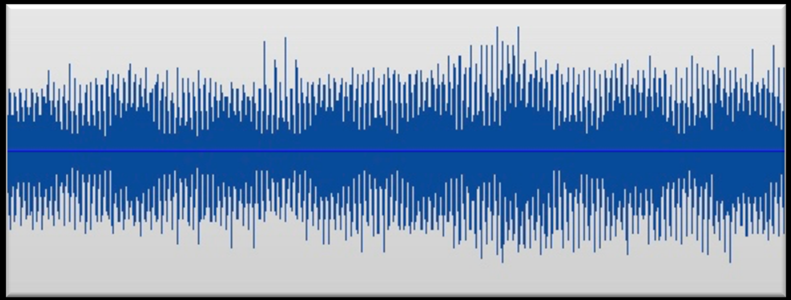
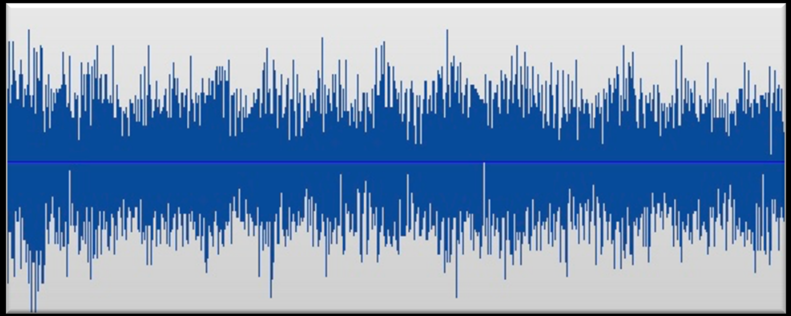
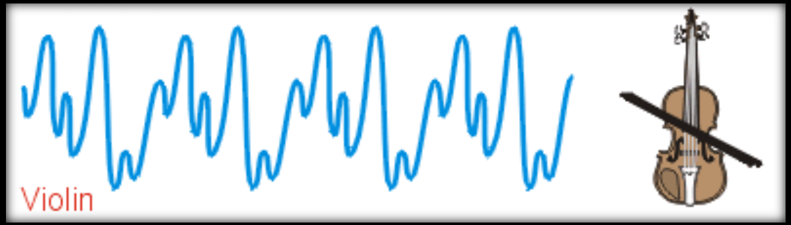
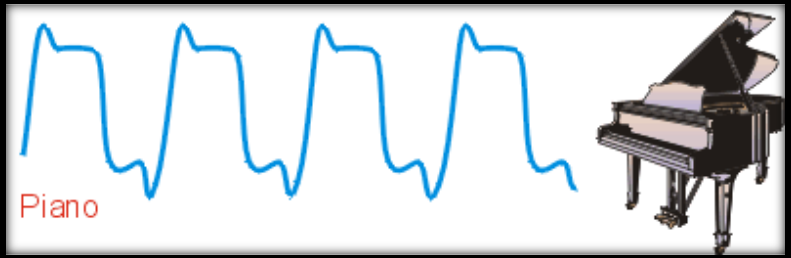
The Sound of Mathematics

Professor Sarah Hart





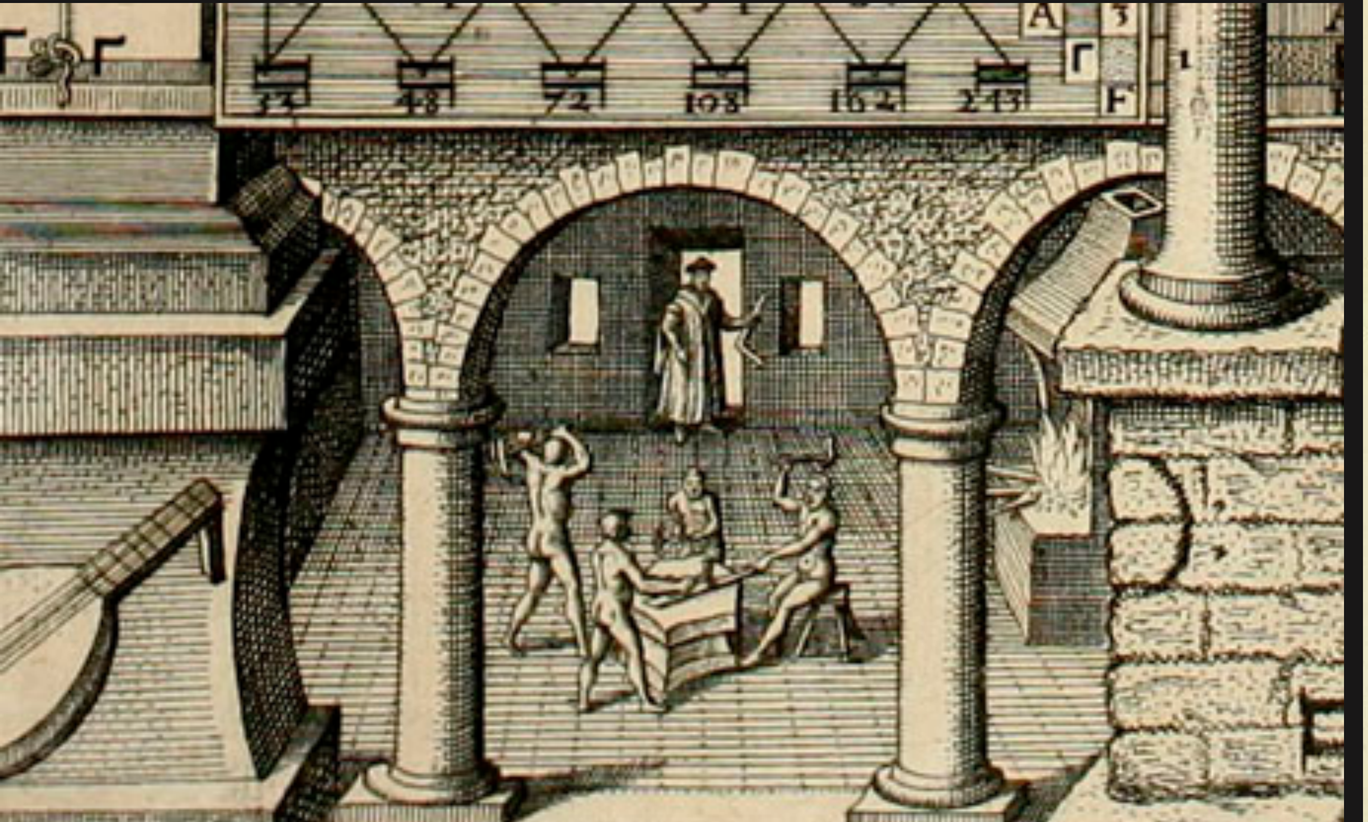




Violin/Piano soundwaves © Benjamin Hollis

Changing the Frequency changes the pitch

What makes sounds harmonious?

