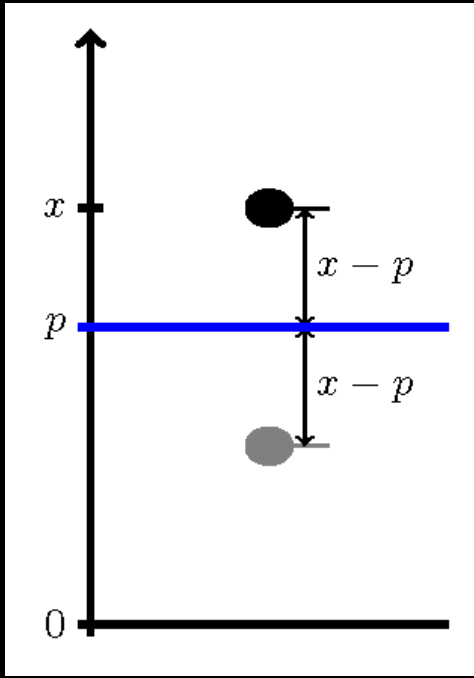


Inversion



- Reflection in horizontal mirror - position matters.
- Inversion I_p about pitch p:

$$\begin{aligned} I_p(x) &= x - 2(x-p) \\ &= 2p - x. \end{aligned}$$



- $I = I_0$. Then $I(x) = -x$ (Eg decree G# is pitch 0)
- $T_{2p}\big(I(x)\big) = T_{2p}(-x) = 2p - x = I_p(x)$.
- All inversions can be made with T and I.
- There are therefore 12 inversions.

Retrogression

Play tune backwards (reflect in vertical mirror). Call this R.

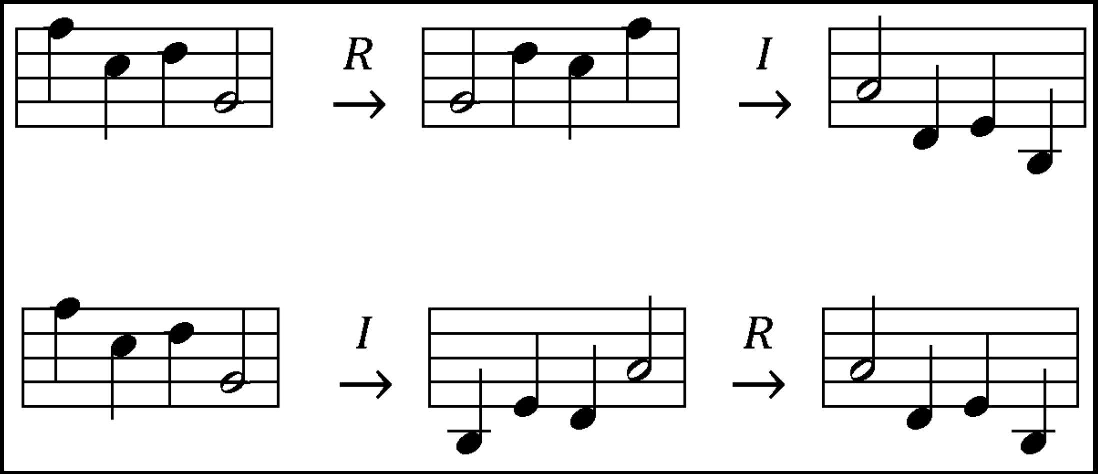
- $RT = TR, RI = IR, IT_n = T_{-n}I$.
- Any sequence of R, T, I can be rewritten as one of

  T_n (translation)

  T_nI (inversion)

  T_nR (retrogression)

  T_nRI (retrograde inversion)
- These are the only interval-preserving transformations.
- 48 in total, forming a “group”.

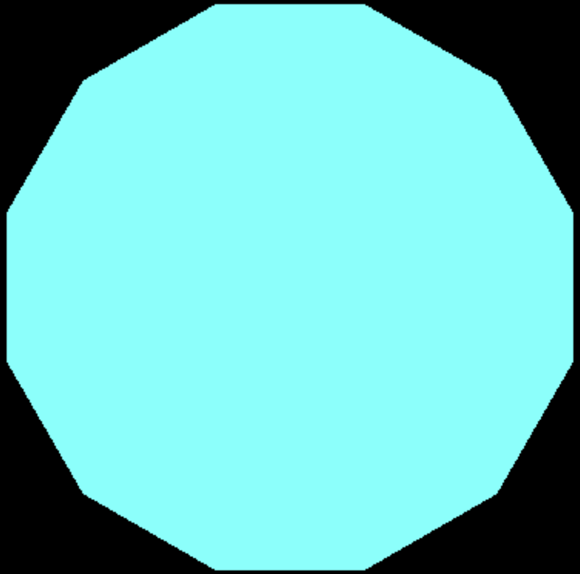


Groups

A group is a set, any pair of whose elements can be combined with some operation to produce another element of the set (the closure property); subject to 3 further rules.

