**The Mathematics of Bell Ringing**
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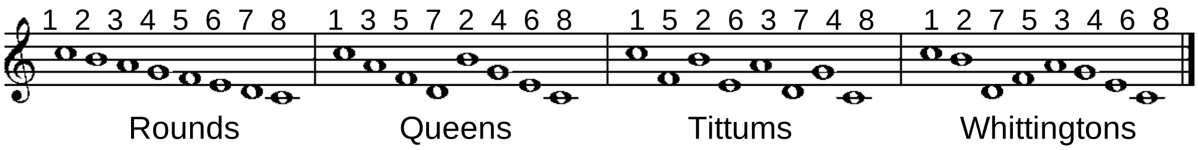
This lecture will look at change ringing, which is ringing a series of tuned bells (as you might find in the bell tower of a church) in a particular sequence, and this has exciting mathematical properties. Along the way we will discuss how bells are made and tuned and give brief highlights from the history of bell ringing.

## Introduction

Bells are the loudest, and one of the oldest, instruments. They are present in cultures across the world. A tolling bell can call people to prayer, warn of emergency, or ring out in celebration. The sounds of bells punctuate our lives, from the handbell in the playground to the chimes of Big Ben. They can mark the end of life too. Ask not for whom the bell tolls, says the poet; it tolls for thee. Today I want to talk about a particularly British (and actually mainly English) thing – the style of bell ringing known as change ringing. It arose here for a variety of reasons which we'll discuss along the way. We'll begin with a question. Suppose you have a set of eight bells (or more properly a *ring*, which is the collective noun) in your church tower, and you want to use them to make a pleasing sound. You could, for example, ring down the notes of the scale repeatedly. Usually, bells in a ring of $n$ bells are numbered highest (treble) to lowest (tenor), 1 to $n$. So, our eight bells rung in order from highest to lowest would be 12345678. This is called *rounds.* After a while that might get a bit boring, so perhaps you want to vary it a bit – maybe you could play something like *Queens, Tittums* or *Whittingtons* (shown).




Because the bells are very heavy objects swinging backwards and forwards, it is very difficult to stop and start them, so our first rule of engagement is that each of the bells must sound in every 'row' of playing. There are other rules, but we will worry about them later. So, the first question is: how many possible rows are there?

The way to answer this is to build up gradually. With just two bells, our possible rows are 12 and 21. With three bells, the first bell can be any of 1, 2 or 3, and then the others can be rung in the two ways we counted for two bells. We get 123, 132, 213, 231, 312, 321. This is six ways, because it is $3 \times 2 \times 1$. With four bells, we assign the bell that goes first, in one of four ways, and then the three remaining bells can be played in six possible orders. So, the number of possible rows, or permutations, of four bells is $4 \times 3 \times 2 \times 1 = 24$. We can see that this idea works in general. The number of possible permutations of $n$ things, in this case the number of possible rows of our n bells, is $n \times (n-1) \times \cdots \times 3 \times 2 \times 1$. This number is known as "$n$ factorial", and written $n!$.

A series of rows consisting of the set of all possible permutations, each played once, is called an *extent*. The table over the page shows the number of rows required to play a full extent on $n$ bells, for increasing values of $n$. The names are the official designations for rings of bells with that number of bells. We'll explain those names later.

| Bells | Name | Number of rows | Time to ring (2 s/row) |
|---|---|---|---|
| 3 | Singles | 6 | 12 seconds |
| 4 | Minimus | 24 | 48 seconds |
| 5 | Doubles | 120 | 4 minutes |
| 6 | Minor | 720 | 24 minutes |
| 7 | Triples | 5040 | 2hrs 48 mins |
| 8 | Major | 40,320 | 22hrs 24 mins |
| 9 | Caters | 362,880 | 8 days 9h 36 mins |
| 10 | Royal | 3,628,800 | 84 days |
| 11 | Cinques | 39,916,800 | 2 years 194 days |
| 12 | Maximus | 479,001,600 | 30 years 131 days |
| 13 | Sextuples | 6,227,020,800 | 395 years |
| 14 | Fourteen | 87,178,291,200 | 5525 years |
| 15 | Septuples | 1,307,674,368,000 | 82,875 years |
| 16 | Sixteen | 20,922,789,888,000 | 1,326,006 years |
| 17 | Octuples | 17! | ≈22.5 million years |
| 18 | Eighteen | 18! | ≈ 406 million years |
| 19 | Nontuples | 19! | ≈7.7 billion years. |

