GRESHAM COLLEGE

The Mathematics of Bell Ringing

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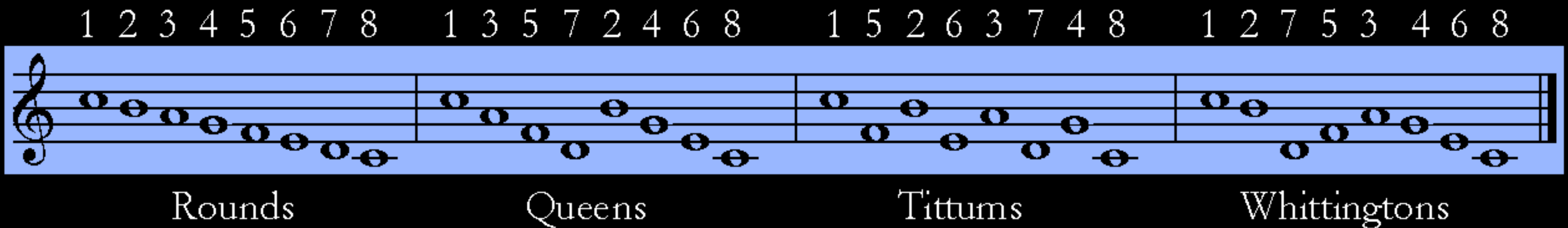
English Church bells

- Bells labelled 1 2 3 … n,
  highest (treble) to lowest (tenor).
- Ringers play *rows*.
- Each bell plays once in each row.



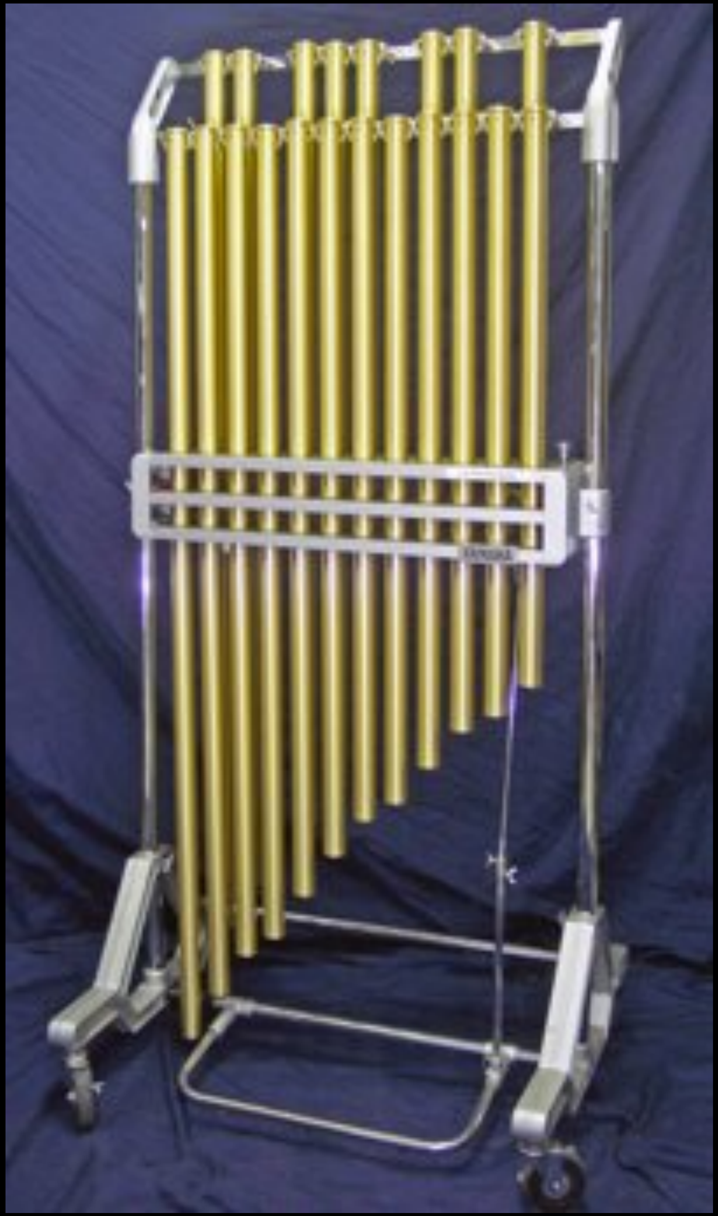
How many possible rows?

- 2 bells: 12, 21
  2 possible rows.
- 3 bells: 123, 132, 213, 231, 312, 321
  $3 \times 2 \times 1 = 6$ possible rows
- 4 bells: $4 \times 3 \times 2 \times 1 = 4! = 24$ rows
- n bells: $n! = n \times (n - 1) \times \ldots \times 2 \times 1$ rows
- An *extent* is a ring of all possible rows, each played exactly once.