Group	**Integers, with the operation +**	$\boldsymbol{M}$ **= {the 48 musical transformations}** with fg defined by $fg(x) = f(g(x))$.
Closure	• If a, b integers, then $a + b$ is an integer.	• If f, g preserve intervals, so does fg
Associative	• $(a + b) + c = a + (b + c)$	• $(fg)h(x) = f(g(h(x))) = f(gh)(x)$.
Identity	• $0 + a = a + 0 = a$ The identity element is 0.	• The identity element is T_0.
Inverses	• $a + (-a) = (-a) + a = 0$ The inverse of a is $-a$.	• Every element has an inverse; $T_n^{-1} = T_{-n}; R^{-1} = R; I^{-1} = I$.

Musical echoes



Many subgroups of M appear in disguise elsewhere.

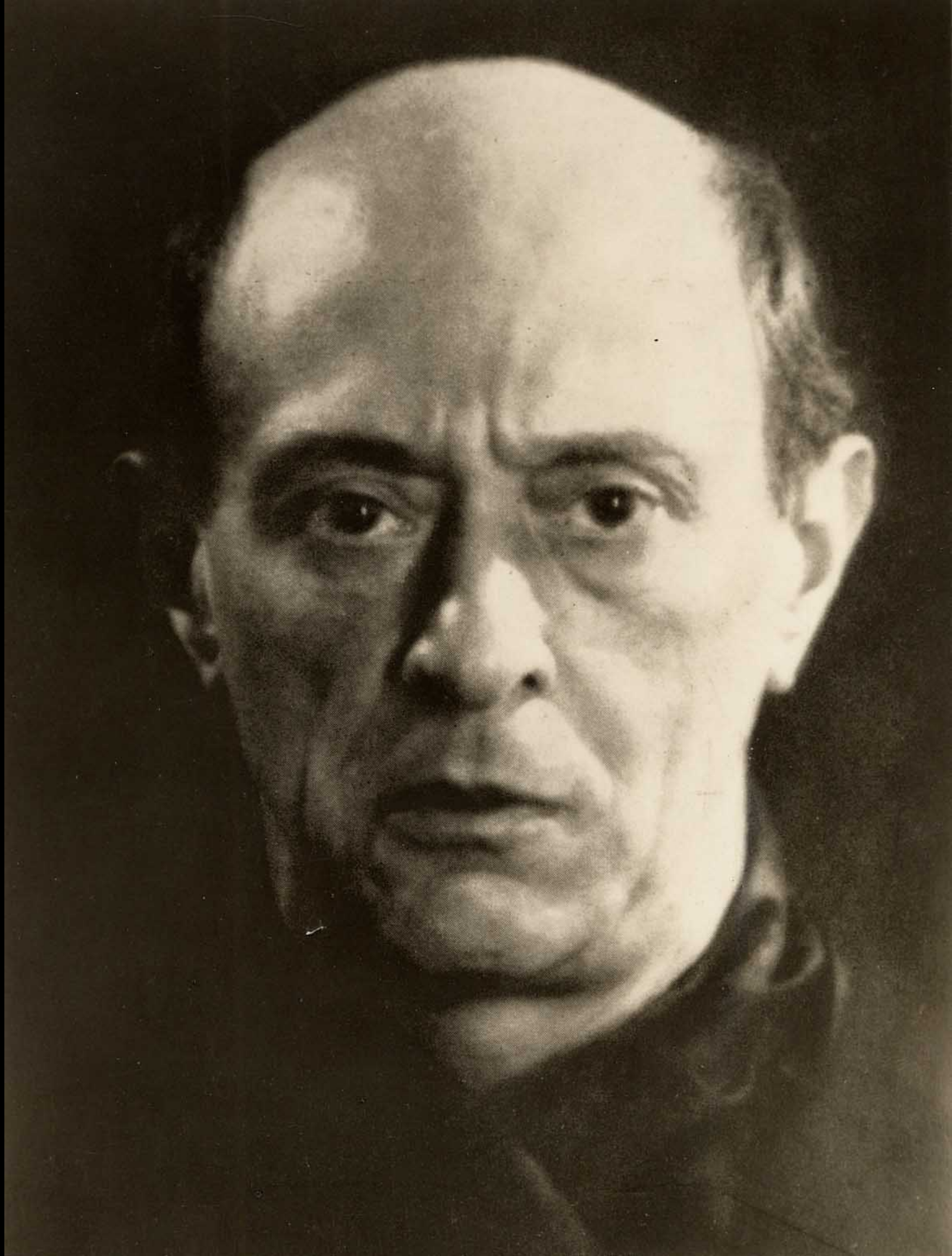
- Translations: Clock group $C_n = \{0, 1, \dots, n-1\}$ when $n = 12$.
  Eg $T_7T_6 = T_1$; $7 \oplus 6 = 1$. One of infinitely many.
- {Translations + Inversions} (24 elements).