It has been estimated that the Sun will engulf the Earth in about 7.5 billion years, after the Sun has entered its red giant phase and expanded to a diameter larger than the Earth's orbit. So, if we start right now and play really fast, we might just be able to do an extent of 19 bells.

It's all very well talking about playing exact rows in a precise order, but the physical set-up of bells is critical to our ability to actually do this, and even in the best-case scenario, we don't have entirely free choice about the order in which we play the rows. To shed some light on this, we need to know a little more about the development of bells and bellringing.

## The Shape of Bells

If you bang a piece of metal, you get a nice loud sound. This is the first step in the evolution of bells – and of course this can develop in a different direction towards gongs and similar instruments. After a while, people worked out that if you bend the flat sheet over, and maybe even join the sides together, you get a very resonant, powerful sound. The next development is to cast metal to form a bell. This was being done in China at least 4,000 years ago. The shape varied but the bells didn't have the 'lip' of a modern European bell. It took longer to start casting bells in Europe, but there is evidence of cast bells from around the fourth or fifth century AD.

Bells must be strong enough to bear their own weight, to be struck with hammer or clapper from the outside or inside, but also elastic enough to be able to vibrate to produce the resonances that give it a full sound. The best material to use is bronze – copper and tin on their own are too soft and would not be able to withstand repeated blows from the clapper.

The context for the use of a bell influences its shape. If bells must sound with other bells to make melodious harmonies, then it is important to be able to design it to play a particular pitch. This precision is hard to achieve and therefore costly. The shape of a cow bell, which just needs to make any old sound for as little expenditure as possible, is thus different from something like the tubular bells used in orchestras. The circular cross-section, both of tubular and standard 'bell-shaped' bells, is likely to have evolved both for strength and because this is one of the shapes that allow most easily for sound waves to travel smoothly around the perimeter of the bell. At a given circumference, particular frequencies and their overtones – multiples of the fundamental frequency – will resonate as standing waves form. The circular cross-section is also something that comes out of the way bell moulds are produced – by rotating a flat shape around in a circle.

There seem to be two main reasons why the bell shape – getting wider at the bottom – has developed. Firstly, the variation in the circumference means that different parts of the bell resonate at different frequencies, which produces a much richer and more complex sound. The shape has evolved over time so that these different frequencies sound harmonious together – a bell is in essence producing a musical chord all on its own. The second reason is to do with the way sound travels. Sound is produced by vibration; for a loud sound we want the sound waves produced by a given musical instrument to get out of the instrument and into the air outside as efficiently as possible. The bell shape is the one that allows the standing waves produced inside the instrument to be transferred to the air by matching the so-called acoustic impedance of the surrounding air. We see exactly the same shape in the brass section of the orchestra, as well as in devices like megaphones, for the same reason. Bells hang up high in the church steeple, so the shape is perfect for spreading the sound as far as possible.

A parish church bell tower might have between five and twelve bells, with eight being fairly typical. The heaviest peal is at Liverpool (Anglican) Cathedral. Thirteen bells, with a combined weight of 16.8 tonnes, are arranged in a circular formation around 'Great George' (too heavy to be part of the peal). The treble weighs 510kg; the tenor weighs 4.1 tonnes. Great George, at 14.7 tonnes, is the second heaviest church bell in Britain, after 'Great Paul', of St. Paul's Cathedral in London (16.8 tonnes). However, the heaviest bell in Britain has, since 2012, been the Olympic Bell, which weighs 22.91 tonnes, and currently resides at Queen Elizabeth Olympic Park. Sadly, it is never rung. The heaviest bell in the world is currently the Tsar Bell, on display at the Kremlin in Russia – it weighs in at 202 tonnes! However, it has never rung and was broken in a fire centuries ago. It would perhaps be more accurate to describe it as the world's heaviest bell-shaped object. The heaviest bell that can actually be rung is the *Bell of Good Luck*, at Foquan Temple, Henan, China, weighing in at 116 tonnes. If you visit you can pay to ring it.