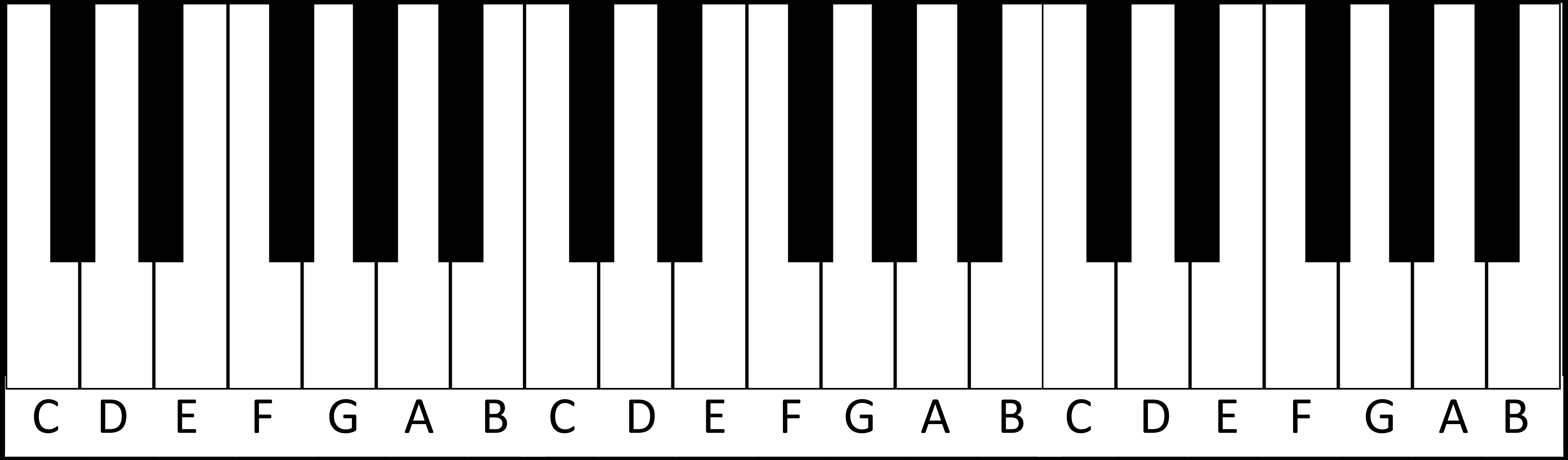


Harmonious Sounds

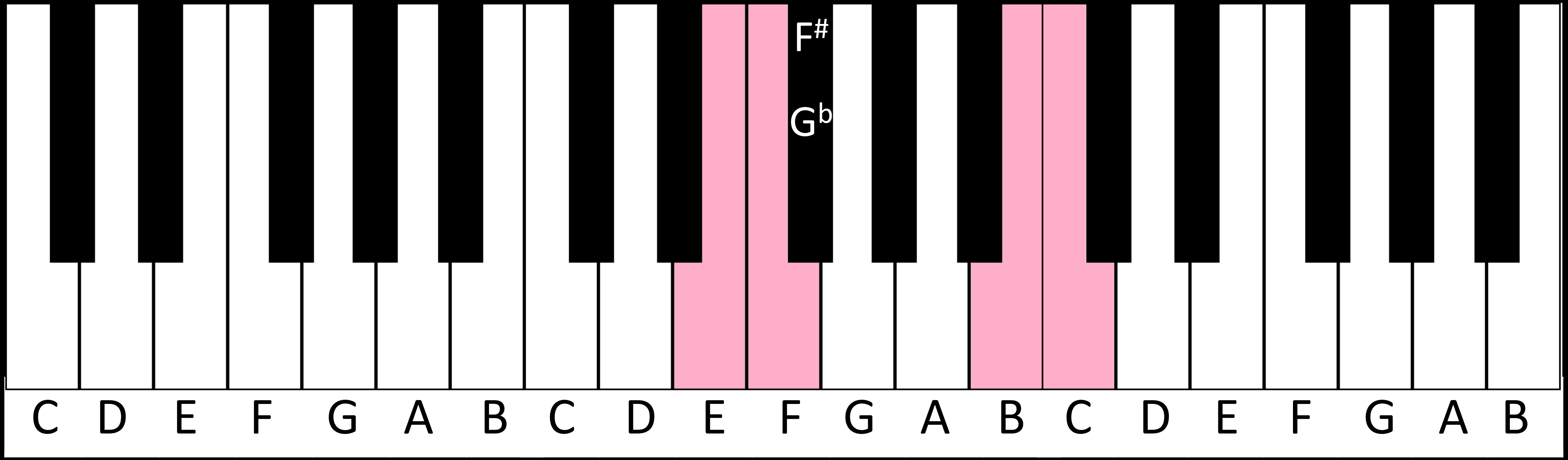
Suppose a string of length l produces a sound of frequency f.

String Length	Frequency	Name
$\frac{1}{2}l$	$2f$	Octave higher
$2l$	$\frac{1}{2}f$	Octave lower
$\frac{2}{3}l$	$\frac{3}{2}f$	Perfect 5th higher
$\frac{3}{4}l$	$\frac{4}{3}f$	Perfect 4th higher