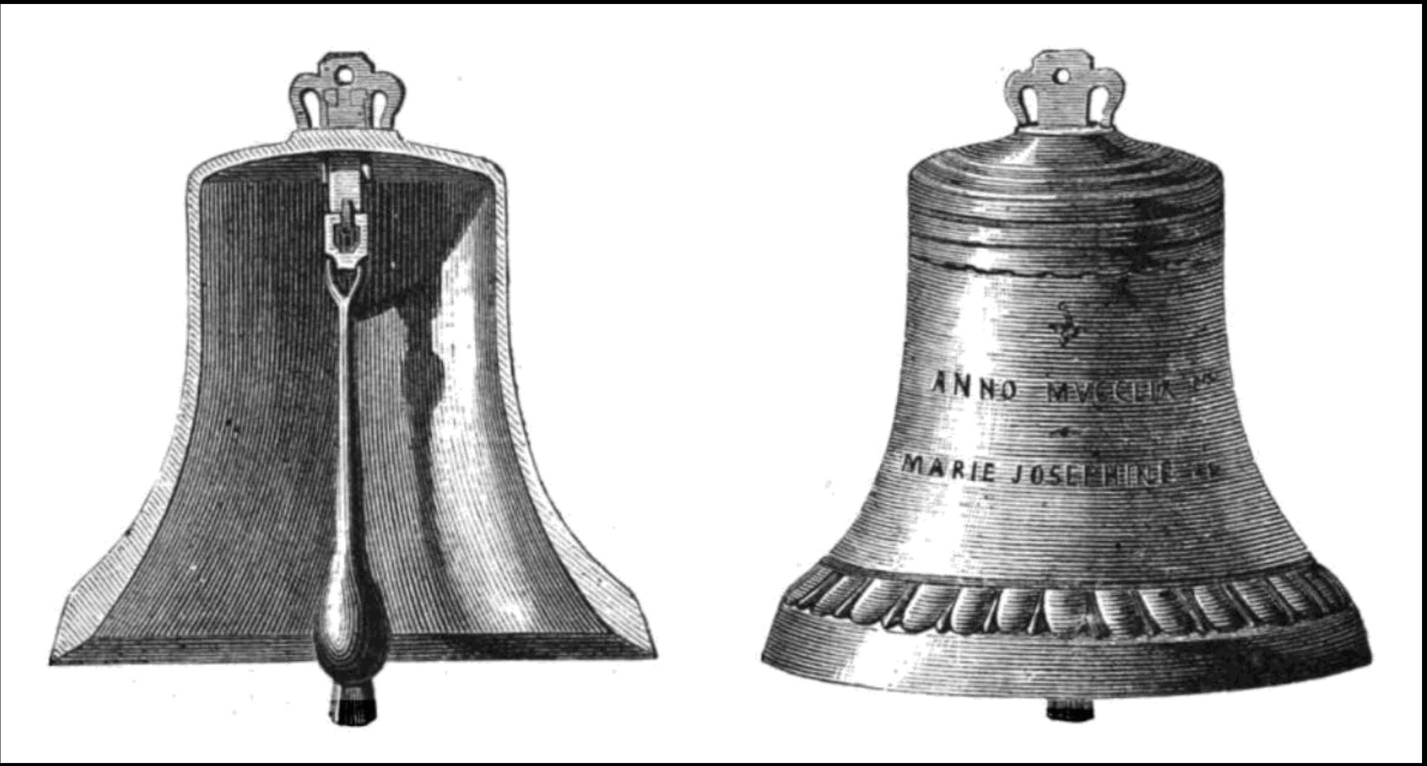
Bells	Name	Number of rows	Time to ring (2s / row)
3	Singles	6	12 seconds
4	Minimus	24	48 seconds
5	Doubles	120	4 minutes
6	Minor	720	24 minutes
7	Triples	5,040	2hrs 48 mins
8	Major	40,320	22hrs 24 mins
9	Caters	362,880	8 days 9h 36 mins
10	Royal	3,628,800	84 days
11	Cinques	39,916,800	2 years 194 days
12	Maximus	479,001,600	30 years 131 days
13	Sextuples	6,227,020,800	395 years
14	Fourteen	87,178,291,200	5525 years
15	Septuples	1,307,674,368,000	82,875 years
16	Sixteen	20,922,789,888,000	1,326,006 years
19	Nontuples	19!	≈7.7 billion years.

The shape of bells

- Must be strong enough to withstand repeated blows from clapper.
- Elastic enough to vibrate to produce resonances.
- Best material is bronze.
- Shape depends on intended use.



The modern church bell shape



- Circular cross-section
- Variation in circumference
- Wider at the bottom

Tuning Bells

Simpson Tuning controls 5 partial frequencies.

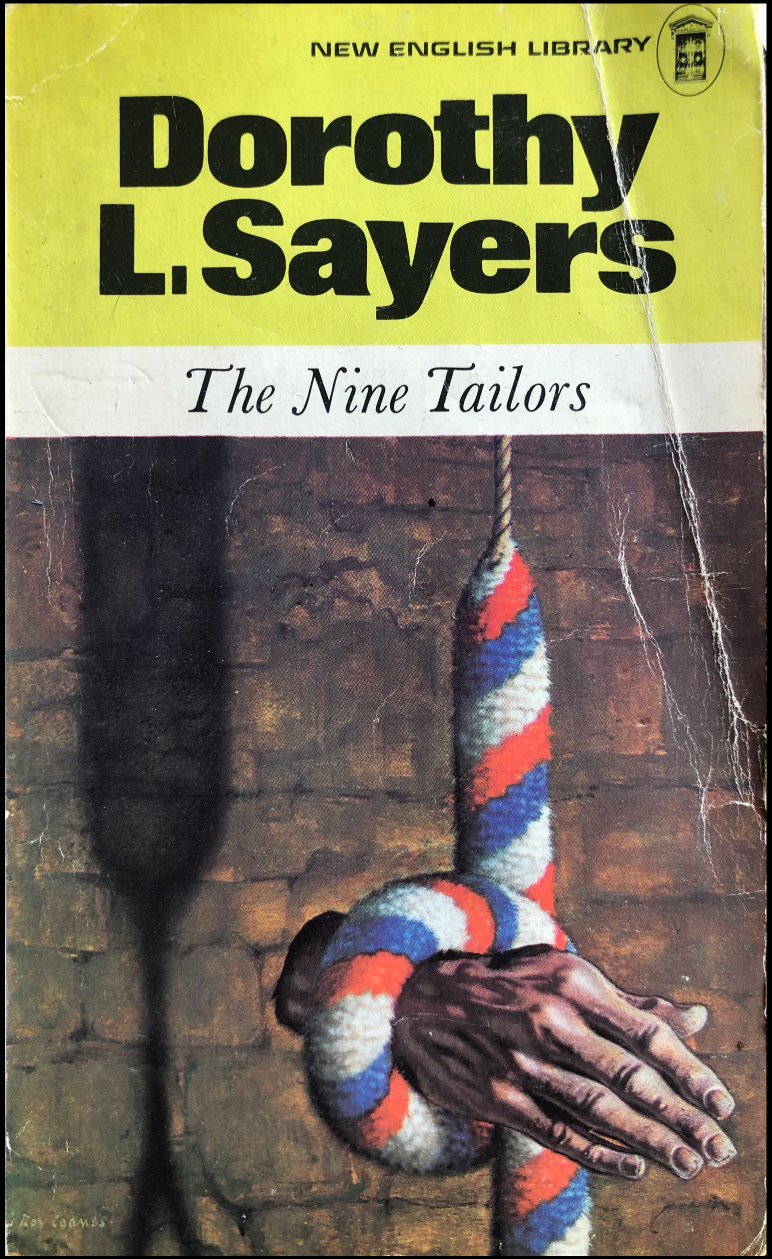
Relative to the note we hear (frequency f):