Dove's Guide for church bell ringers gives only five rings having more than twelve bells, with the highest number being sixteen; these rings of sixteen bells can be found in Birmingham, Dublin and Perth (Australia, not Scotland). Even a ring of sixteen bells cannot compete with a carillon. A carillon is a large set of stationary bells, played via a keyboard – to qualify there must be at least 23 bells (anything less is a *chime*), but often there are many more, and their combined weight can approach 100 tonnes. The carillon at the Palace of Mafra in Portugal has 120 bells!

## <u>Making and Tuning Bells</u>

Making a bell in a foundry is a lengthy process, as both the inner and outer profile of the bell must be just right. To do this, a mould is made in two stages. First, a roughly bell-shaped brick core is built on a base of metal. This is covered with loam – a mixture of clay, sand, straw, horse manure and animal hair – that is formed by hand into the required shape for the inside of the bell. The shape of the outside of the bell is similarly formed from loam, packed into an iron shell called a cope. The straw and animal hair helps to strengthen the loam mixture, but also, by burning away during heating, leaves tiny pathways for air to escape, which improves the cooling process. The moulds are finished using a 'strickle', which is a board cut accurately into the required bell shape and rotated by machine to ensure an accurate and symmetrical finish. Any decoration for the outside of the bell can then be added, by making appropriate indentations in the outer mould. The whole thing is then heated, dried, and coated in graphite. Finally, the inner and outer parts are joined together, and, in the case of large bells, buried in a pit to support the structure under the weight of all that bronze. Finally, the molten bronze is poured in, and the bell is left to cool, usually for several days. At the peak of bellringing, there were 60 bell foundries, but for most of the last century there were only two – the Whitechapel Bell Foundry, and John Taylor in Loughborough. Sadly, the Whitechapel Bell Foundry, which was England's oldest manufacturing business, dating from 1570, closed recently, leaving just John Taylor Bell Foundry.

A lecture by W. W. Starmer given at the Royal Musical Association on December 10th, 1901, states the rules for proportions, where here the 'sound bow' is the widest part of the bell. The proportions of the bell, '*taking thickness of sound bow as the unit, are: thickness of sound bow, 1; diameter of mouth, 15; diameter of shoulder, 7½, height, 12*'. The frequency of the main 'strike note' of the bell, along with the other tones it produces, will vary as we scale it in size. This is complicated by the fact that its thickness will of course also vary as we do this. The frequency is inversely proportional to the diameter of the bell, but the thickness comes into play too. Church bells do not precisely scale up – the smaller bells have a higher relative thickness in order to prevent their sound being drowned out by the heavier, larger bells with which they are competing.

A bell can be tuned to some extent once cast, but only a little, and only by removing metal. If metal is removed from the inside of the bell, it has the effect of increasing the internal diameter slightly, and thus lowering the frequency – flattening the tone. In the past this was done by 'chip tuning' when the founder literally chipped away at the inside of the bell to remove small pieces of metal. Nowadays a vertical lathe is used to remove metal in a much more uniform way. Bells must be in tune both with themselves and with the other bells in their ring, so this is an extremely complicated process. The science behind bell tuning was first described by Canon Arthur B. Simpson in the 19th century, and British bell foundries soon adopted the 'Simpson Tuning'. It controls for the five main partial frequencies. These are known as the nominal, quint, tierce, prime, and hum notes, as shown below, relative to the prime pitch $f$.