Some Musical Notation

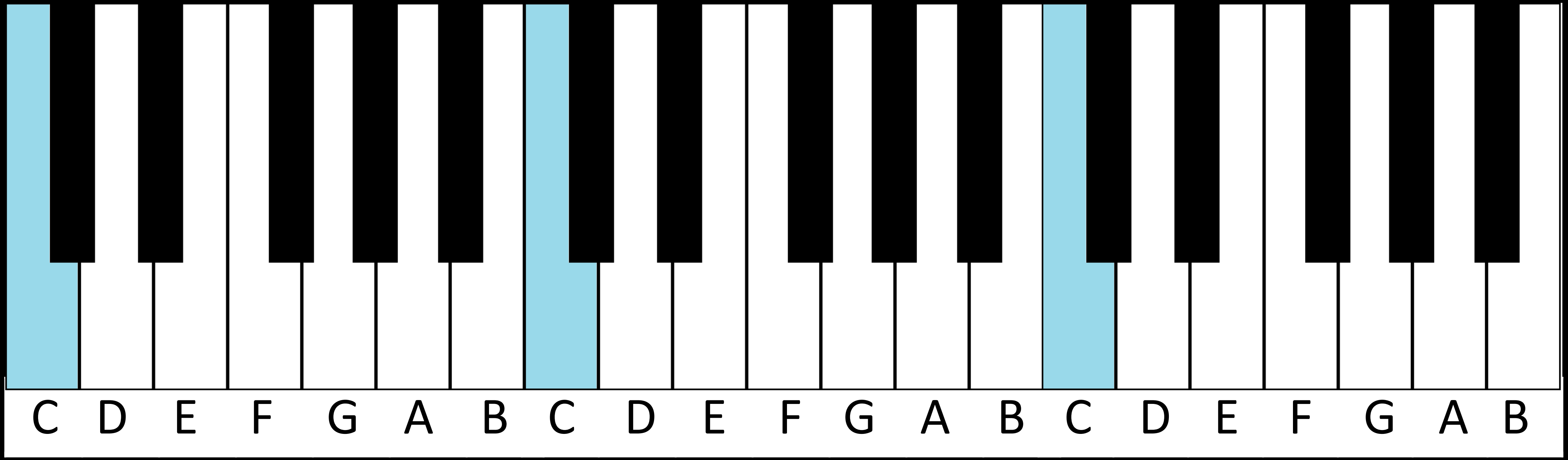


Semitones

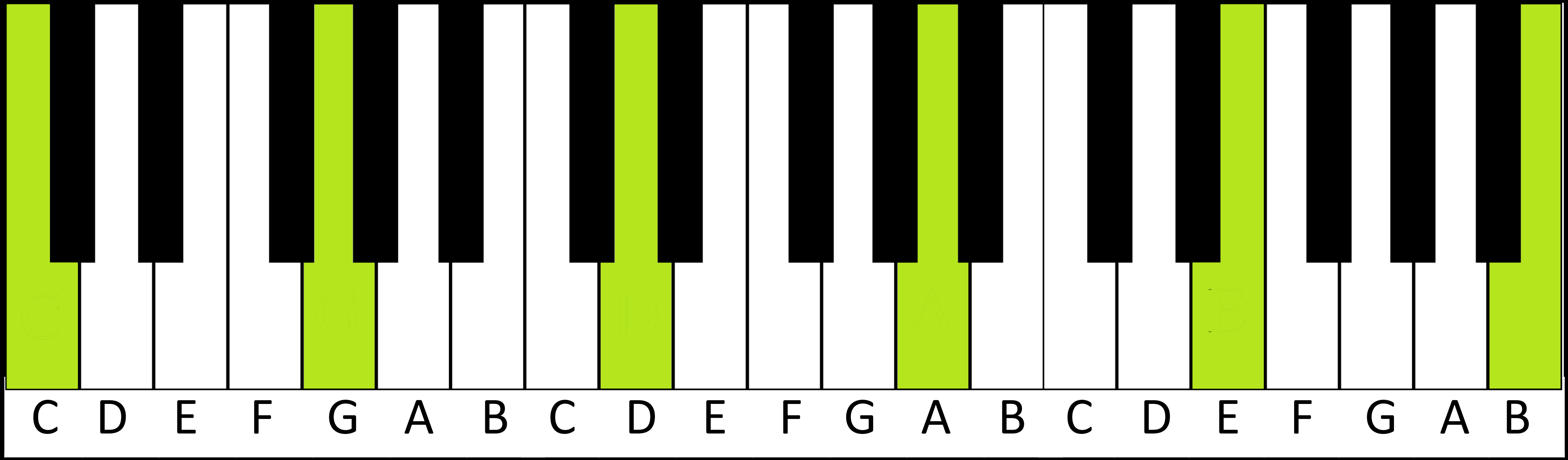


Octaves

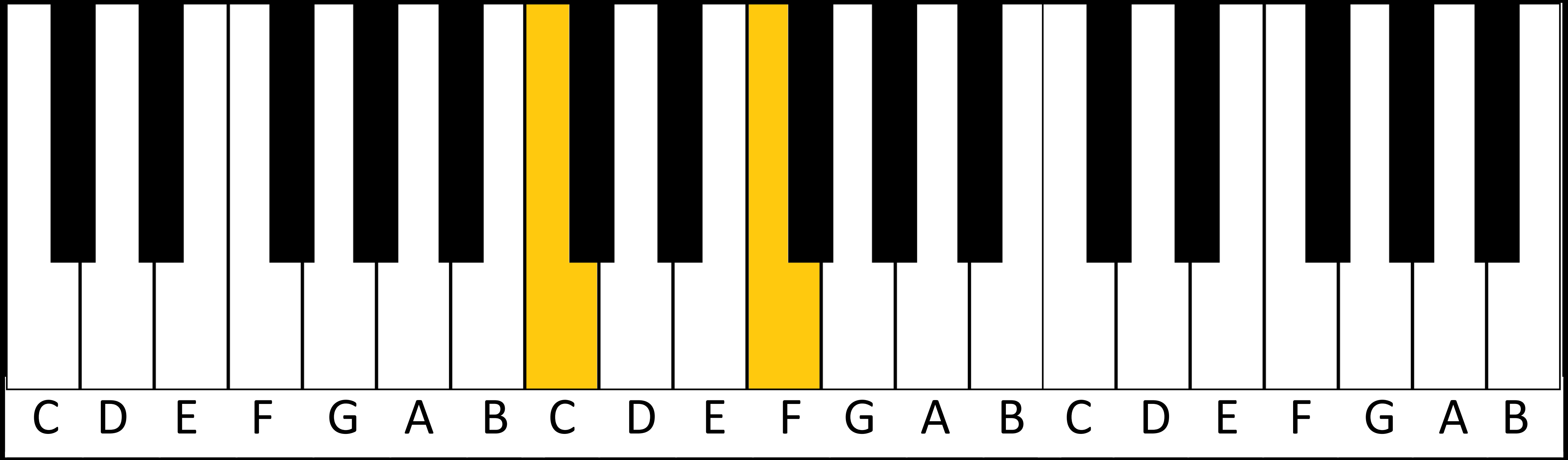
12 semitones in an octave; 7 notes in the diatonic scale



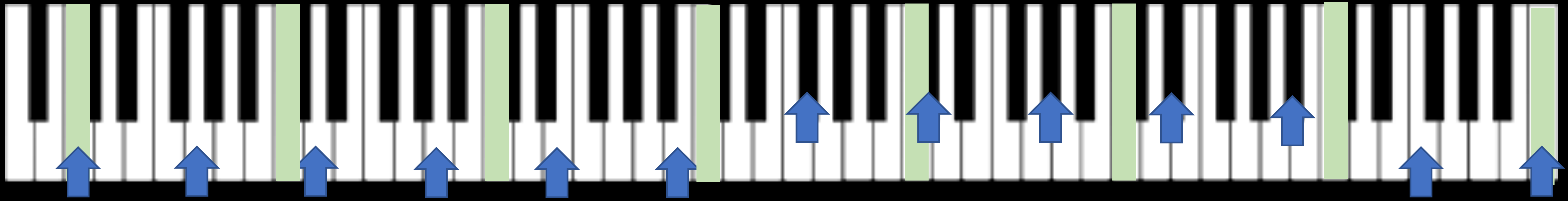
Fifths

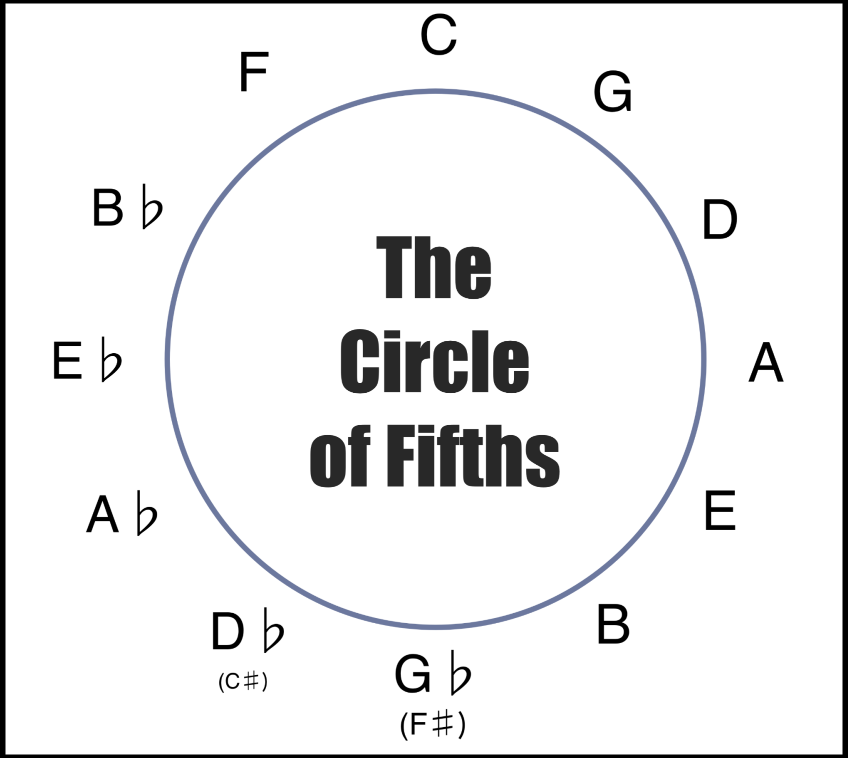


Fourths



Circles and Spirals

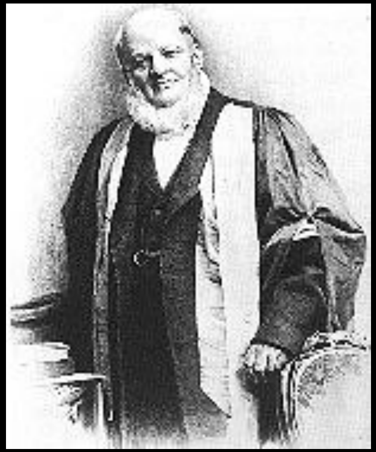




The Problem

- Start at f
- 7 octaves: $128f$
- 12 fifths: $\left(\frac{3}{2}\right)^{12} f \approx 129.7f$.