  Translation $\mathrm{T}_1 \mapsto$ rotation $30°$; $I \mapsto$ reflection.

  Symmetry group of the dodecagon!
- $\{T_0, I, R, RI\}$ is a subgroup – the mattress-turning group (also known as the Klein 4-group).

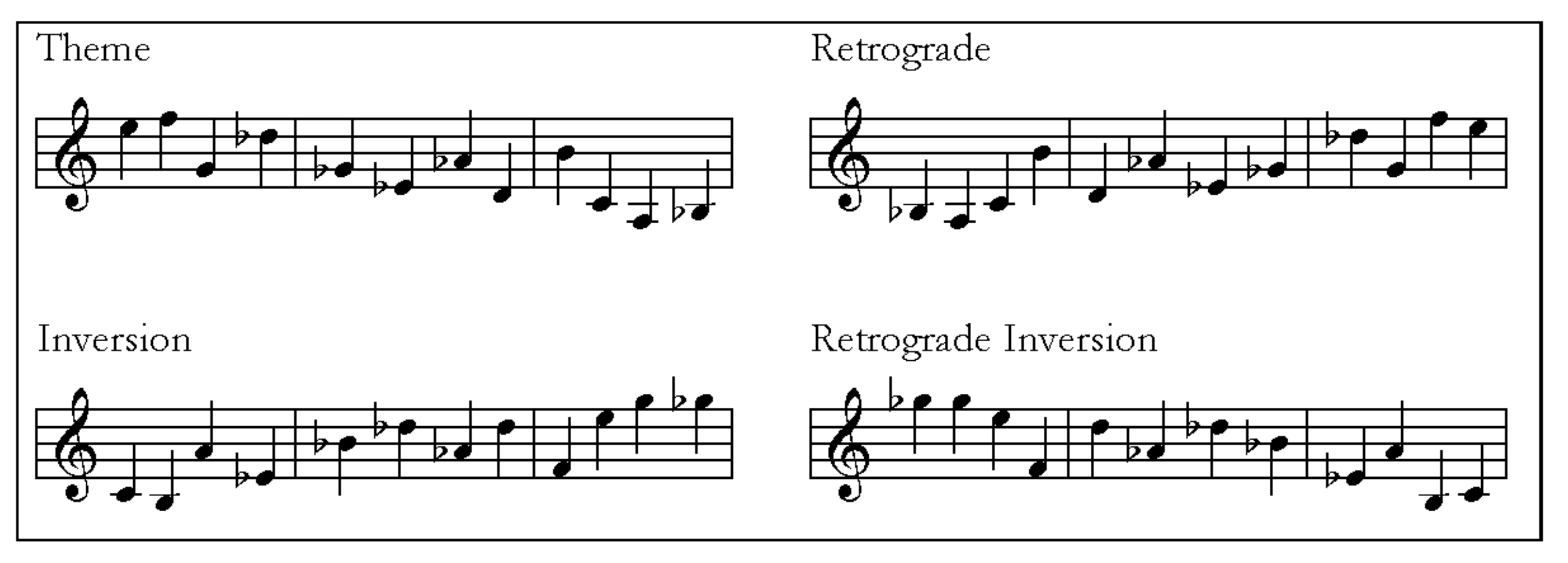
Groups and Tone Rows

- Schoenberg’s Twelve-Tone Method
- No key signature: instead a row with each pitch class represented exactly once in a defined order.
- Only the pitch class matters, not the register.
- We may transform the tone row but only using interval-preserving maps – our group M of 48 translations, inversions, retrogressions, and retrograde inversions.
- Ensures the transformed row still has twelve distinct tones.