- Nominal $2f$ (an octave above)
- Tierce $\frac{6}{5}f$ (minor 3rd above)
- Quint $\frac{3}{2}f$ (perfect 5th above)
- Prime f
- Hum $\frac{1}{2}f$ (octave below)

A few records

- World's largest bell: Tsar Bell, 202 tonnes.
- Largest working bell: Bell of Good Luck, China, 116 tonnes.
- Largest bell in UK – Olympic Bell, 23 tonnes.
- Largest church bell in UK – Great Paul (St Paul's) - 16.8 tonnes.
- Bell towers have up to 16 bells; normally 8 – 12.

Full-circle Ringing



Change Ringing

"The art of change ringing is peculiar to the English, and, like most English peculiarities, unintelligible to the rest of the world. To the musical Belgian, for example, it appears that the proper thing to do with a carefully tuned ring of bells is to play a tune upon it. By the English campanologist, the playing of tunes is considered to be a childish game, only fit for foreigners; the proper use of bells is to work out mathematical permutations and combinations."

— Dorothy L. Sayers, The Nine Tailors (1934)

The rules

- Bells are numbered 1 to n.
- Bells sound in a sequence of *rows*.
- Movement between rows is a *change*.
- Each bell sounds once in each row.
- Each bell can move at most one position in each change.
- An *extent* on n bells starts and ends with rounds: $1\ 2 \cdots n$. Every possible row played exactly once (apart from final rounds).
- A *peal*: over 5000 different rows (start and end with rounds).

Example on 4 bells

- 1234 (rounds)
- 2134 (a plain change)
- 1243 (a cross peal)
- 1234 (back to rounds)

Example of illegal move:

$1234 \rightarrow 4321$

Extents on 3 bells

Is an extent on n bells always possible?

A	B	C	
1	2	3	
			(AB)
2	1	3	
			(BC)
2	3	1	
			(AB)
3	2	1	
			(BC)
3	1	2	
			(AB)
1	3	2	
			(BC)
1	2	3	

A	B	C	
1	2	3	
			(BC)
1	3	2	
			(AB)
3	1	2	
			(BC)
3	2	1	
			(AB)
2	3	1	
			(BC)
2	1	3	
			(AB)
1	2	3	

More bells – plain hunting

- Method for $n = 2k$ bells (or $2k + 1$ with tenor covering)

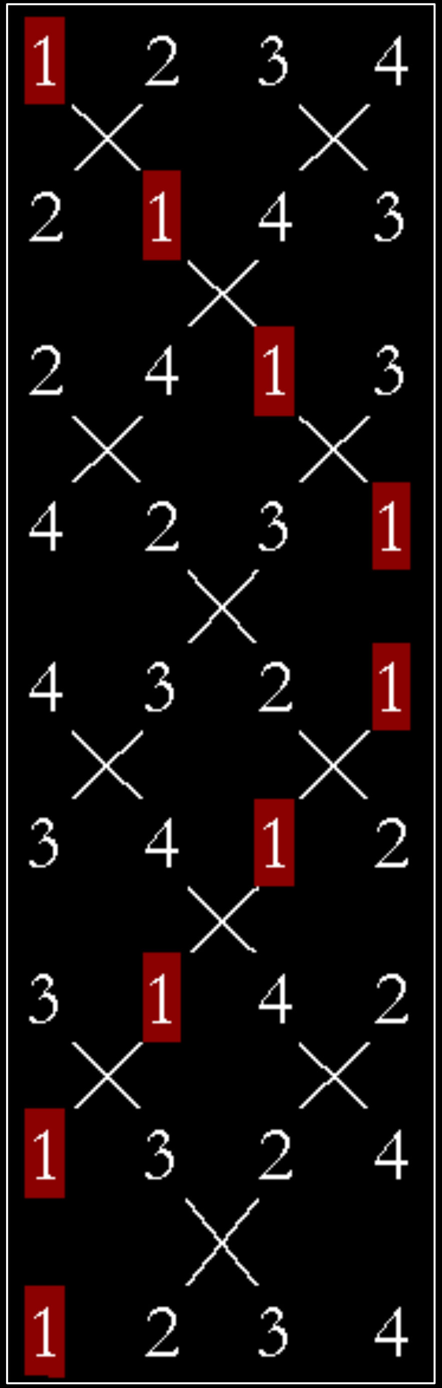
- r : k swaps
- s : fix bell 1 and n, do $k - 1$ swaps.