Cents



- 100Hz to 200Hz is an octave, but so is 200Hz to 400Hz.
- Ratio $\frac{f_2}{f_1}$ of two frequencies is equal to c cents, where

$$\frac{f_2}{f_1} = 2^{c/1200}.$$

- Frequency ratio r corresponds to $1200 \log_2 r$ cents (¢).
- Suppose ratio r_1 that is a_1¢, is followed by ratio r_2 that is a_2 ¢.
- Ratio of the outcome is $r_1 r_2$, but the new cent value is:

$$1200 \log_2 r_1 r_2 = 1200(\log_2 r_1 + \log_2 r_2) = 1200 \log_2 r_1 + 1200 \log_2 r_2 = a_1 + a_2.$$

The Pythagorean Scale

C	D	E	F	G	A	B	C
1	$\frac{9}{8}$	$\frac{81}{64}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{27}{16}$	$\frac{243}{128}$	2

- After 12 pitch classes, next is *almost* the same as one you have – hence 12 "notes".
- 53 perfect 5^{th}s $\approx$ 31 octaves more closely (3.6 cents)
- Can never complete the circle: $\frac{3}{2}^{m} = 2^{n}$ implies $3^m = 2^{m+n}$.

What went wrong?

- Singing “in a womanish manner with tinkling”… “as if imitating the wantonness of minstrels” (1132)
- Licentious modulations! Mountainous collections of cacophonies!
- Instruments with fixed keys by c1400 (shown: spinetta, 1540)
- “English” harmonies: 3rd and 6th.

Just Intonation

- Pure major 3rd: 5/4 (386¢). Pure minor 3rd: 6/5
- Pythagorean major 3rd: 81/64 (408¢).
- Bartolomeo Ramos de Pareja in 1482 suggested "just-intonation."
- C to E, F to A, G to B are pure major thirds.

	C	D	E	F	G	A	B	C
Pythag.	1	$\frac{9}{8}$	$\frac{81}{64}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{27}{16}$	$\frac{243}{128}$	2
Ramos	1	$\frac{9}{8}$	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{15}{8}$	2

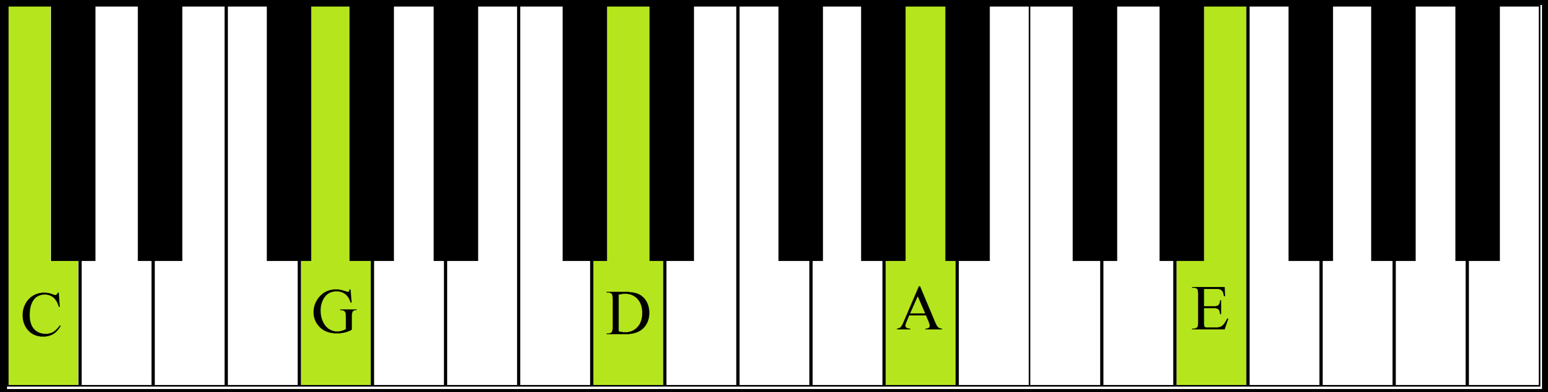
- D to A is now $\frac{5}{3} \div \frac{9}{8} = \frac{40}{27}$.
- over 22¢ away from a pure perfect 5th.

Franchinus Gaffurius

Practica Musicae (1486)

A practical solution

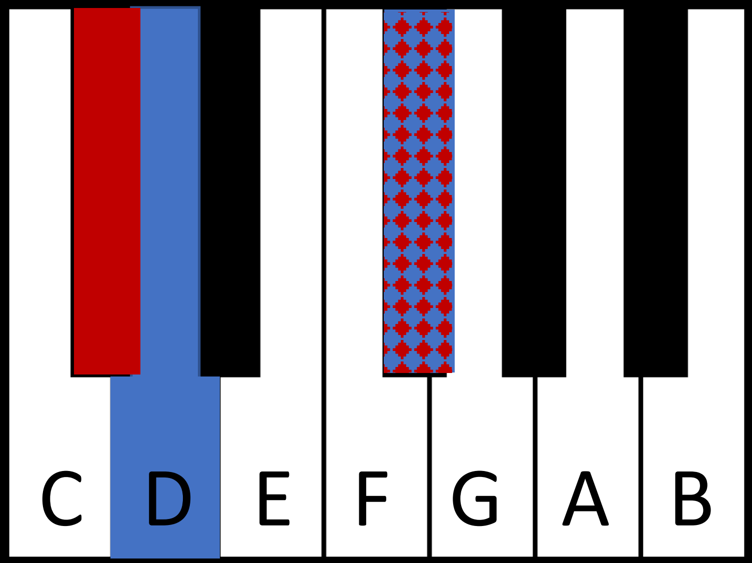
- Organ makers adjusting lengths of pipes to temper the ratios of their fifths – *temperament* or *participata*

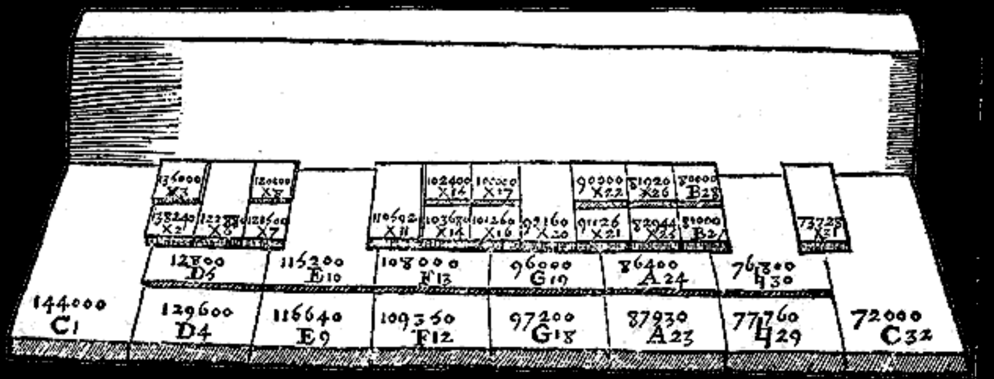


- 4 pure 5ths $\left(\frac{3}{2}\right)^4 = \frac{81}{16}$; want pure 3rd: $4 \times \frac{5}{4} = 5$.
- Want "tempered" 5th to be x such that $x^4 = 5$.
- New "G" is 697¢; the pure "G" is 702¢.
- Called **Mean-Tone Temperament**