Example: Schoenberg's Suite for Piano (Op.25)

First work composed entirely using tone rows.



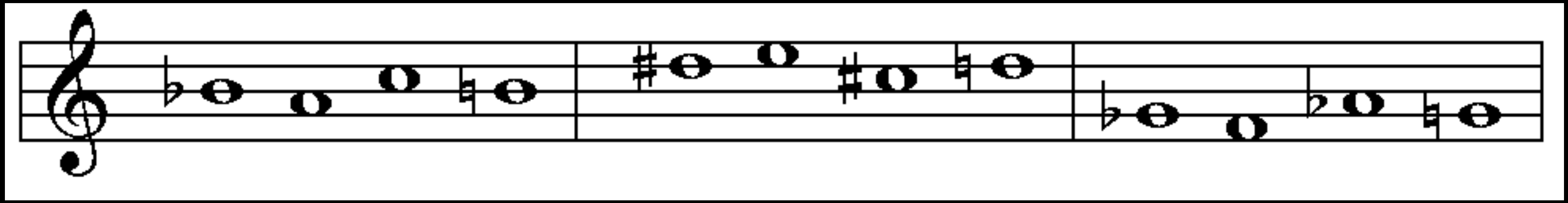
- In German musical notation, B stands for B♭, with H signifying B natural.
- The first four notes of the retrograde theme are B-A-C-H!

How many tone rows?

- How many scales? 12 major, 12 minor – 24 keys.
- How many rows?

  $12 \times 11 \times \cdots \times 1 = 479$ million
- BUT this overcounts – we are allowed to transform with elements of M.
- Why can't we just divide by 48?

- Webern uses a tone row W made of three 'BACH's in his String Quartet Op. 28; plain, inverted, retrograde inverted.
- Turns out $IT_{-1}R(W) = W$.
- So W is not in a set of 48 equivalent tone rows!



Group Theory to the Rescue

- Group G permuting elements of some set X.
  (eg M permutes the set X of possible tone rows.)
- Elements of X are in same "class" if they are mapped to each other by any element of G.
- For g in G, write Fix(g) for the set of elements of X that are left fixed by g. (eg $W \in \mathrm{Fix}(IT_{-1}R)$).
- Burnside's Lemma: the total number of different classes of elements of X is given by

$$\frac{1}{|G|}\sum_{g\in G} |\mathrm{Fix}\,(\mathrm{g})|$$

- Instead of working through 479 million tone rows, we can work through just 48 elements of M. Eg Fix$(T_1) = \emptyset$.
- There are 9,985,920 tone row classes!