On 6 bells: $r = (AB)(CD)(EF)$

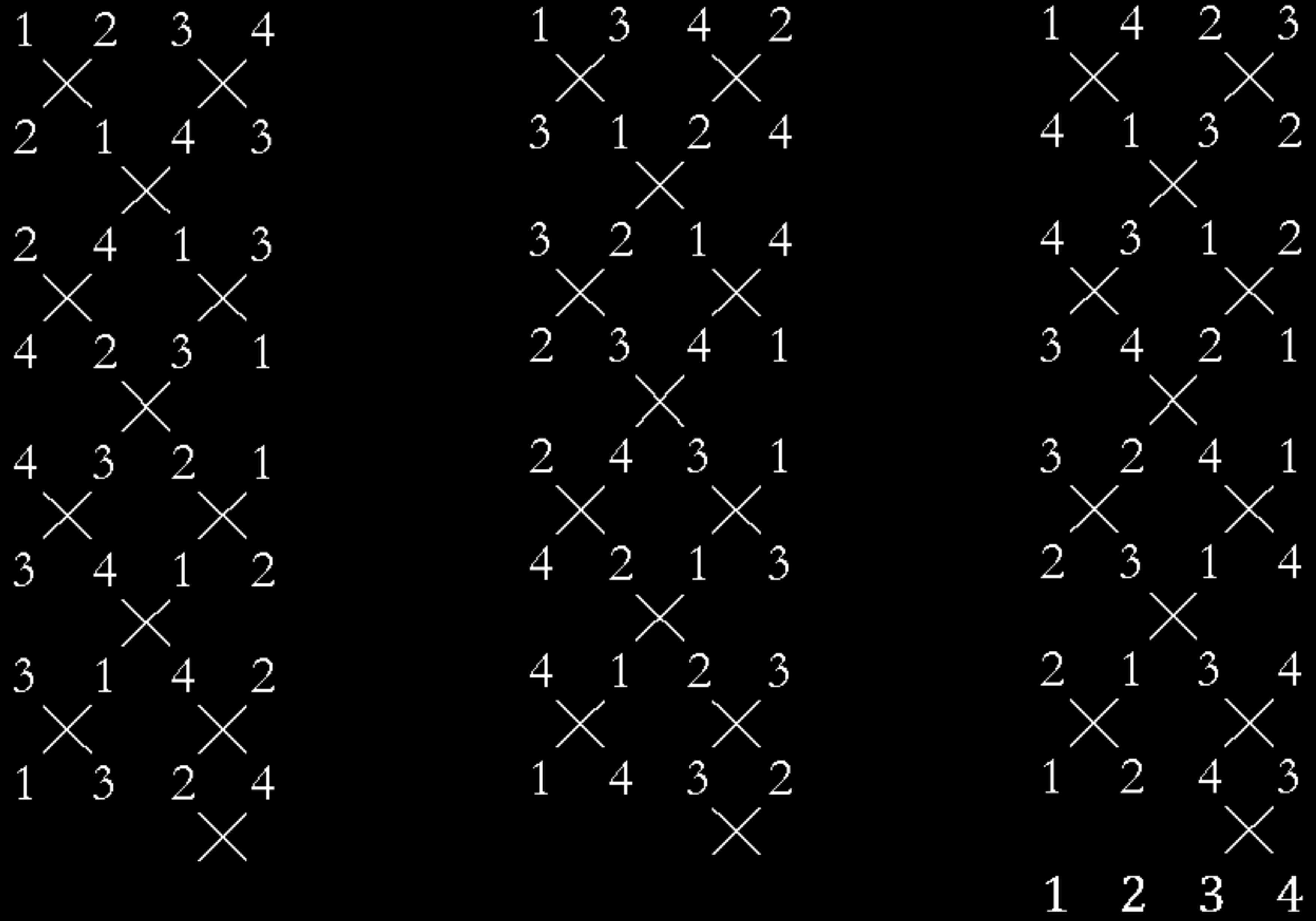
$s = (BC)(DE)$

On 4 bells: $r = (AB)(CD)$

$s = (BC)$



Plain Bob Minimus – extent on 4 bells.