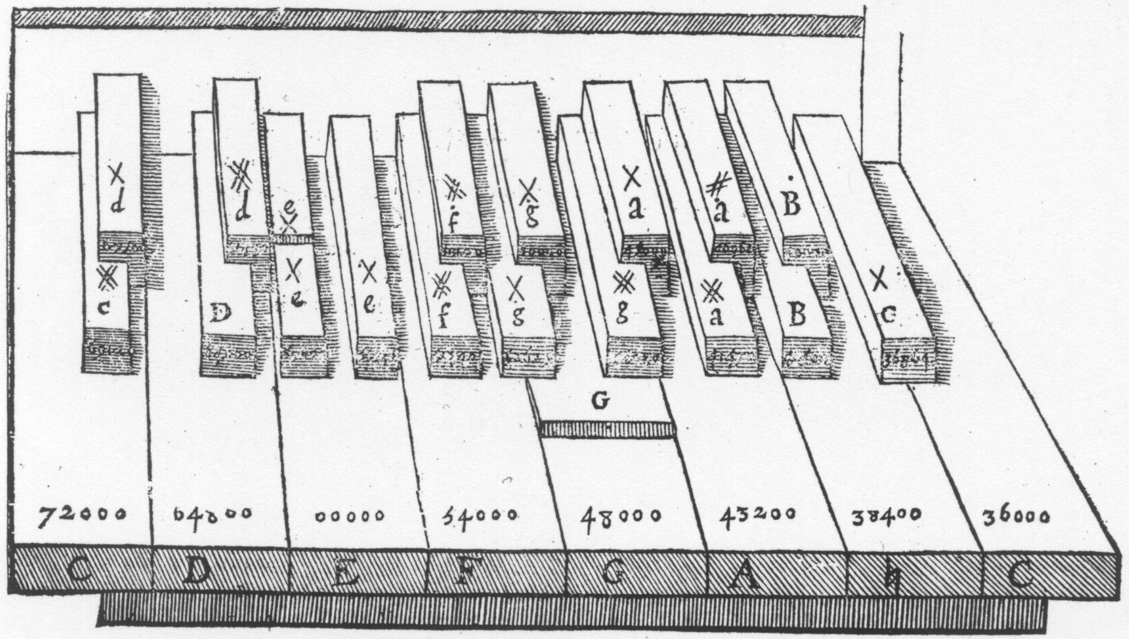
Mean-tone Temperament

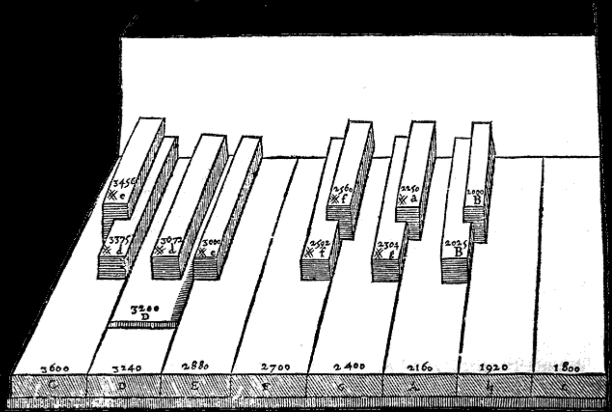
- Suppose we use mean-tone temperament to tune our keyboard instrument.
- In D major, the key between F and G needs to be F$^{\#}$, a major third above D.
- In D$^{\flat}$ major, the key between F and G needs to be G$^{\flat}$, a perfect fourth above D$^{\flat}$.
- You can't do both!



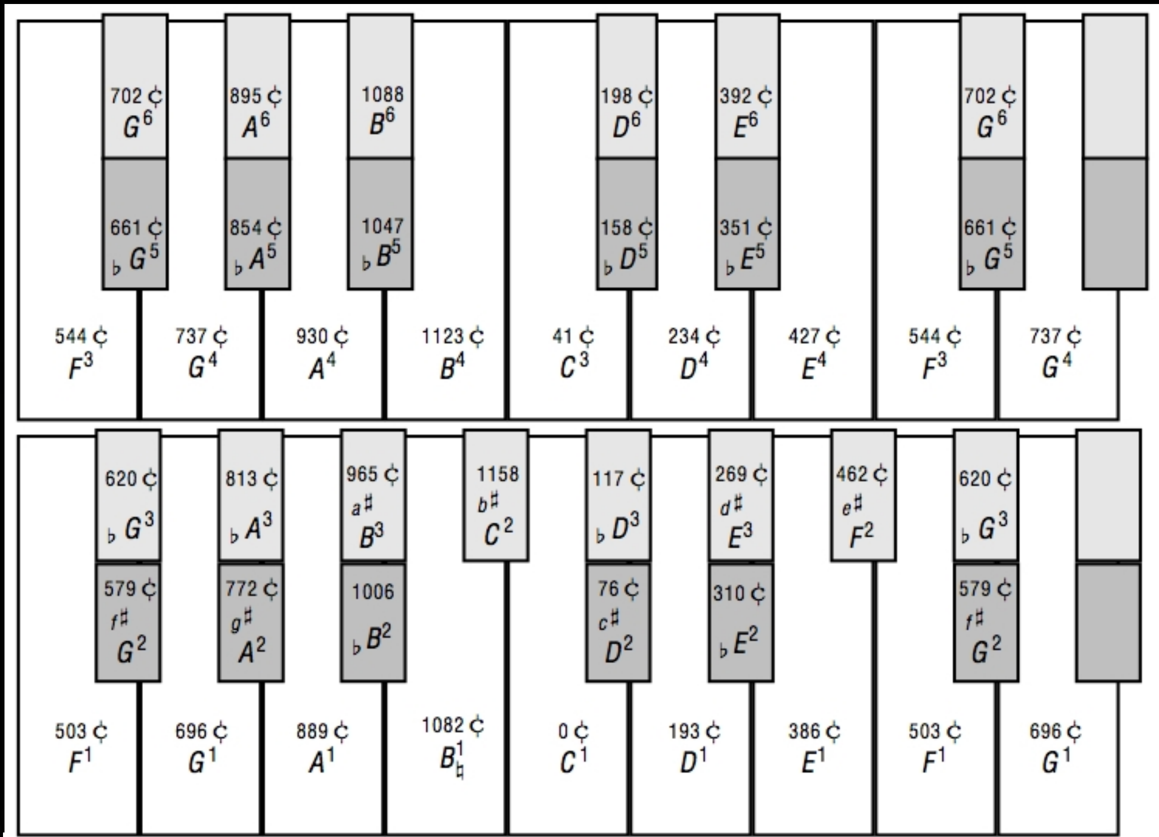


Clauier parfait de vingt-sept marches sur l'Octaue, respondant à la Table precedente.