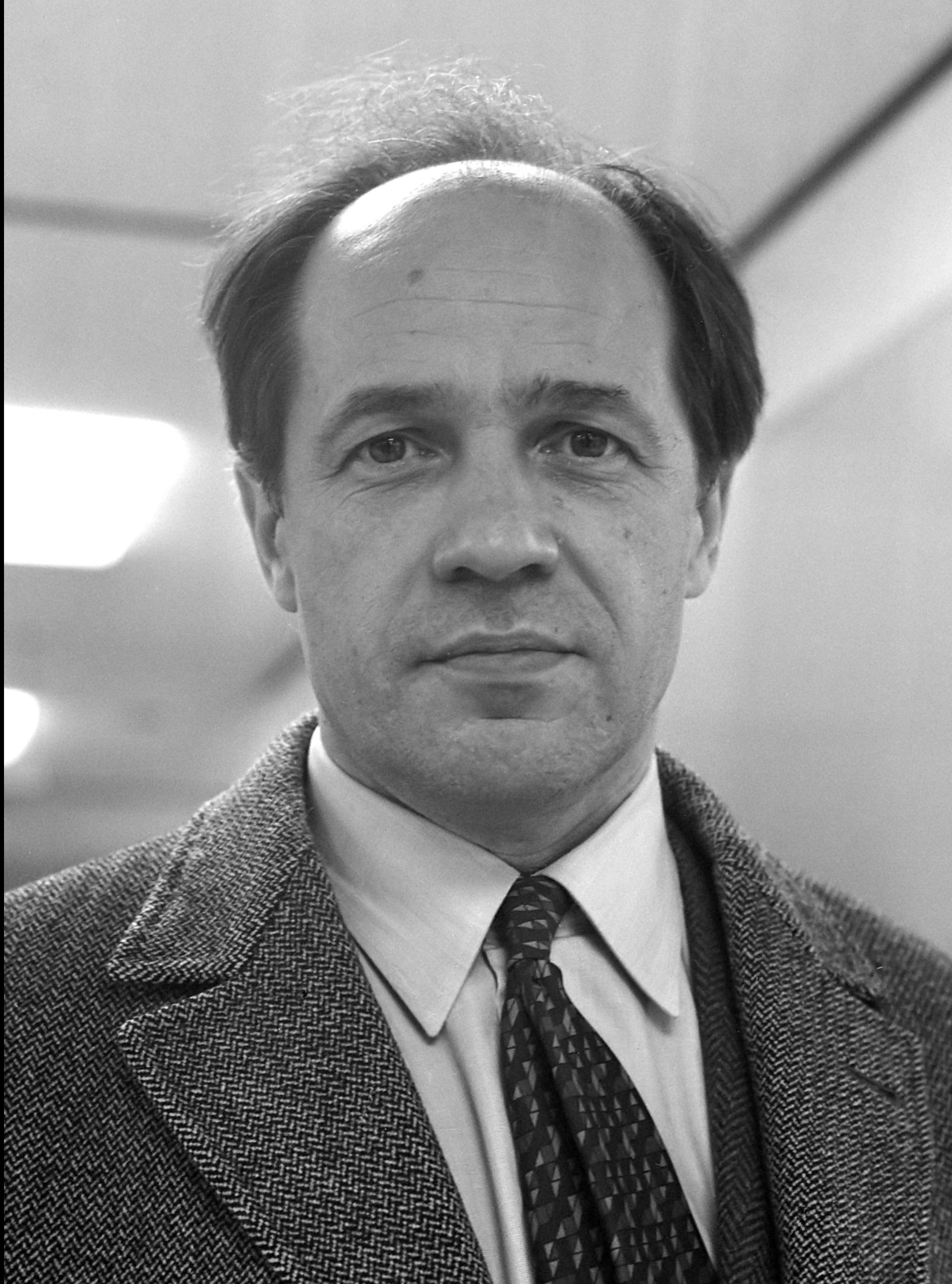
Total Serialism

- Every aspect of the music defined by formal structures.
- Pierre Boulez: *Structures* (1952) for two pianos.
- Tone row and 48 images were used to define notes, orders, note durations, dynamic instructions and mode of attack!

“It was the period. It was immediately after the war - and we wanted a *tabula rasa*. [..] We wanted to do something new. So in *Structures, Book I* (1951–52), where the responsibility of the composer is practically absent, I was extreme on purpose. Had computers existed at that time I would have put the data through them and made the piece that way. But I did it by hand. I was myself a small and very primitive computer. It was a demonstration through the absurd.”

Pierre Boulez, interviewed in 2011

(Picture credit: Dutch National Archive 1968)



Conscious use of mathematical forms

- Sets and Groups
- Magic Squares
- Infinite Sequences
- Fractals (eg Kaija Saariaho)
- Probability and Randomness

- Much more detail in the transcript!