Change ringing Methods

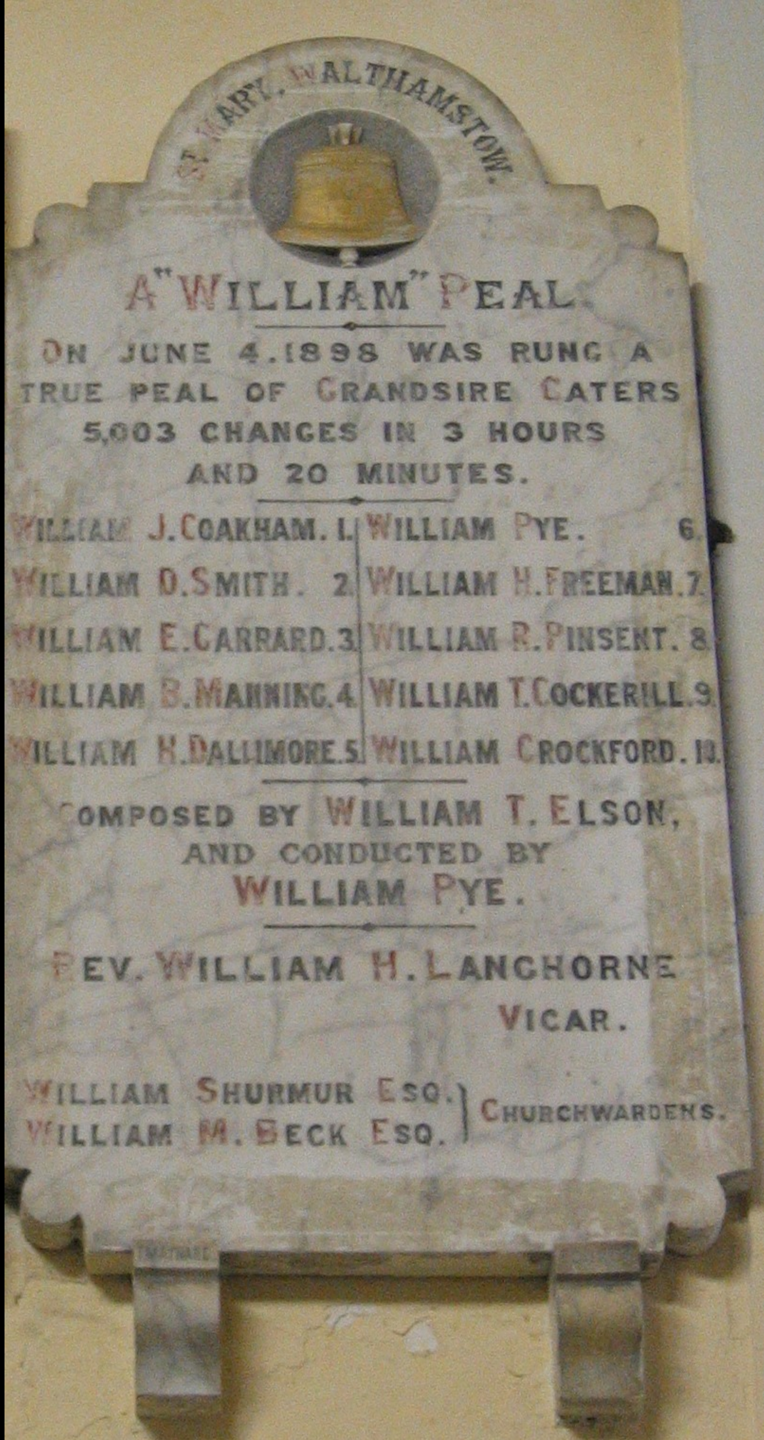
- Ringers follow the set pattern; a call tells ringers what to do to switch to the next method.
- 3, 5, 7, 9, 11 bells are: Singles, Doubles, Triples, Caters, Cinques
- 4, 6, 8, 10, 12 are: Minimus, Minor, Major, Royal, Maximus

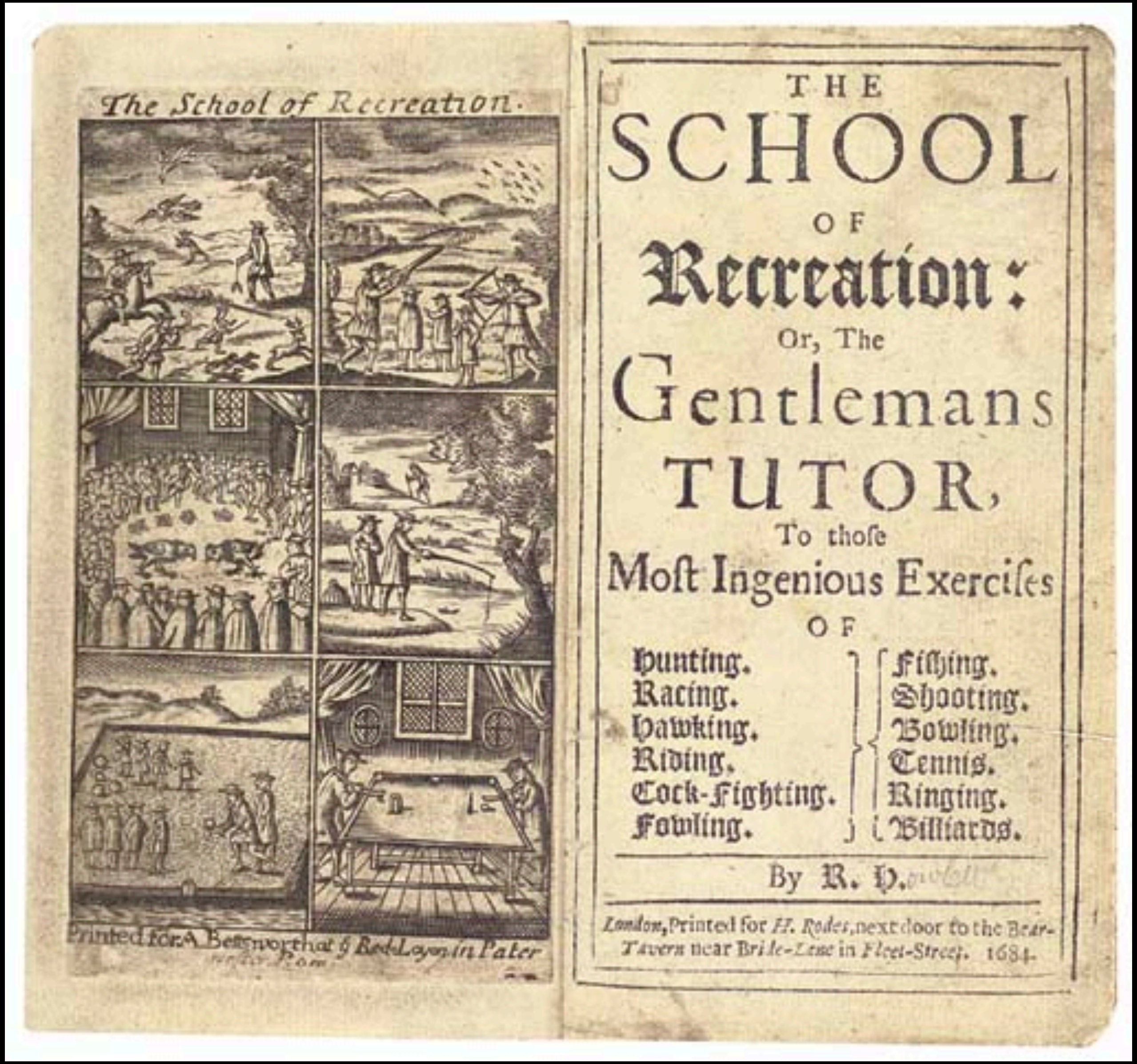
Examples

- Plain Bob Minimus
- Stedman Doubles
- Grandsire Triples
- Primrose Surprise Major
- Avon Delight Maximus