Nicola Vicentino (1511-1575) and his Archicembalo



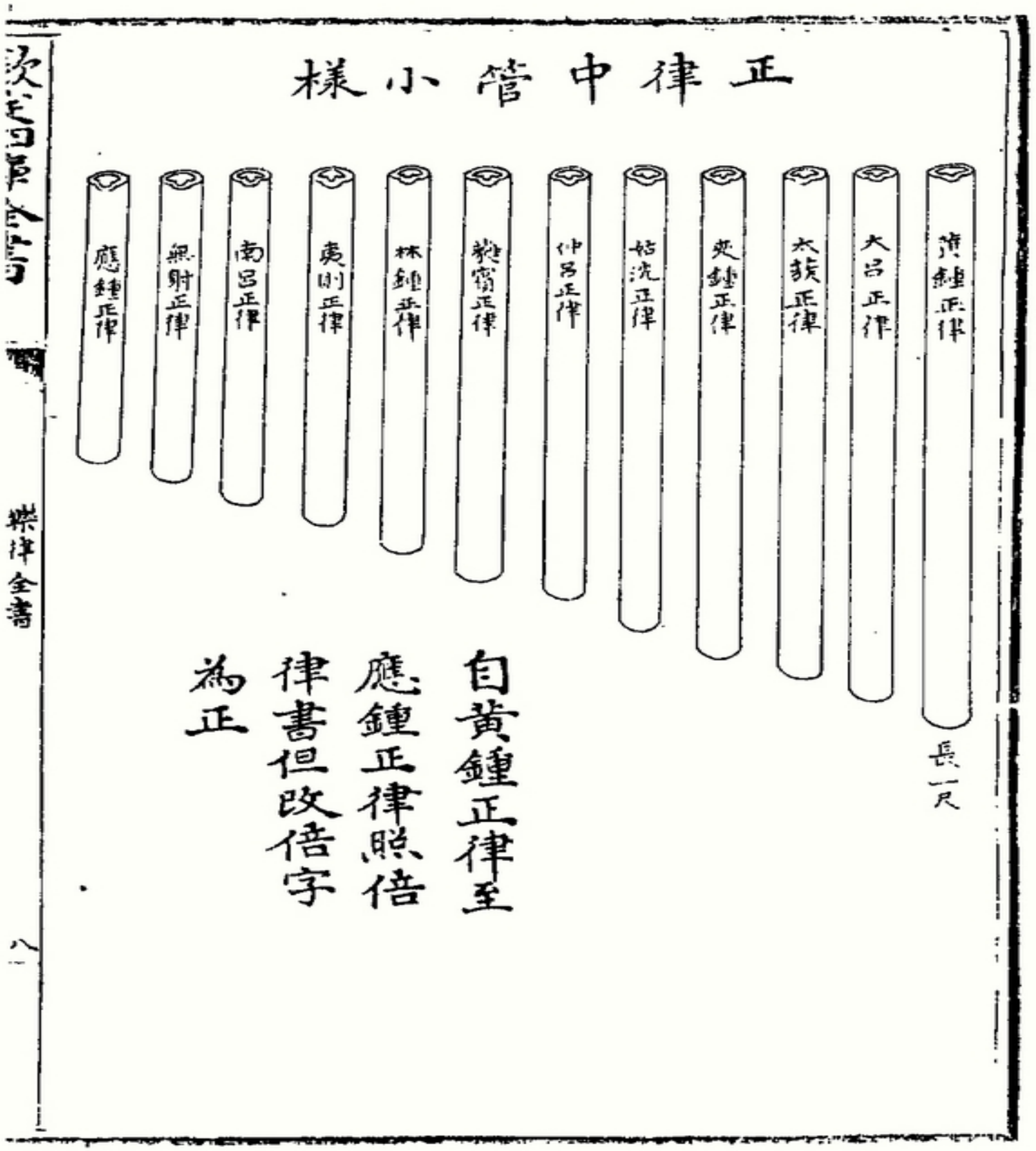
Equal Temperament

- Simon Stevin (1548-1620), a strong proponent of equal temperament.
- Make all twelve tones equally spaced.
- Psychologically challenging!
- Each semitone must be $\sqrt[12]{2}$ times the last.

Cent Values	**Third (5/4)**	**Fourth (4/3)**	**Fifth (3/2)**	**Octave (2)**
Pure	386	498	702	1200
Pythagorean	408	498	702	1200
Mean-Tone	386	488	697	1200
Equal	400	500	700	1200

The Pitch-Pipes of Chu Tsai-Yu

- Described pipes in 1584 treatise.
- $\frac{749}{500}$ for perfect 5th.
- This is 699.65¢.

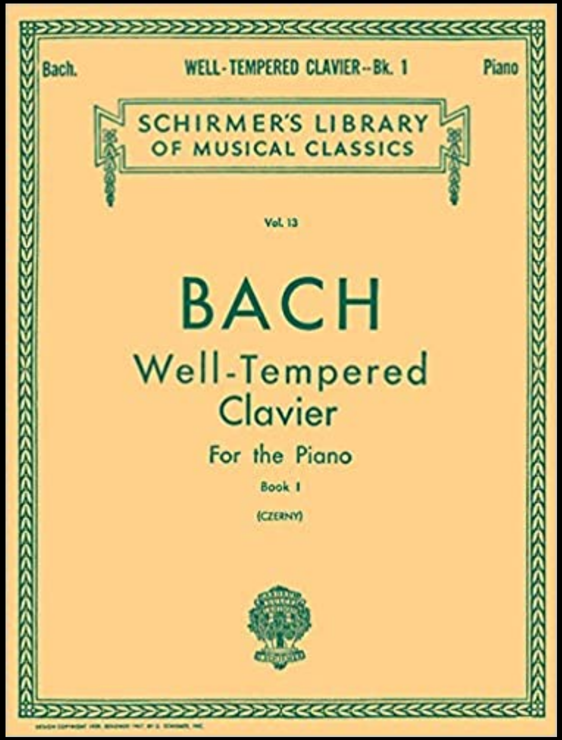


Practical Solutions

- Vincenzo Galilei (1520 – 1591)
- Used Boethius' 18:17 semitone (99¢)
- Perfect 5th is 693¢
- Strings from nut to bridge, say 1m.
- Nut to 1st fret: 1/18th total, ≈5.56cm
- 1-2nd fret: $\frac{1}{18} \times \left(1 - \frac{1}{18}\right) = \frac{17}{324}$ ≈5.25cm
- 2nd -3rd fret: ≈ 5.00cm etc

Other Solutions

- Andreas Werckmeister (1681) “Well-Temperament”
- Did Bach use it??



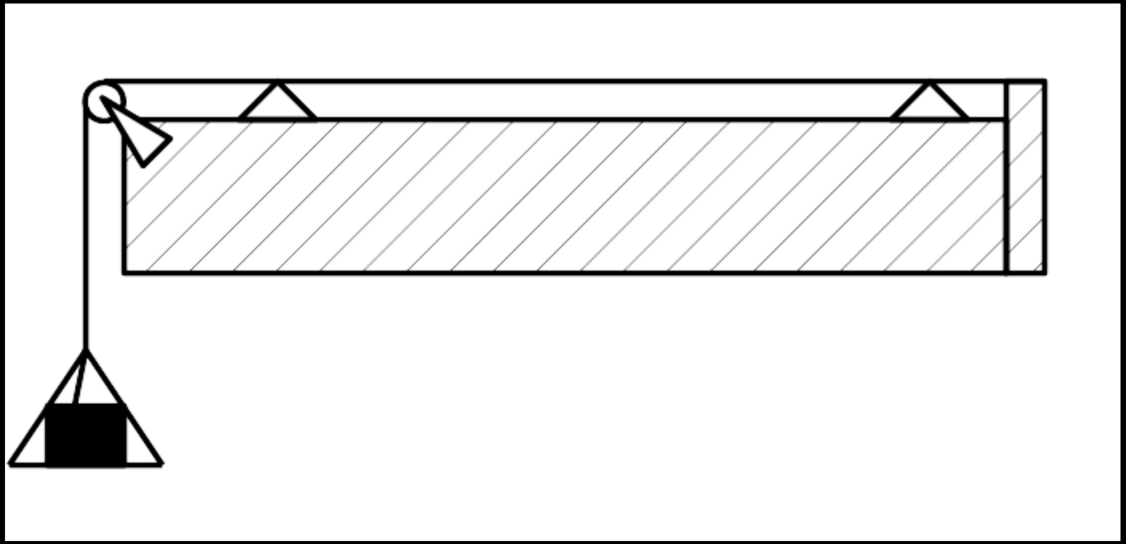
The Mathematics of Frequency

- Vincenzo Galilei discovered for vibrating strings that

$$\text{pitch} \propto \sqrt{(\text{Tension})}$$

- Galileo Galilei stated further laws.
- Mersenne's own experiments led to Mersenne's laws:

$$f = \frac{1}{2l}\sqrt{\frac{T}{\mu}}$$



- Mersenne measured frequency by doubling lengths repeatedly until he could see and count vibrations.

Why are these ratios harmonious?

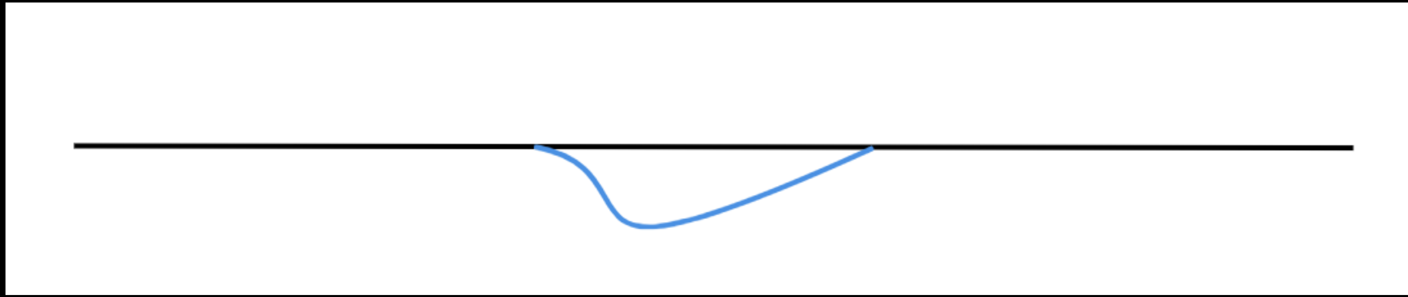
- April 1668: *“with Lord Brouncker to the King’s Head Taverne by Chancery Lane, [..] I did hear of Mr Hooke and my Lord an Account of the reason of concords and discords in musique, which they say is from the equality of vibrations, but I am not satisfied in in, but will at my leisure think of it more.”*
- *Next day: “by coach to Duck Lane, to look out for Marsanne, in French, but it is not to be had”.*
- January 1669: [I am] *“in the right way of unfolding the mystery of this matter, better than ever yet”.*

The Wave Equation

- Take a string fixed at both ends (eg a violin string).
- Disturb it at time $t = 0$. The vertical displacement y at a point x along the string depends both on x and t.

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{\mu} \times \frac{\partial^2 y}{\partial x^2}$$

- Jean-le-Rond D'Alembert (1717-1783) found a method to solve this.



Wave $A \rightarrow$ Wave $B \leftarrow$

- Solution is wave A + wave B

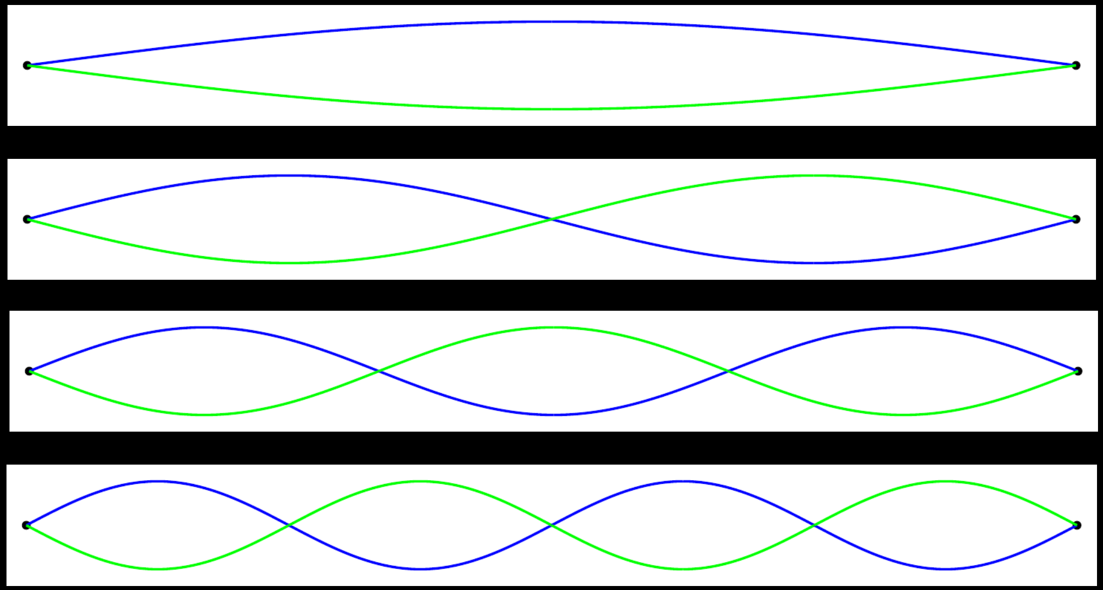
$$A(t + 2l) = A(t)$$

$$B(x) = -A(l - x)$$

- Periodic with period $2l$.

Fourier's breakthrough

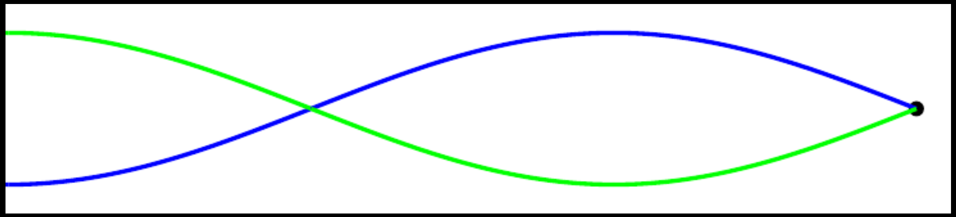
- ANY periodic function can be broken up into a combination of sine waves!
- Every solution of the wave equation for string fixed at both ends is a sum of sine waves of period $2l$ (if l is the length of the string).
- Corresponds to frequencies $f, 2f, 3f$ etc.