Kaija Saariaho: Image credit Andrew Campbell

Sets and Groups – Milton Babbitt

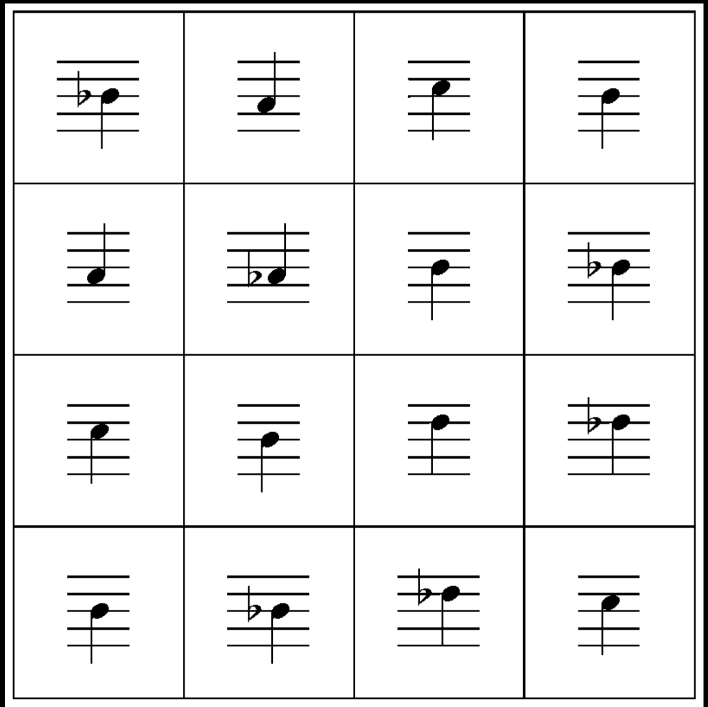
- **Babbitt's Theorem**
  Given any set of tones, the multiplicity of occurrence of a given interval in the set determines the number of tones in common between that set and its translations by that interval.
- Eg: if interval of 5 semitones occurs 3 times, then if you translate the set through 5 semitones, the new set has exactly 3 notes in common with the original set.
- Sets which have each interval the same number of times (difference sets) will intersect all their translations the same number of times.
- Can't have all 12 intervals exactly once as tritone must appear an even number of times.

Milton Babbitt, American composer, pioneer of serialism and electronic music

Magic Squares – Peter Maxwell Davies

- In *A mirror of whitening light* (1976-7), Gregorian chant was "processed" through a magic square. Composition then used paths through the square.

Bach (B♭, A, C, B) processed through Dürer



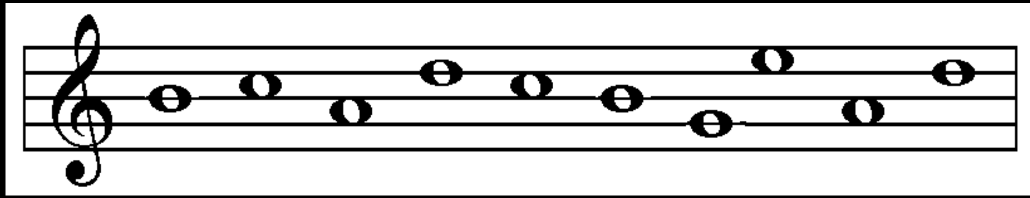
1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

Infinite Sequences – Per Nørgård

- Danish composer Per Nørgård has used his "infinity series" in much of his work since the 1960s.
- Defined with a recurrence relation, like the Fibonacci sequence.
- Fibonacci: 1, 1, 2, 3, 5, 8, 13, …. Given by $f_n = f_{n-1} + f_{n-2}$.
- Infinity Series: Start with pitches a_1 and a_2. Then:

$$a_{2n+1} = a_{2n-1} - (a_{n+1} - a_n)$$
$$a_{2n+2} = a_{2n} + (a_{n+1} - a_n)$$