Mathematical Questions

- Is an extent possible on $5, 6, 7, \ldots n$ bells?
- Can we design "interesting" peals – no bell in the same place for more than two rows.
- Can we design memorable methods?
- How do we *prove* we have a true extent/peal?




The School of Recreation.

THE
SCHOOL
OF
Recreation:
Or, The
Gentlemans
TUTOR,
To thoſe
Moſt Ingenious Exerciſes
OF

Hunting.	Fiſhing.
Racing.	Shooting.
Hawking.	Bowling.
Riding.	Tennis.
Cock-Fighting.	Ringing.
Fowling.	Billiards.

By R. H.

London, Printed for H. Rodes, next door to the Bear-Tavern near Bride-Lane in Fleet-Street. 1684.

The 17th century

- Ringing a popular sport - see *The School of Recreation* (1684)
- Many clubs, societies, rival parish teams
  - The Scholars of Cheapside (1603)
  - Society of Colledg [sic] Youths (1637)
  - Society of Cumberland Youths (1747)
- Extents on 4, 5 and 6 bells all achieved.

Fabian Stedman (1640-1713)

- London printer and bell-ringer.
- Arranged printing of Richard Duckworth's *Tintinnalogia – or, the Art of Change Ringing* (1668)
- Wrote *Campanalogia – or the Art of Ringing Improved* (1677)
- Contains 53 "London Peals" invented by Stedman, including "Stedman Doubles".
- Very clear explanations of why the peals are "true".

Stedman plaque at St Andrews Undershaft, London

Groups

- A group is a set of objects along with a way of combining them such that the combination of any two objects is another object in the set *(closure)*, subject to 3 rules.

Positive real numbers, with $\times$

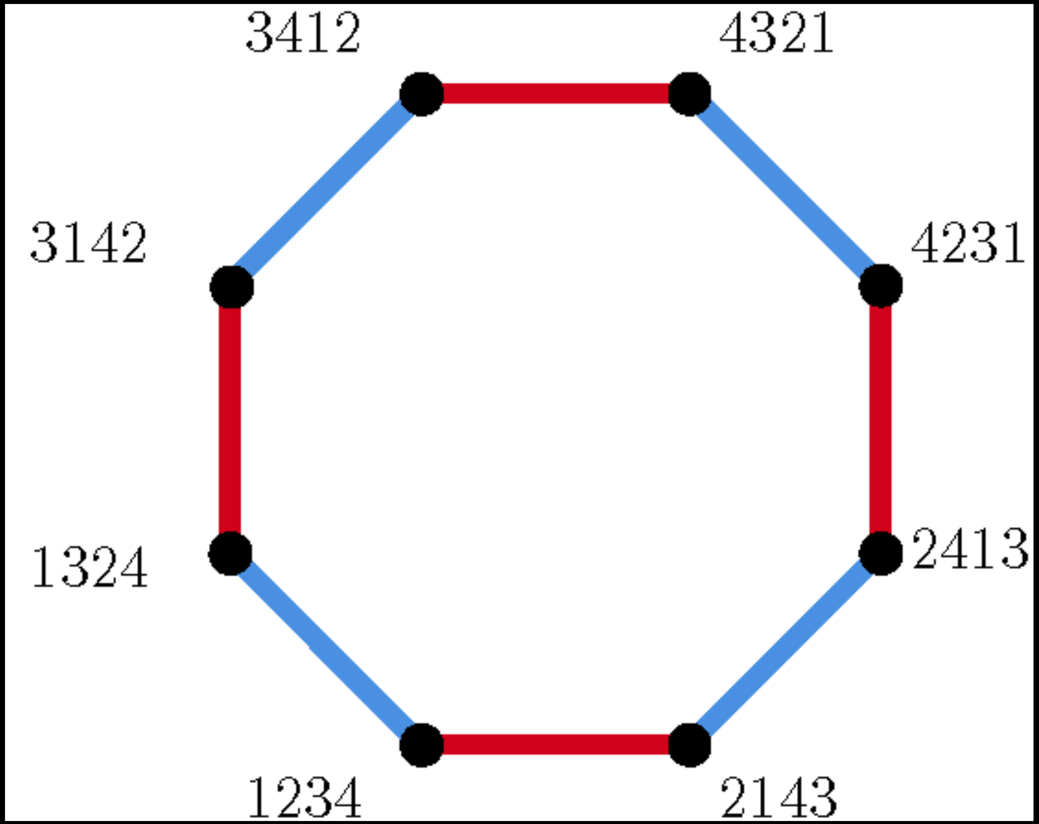
- *Closure*: $a \times b > 0$
- *Identity*: $a \times 1 = 1 \times a = a$
- *Inverse* of a is $a^{-1} = \frac{1}{a}$
- *Associative*: $(a \times b) \times c = a \times (b \times c)$

- S_n: the set of permutations of n objects in positions $A, B, C, \ldots$
- $(AB)(CD)$ sends $12345 \cdots n$ to $21435 \cdots n$
- Can check the rules hold.
- Permutations correspond to rows. There are $n!$ permutations.
- S_4 has 24 elements.