| Nominal | Quint | Tierce | Prime | Hum |
|---|---|---|---|---|
| $2f$ | $3f/2$ | $6f/5$ | $f$ | $f/2$ |
| Octave above | 5th above | Minor 3rd above | – | Octave below |

When a bell is struck, we do not hear a chord. The brain processes the sounds in a rather curious way. The sound most people actually hear, the 'strike note' is not the nominal pitch, but an octave below it. (This may coincide with the prime partial, if it is tuned to be an octave below the nominal, but occurs even if this is not the case.) When the partial frequencies in a sound are all common multiples of a frequency that is not played, then this 'missing fundamental' frequency is often heard anyway – it is called the virtual pitch. This is why we hear the octave below the nominal. The same Starmer lecture mentioned above includes a reference to this phenomenon being demonstrated by the then Gresham Professor of Music: '*Sir Frederick Bridge, at one of his Gresham Lectures, had two anvils brought in. One sounded a fifth above the other. He said, "When you hear these two struck together you will hear the octave below the lower one." He struck them several times, but no lower note came. But at last he got a splendid low note'.* This is because a perfect fifth corresponds to a 3:2 frequency ratio. So, if the octave below the lower anvil corresponds to frequency $f$, then the two anvils had frequencies $2f$ and $3f$. What is happening with the bell is a similar phenomenon.

## Ringing the Bells

This is where we get to the crux of what makes English bell ringing different. You can mount your bells in a fixed position and ring them from the outside with hammers or clappers. This is what is done in carillon bells. These can be timed very well, but the sound does not carry as far and is not as resonant as with swinging bells with internal clappers. The next option is 'chiming', where the clapper is inside the bell – think of that handbell used in the school playground. You swing it up and down, and the clapper hits the bell at the top and the bottom of its arc of motion. This can be extended to church bells by means of a rope fixed to a lever which pulls the bell up and down.

In England, monasteries had bells by around the $8^{th}$ century, to signal times of prayer, mealtimes and so on. Gradually, the idea spread to normal parish churches, and the number of bells increased, with different bells or different combinations of bells sometimes having different meanings. One of the things that may have encouraged this spread was a decree in 750AD which said that a Saxon could become a Thane if his estate was at least 500 acres in size and contained a church with a bell tower. By Tudor times, it was not unusual for churches to have at least three or more bells. Ringers started to experiment with ringing several bells together in different combinations during celebrations. This requires teamwork, practice and skill.

At the same time the technology was developing. With the simple lever, you have no control on the timing of the down swing. Gradually, as bells were swung higher, the ringers gained a little more control, and time to change the orderings of rows. The lever became a quarter wheel, then a half wheel. The key development, occurring at some point in the 1500s, was to have a full wheel with the rope able to wind both ways around it. This means the bell can swing all the way round, in what is known as full-circle ringing. If we think about pendulums, the rule is that the period of the swing is the same no matter what the amplitude. But this breaks down for larger swings. The bell swings more slowly as the swing gets larger – we have all felt that moment of weightlessness when at the top of the arc of motion when we are swinging on a swing. It's the same with bells. That moment at the very top of the swing when the bell is moving slowly is a brief instant when a small change by the ringer can have a large effect on the timing of the swing. It allows two bells to swap order in the row. However, you cannot change the ordering too much. The bell that rang eighth in one row could not immediately ring first in the next row. A final development was adding a 'stay' so that bells could be kept in the mouth up position. We should also note that these large bronze bells swinging up and down put a lot of stress on the surrounding structures. During the swing, a bell generates forces of up to four times its weight downwards and twice its weight laterally. Bell frames had to be strong enough to withstand the forces, and often you will see a mixture of angles of swing – some bells swinging at right-angles to others, to ensure a more even distribution of the stresses.

Everything was in place, then, by the 1600s. There were more bells – and these weren't destroyed during the Reformation like so much else, probably because of their secular uses, such as for particular events, celebrations, or even for sounding curfews. The bell-ringers had finer control over the timing of the bells' swings, and 'ringing the changes' (a phrase first recorded in 1614) became a very popular pastime – even a sport.