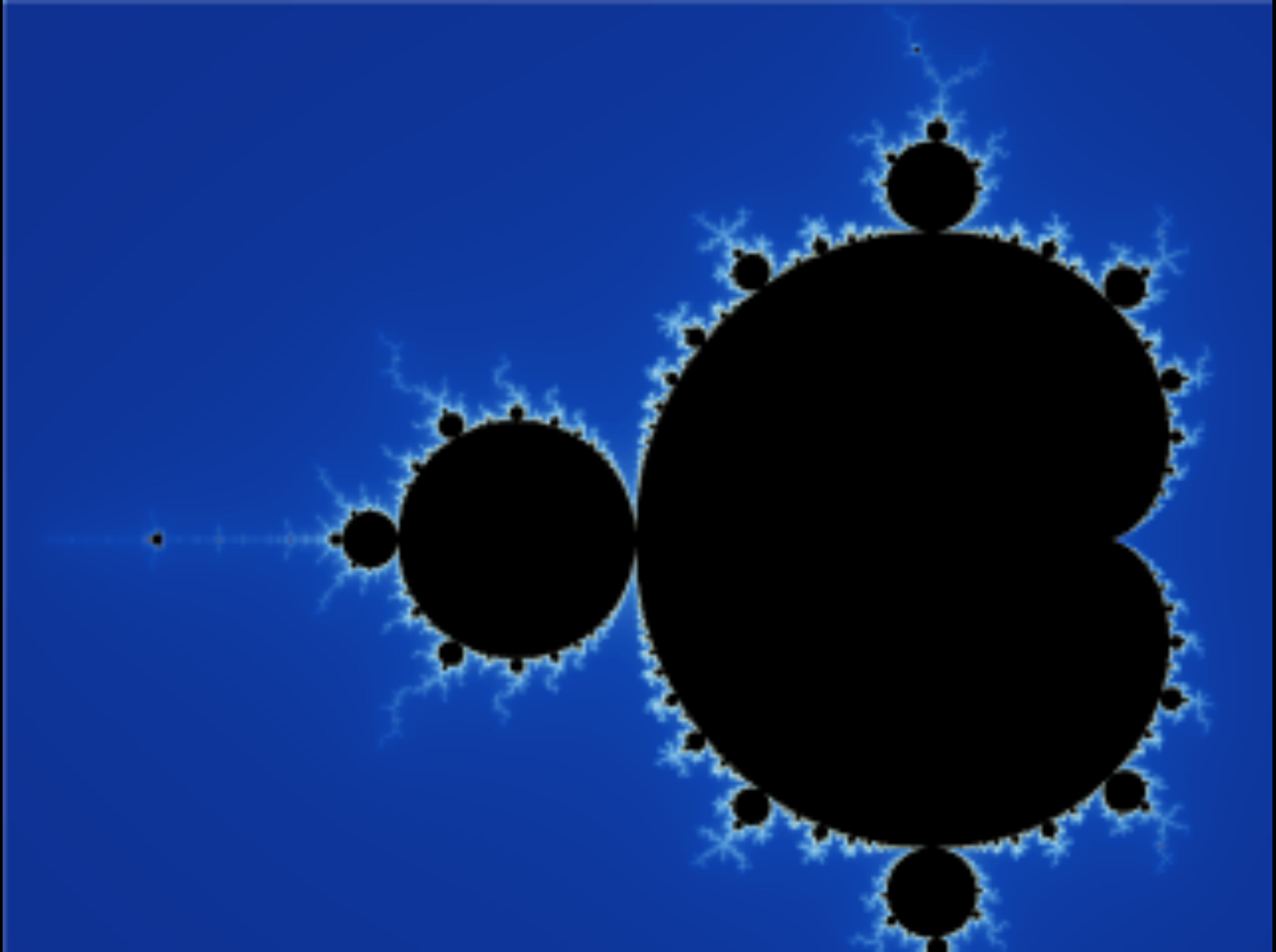
- Eg start with B and C ($a_1 = 0$, $a_2 = 1$), use the notes of the C major scale.



Fractals – Kaija Saariaho (greatest living composer!)

Nymphéa (Jardin secret III) (1987), for string quartet and electronics, was partially written with the aid of a fractal generator.

‘The basic material for the rhythmic and melodic transformations are computer-calculated in which the musical motifs gradually convert, recurring again and again.’



Fractal Music Generators

- Fractal music has self-similarity at different scales.
- Fractal music generators use fractal sequences to produce infinite sequences of numbers that can be converted into note pitches, durations, dynamics, etc.
- Eg: Thue-Morse sequence
  - Convert 1, 2, 3, 4, … into binary:

    1, 10, 11, 100, 101, 110, 111, 1000, 1001, 1010, 1011, 1100, …
  - Sum the digits of each term:

    1, 1, 2, 1, 2, 2, 3, 1, 2, 2, 3, 2, …
  - Pick out alternate terms, starting from second term.