The hunting group

- Not all permutations of S_n are legal changes, but they all can be obtained via a sequence of legal changes.
- H is a *subgroup* of order 8 in S_4.
- Composing an element x in turn with everything in a subgroup H, gives a "coset", xH.

- Recall $\boldsymbol{r} = (\boldsymbol{AB})(\boldsymbol{CD})$, $\boldsymbol{s} = (\boldsymbol{BC})$.
- H = everything we can make from $\boldsymbol{r}$ and $\boldsymbol{s}$.

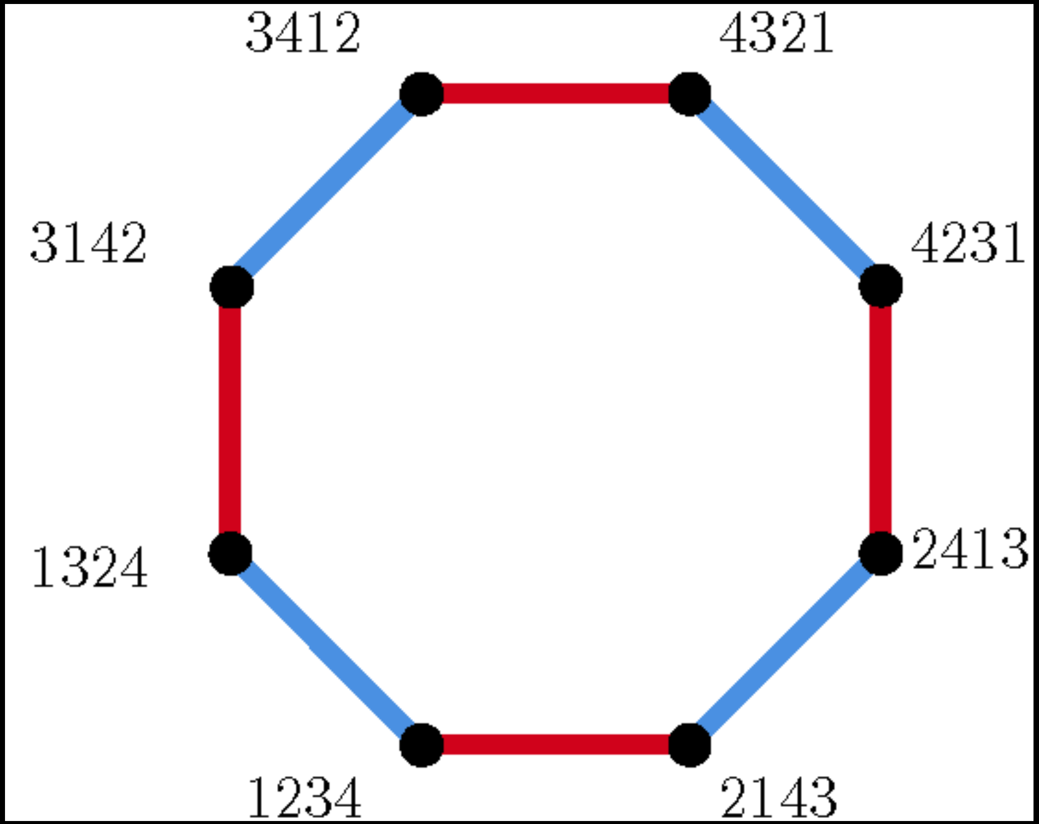


The hunting group

Let $t = (CD)$.

1234	$= e$	1342	$=(st)e$
2143	$= r$	3124	$=(st)r$
2413	$= rs$	3214	$=(st)rs$
4231		2341	
4321		2431	
3412	$\vdots$	4213	$\vdots$
3142		4123	
1324	$= s$	1432	$=(st)s$

- Recall $\boldsymbol{r} = (\boldsymbol{AB})(\boldsymbol{CD})$, $\boldsymbol{s} = (\boldsymbol{BC})$.
- H = everything we can make from $\boldsymbol{r}$ and $\boldsymbol{s}$.



Plain Bob is a Compilation of Cosets!

1234	$= e$	1342	$=(st)e$	1423	$=(stst)e$
2143	$= r$	3124	$=(st)r$	4132	$=(st)^2r$
2413	$= rs$	3214	$=(st)rs$	4312	$=(st)^2rs$
4231		2341		3421	
4321		2431		3241	
3412	$\vdots$	4213	$\vdots$	2314	$\vdots$
3142		4123		2134	
1324	$= s$	1432	$=(st)s$	1243	$=(st)^2s$
			and finish with	1234	$=(st)^3$

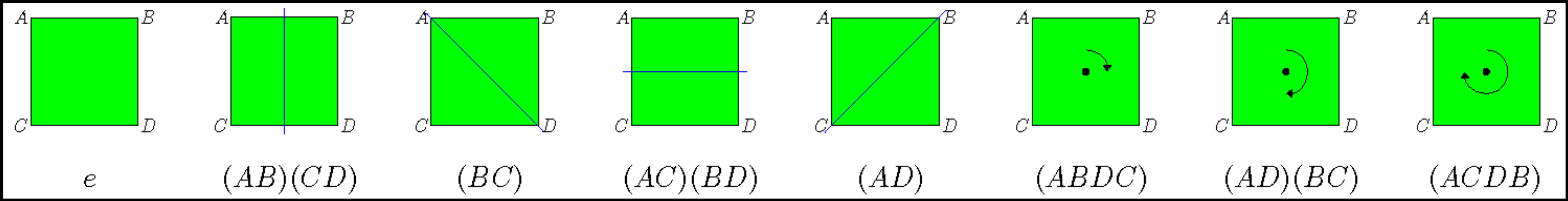
S_4 is the union of:

H

$(st)H$

$(st)^2H$

The hunting group in another context



- Hunting group on four bells "isomorphic to" group of symmetries of the square, with $r = (AB)(CD)$, $s = (BC)$.
- True for n bells and regular n-gons (n even).