## Change Ringing

If we forgive the mild xenophobia(!), the following, from an old Dorothy Sayers mystery novel featuring her detective Lord Peter Wimsey, is an amusing description of change ringing.

"*The art of change ringing is peculiar to the English, and, like most English peculiarities, unintelligible to the rest of the world. To the musical Belgian, for example, it appears that the proper thing to do with a carefully tuned ring of bells is to play a tune upon it. By the English campanologist, the playing of tunes is considered to be a childish game, only fit for foreigners; the proper use of bells is to work out mathematical permutations and combinations.*"

— Dorothy L. Sayers, The Nine Tailors (1934)

In change ringing, each bell in a ring is numbered from 1 (the highest) to $n$ (the lowest). The bells sound in a sequence of *rows*. The movement between one row and the next is called a *change*. There are two rules.

1. Each bell sounds exactly once in each row.
2. In any change, each bell can move at most one position.

These rules are in place because this is the most amount of variation that the physical set-up allows.

A change-ringing performance in which each possible row is played exactly once, starting with $123\ldots n$ (and by convention adding a final $123\ldots n$ to finish – the only repeated row in the performance), is called an *extent*. The $123\ldots n$ row, recall, is called *rounds*. A *peal* initially meant any change-ringing performance – with a *true peal* meaning one where no row was repeated. Nowadays, a peal is a performance that has over five thousand different rows, starting and ending with rounds. Often, the lowest, or *tenor*, bell is kept in position and played last in each row – this sonic punctuation gives structure to the music. Of course, one cannot get an extent on the full set of $n$ bells like this: instead, one can aim for an extent on $n-1$ bells.

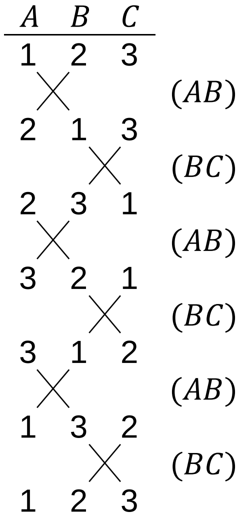
Early ringers used just 'plain changes', where only one pair of bells is swapped between rows. But during the 17th century the more exciting 'cross peals' came into use, where several pairs of bells are swapped at the same time. These must be adjacent pairs, of course, because each bell can move at most one position. In fact, Rule 2 means that the only possible changes are combinations of swaps of adjacent bells. This is because if we suppose in a given change that position $P_k$ is the first position where the bell moves (so the bells in positions $P_1$ to $P_{k-1}$ stay fixed), then that bell must go to position $k-1$ or $k+1$. But position $k-1$ is already full, because the bell there doesn't move. So, the bell in position $k$ must go to position $k+1$. This leaves a gap at position $k$. It can only be filled by the bell at position $k+1$. Therefore, we have a swap; the notation for this is $(P_k\ P_{k+1})$. Then we move to the next position where a bell moves and repeat the same reasoning; another swap must take place there, and so on.

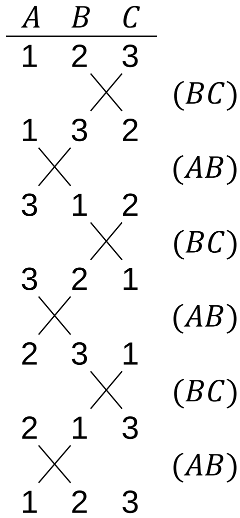
We now arrive at the key question: is it possible to work through all possible rows exactly once, starting and finishing with rounds, such that between rows each bell moves at most one position, to produce a true peal, or even an extent?

Let's see what happens with 3 bells. Performances with three bells are known as *singles*, because only a single swap is possible at a time. We start with rounds, 123. Let us call the positions of the bells $A, B$ and $C$. Then $(AB)$ means swap the bell in position $A$ with the bell in position $B$, for example.[1] The only possible changes are $(AB)$ and $(BC)$. Suppose we choose $(AB)$. Then we get

[1] Mathematicians might feel that it would be better to label the bells $A, B, C$ and the positions 1, 2, 3, so that we can write permutations such as (12) and (23). The problem with this is that since bells produce musical notes, we will tie ourselves in knots in a different way by potentially labelling bells, whose pitch is one of the notes A, B, C, D, E, F or G, with a letter signifying something quite different!