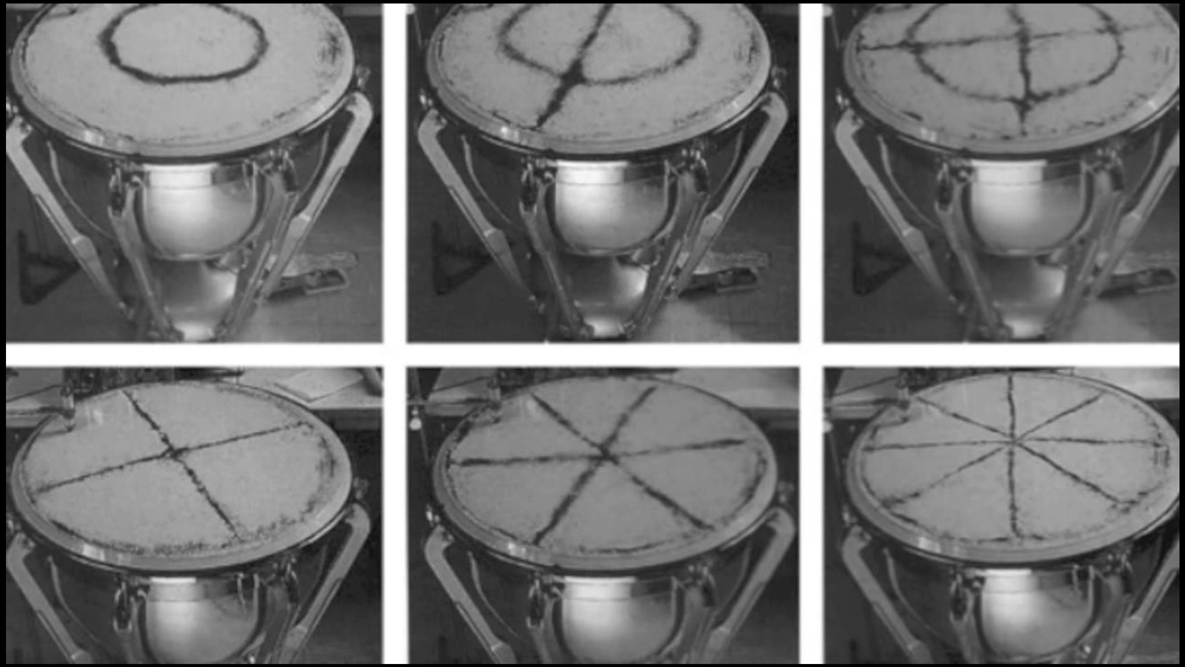
- Instruments have different combinations of these waves.
- Initial "transient sound" is also important.

Other instruments

- Flute: open at both ends so pressure there equals ambient pressure. Same solutions as string.
- Clarinet: closed at one end. Maximum pressure at closed end, favours odd multiples of fundamental frequency.

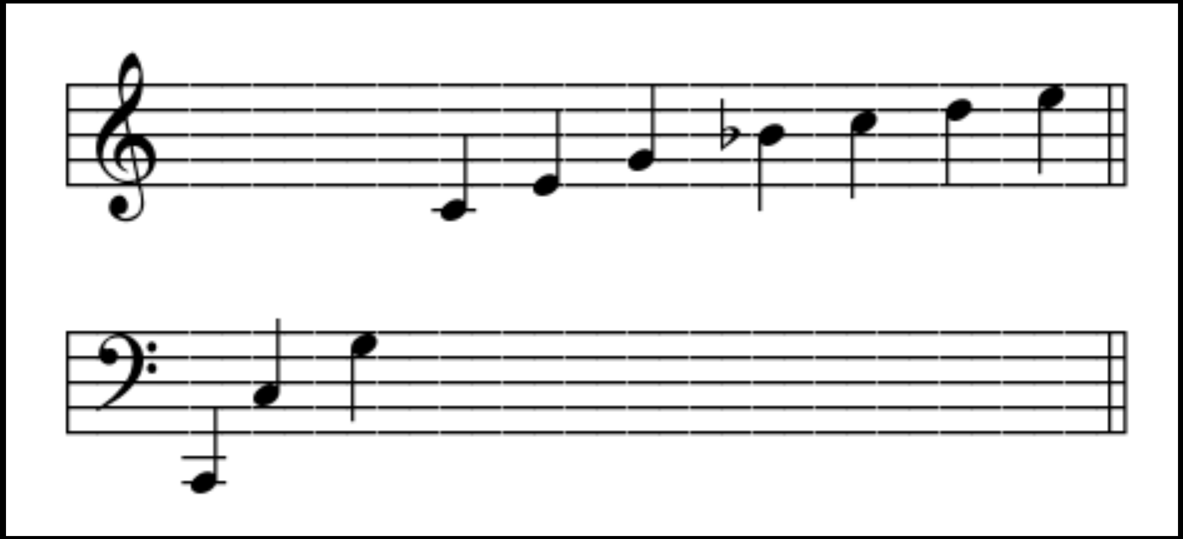


- Drums: two dimensional wave equation.



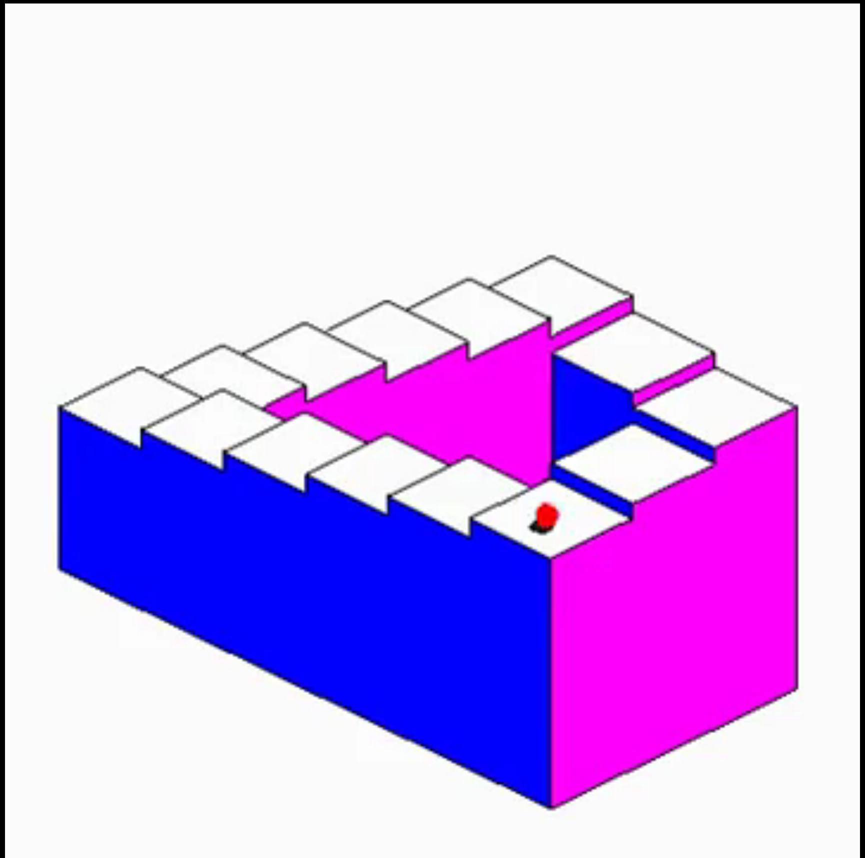
Overtones and harmonies

- A note played on a musical instrument has:
- a fundamental frequency f
- overtones, or harmonics, that are integer multiples of f.



- Harmonics of $2f$ are harmonics of f.
- $\frac{3}{2}f$ (a perfect 5th) has many harmonics in common with f.

And finally…



Video created by Howard Freeland

- Understanding how notes are made allows us to create aural illusions.
- See work of Roger Shepard and Diana Deutsch.
- Create tones out of harmonics spaced one octave apart with middle pitched ones, not lowest, being loudest.

GRESHAM COLLEGE

The Mathematics of Bell Ringing

January 5th, 1pm

Gresham.ac.uk
@greshamcollege

Birkbeck UNIVERSITY OF LONDON

BRITISH SOCIETY FOR THE HISTORY OF MATHEMATICS