Cosets and truth

- Different cosets are disjoint.
- For a "true" performance, proof reduced to checking first entry in each column.

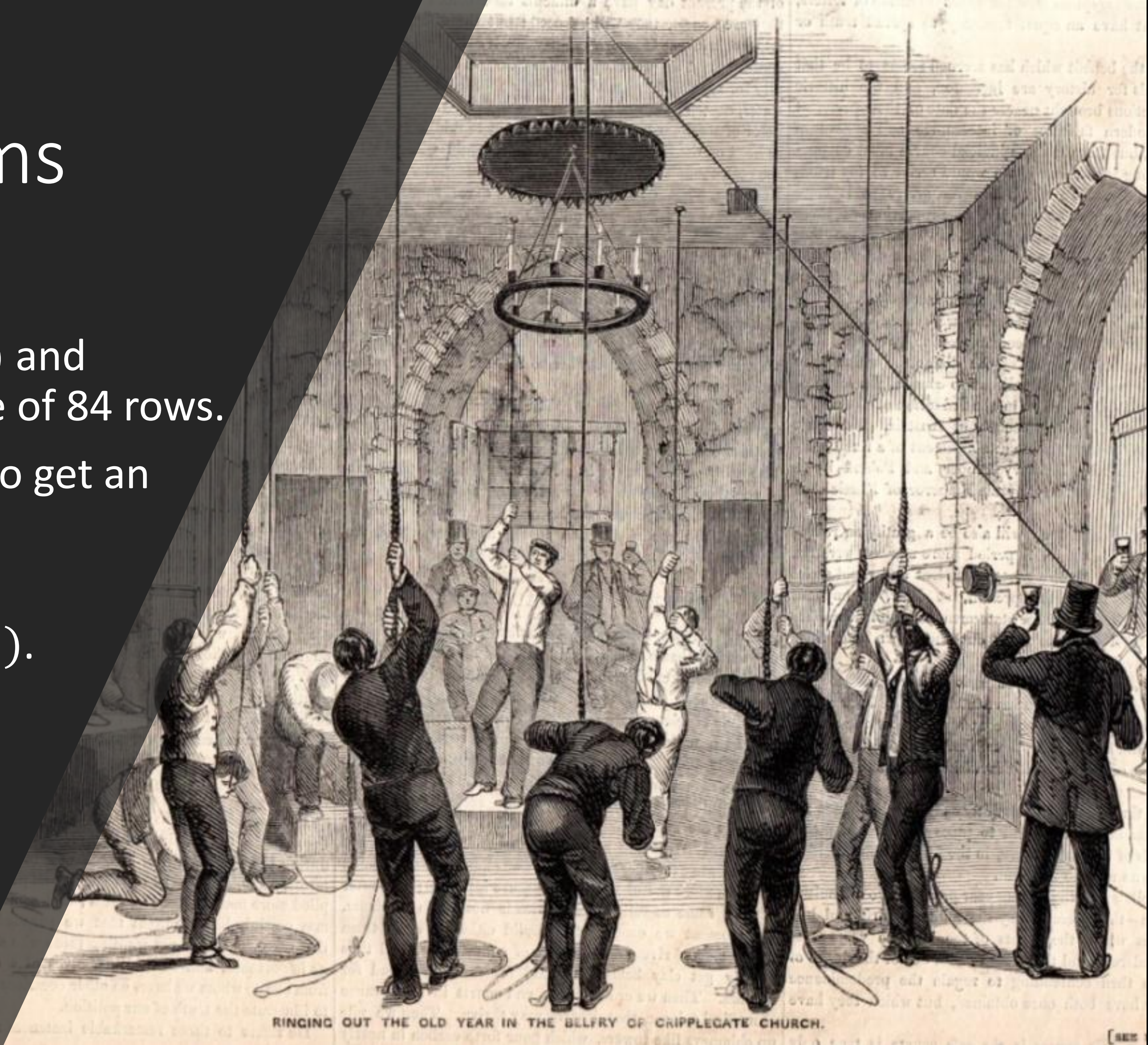
Five bells (120 rows)

Stedman doubles

- Double swaps $(AB)(DE), (BC)(DE)$, $(AB)(CD)$, combine to produce "plain course" of 60.
- This is a subgroup of S_5 called A_5, the "even" permutations.
- After plain course do eg (AB), then repeat plain course: an extent!

Appealing problems

- Stedman Triples on 7 bells.
- $(AB)(DE)(FG)$, $(BC)(DE)(FG)$ and $(AB)(CD)(EF)$ give plain course of 84 rows.
- Need 60 different cosets of this to get an extent of 5,040 rows.
- Can be done using combination of $(AB)(CD)(FG)$ and $(AB)(CD)$. But $(AB)(CD)$ less desirable.
- Stedman conjectured it could be done only with triple swaps.
- Only solved in 1994!



Where next?

- Computers used since the 1950s, but human involvement still needed!
- For those without a handy belfry, try handbells.
- In 2007, Philip Earis, Andrew Tibbetts and David Pipe, of the Ancient Society of College Youths, rang 100 different extents on six handbells.
- They rang for 24 hrs, 9 minutes.


Thank you!

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Mathematical Journeys into Fictional Worlds

February 9^{th}, 1pm

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