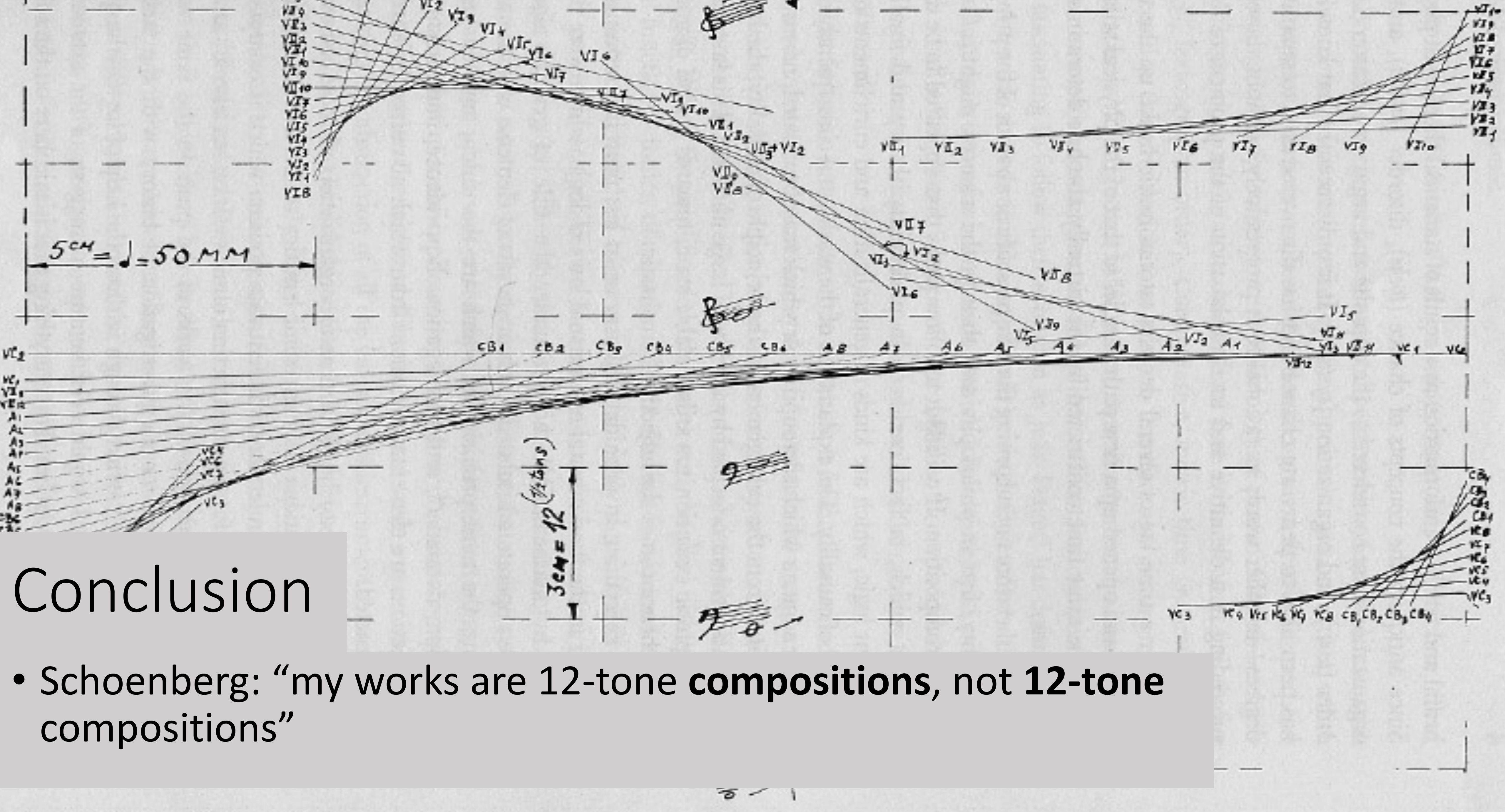
    1, 1, 2, 1, 2, 2, ….
  - Self similarity – and can repeat indefinitely!

Probability and Randomness

- Iannis Xenakis used a probability distribution for *Achorripsis*
- Poisson Distribution: if average number of events per unit time is λ, then the probability P_k that the event will happen k times in that time is
$$P_k = \frac{\lambda^k}{k!}\, e^{-\lambda}$$
- Performance divided into 196 cells: 28 units of time for 7 instruments.
- Eg Number of cells in which 2 events happen is $196P_2$. With $\lambda = 0.6$, we get 19. So in 19 of the cells, 2 events occur.
- He calls the events "clouds of sound"…

Un infinito Numero di Minuette Trio

- You can own a unique Haydn minuet!
- Game circa 1790.
- In each of 16 bars, roll a die to select one of six options.
- Not infinite – “just” 6^{16}. Actually, only 4 different 8^{th} bars, and 3 different 16^{th} bars.
- So only 940 billion minuette trio.



Conclusion

- Schoenberg: “my works are 12-tone **compositions**, not **12-tone** compositions”

Fig. I–2. String Glissandi, Bars 309–14 of *Metastasis*

Thank you for listening!

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