213. If we do $(AB)$ again, we will get back to 123, repeating a row. So, we are forced to do $(BC)$; by the same token, if the subsequent move were $(BC)$ again, we would return to the previous row. The net effect is that we must alternate $(AB)$ and $(BC)$. This gives 123, 213, 231, 321, 312, 132, 123. We have created an extent on three bells! The only other way to do this is by starting with $(BC)$ and alternating. We get 123, 132, 312, 321, 231, 213, 123. Both methods are shown below, with × indicating a swap between given bells.




On higher numbers of bells, just swapping one pair at a time gets a bit boring, and generally we try to swap as many pairs as we can as often as we can. Change ringing performances must be learnt by the ringers and tend to involve combinations of set sequences called methods. The simplest method is 'plain hunting', which is done on an even number of bells (or an odd number with the tenor removed from the equation by always playing last or *covering*). To describe plain hunting, suppose we have $2k$ bells. Our first move is the change with $k$ swaps (the maximum possible). We call this move $r$. The second change, which we call $s$, is the $k-1$ swaps just involving the inner $2k-2$ bells. We then alternate $r$ and $s$ until we return to rounds. On six bells in positions $A$, $B$, $C$, $D$, $E$ and $F$, this means we alternate $r=(AB)(CD)(EF)$ with $s=(BC)(DE)$. On four bells, with positions $A$, $B$, $C$ and $D$, we alternate $r=(AB)(CD)$ with $s=(BC)$. Thus, plain hunt on four bells would give 1234, 2143, 2413, 4231, 4321, 3412, 3142, 1324, 1234.

In plain hunting, the treble (bell 1) is 'hunting' through the other bells – it moves from position 1 to position $n$ and back again. An extent on four bells would have 24 rows, so we cannot achieve it through plain hunting alone. However, we can tweak things slightly. If every time the treble (bell 1) returns to the lead position, our next move is not the usual $(BC)$ but, as a one-off, $(CD)$, then we can launch another round of plain hunting from a different starting point. This is called Plain Bob. Here is an example on four bells. Each column is a course of plain hunting; to get from one column to the next we perform $(CD)$ on the final entry. This, for instance, sends 1324 to 1342.

| | | |
|---|---|---|
| 1234 | 1342 | 1423 |
| 2143 | 3124 | 4132 |
| 2413 | 3214 | 4312 |
| 4231 | 2341 | 3421 |
| 4321 | 2431 | 3241 |
| 3412 | 4213 | 2314 |
| 3142 | 4123 | 2134 |
| 1324 | 1432 | 1243 |
| and finish with 1234 | | |

In general, for plain hunting on $n = 2k$ bells, you alternate $(P_1P_2)(P_3P_4)\cdots(P_{2k-1}P_{2k})$ with $(P_2P_3)\cdots(P_{2k-2}P_{2k-1})$. The net effect is $(P_1P_3P_5\cdots P_{2k-3}P_{2k-1}P_{2k}P_{2k-2}\cdots P_4P_2)$. This permutation has 'order $n$', meaning that $n$ repetitions of it gets you back to the arrangement you first started with. Thus, there will be $2n$ rows.

There are thousands of change ringing methods. Each is given a three-part name, comprising the specific name, the 'family' name, and the 'stage' name. The specific name is chosen by the creator. The family name relates to the structure – the pattern of changes. The stage name specifies the number of bells. Because of the importance of the number of possible swaps, this is reflected in the stage name. For 3 bells, it is Singles. For 5, 7, 9 and 11 respectively, it is Doubles, Triples, Caters and Cinques. If we keep the habit of leaving the tenor bell fixed, these odd numbers are the first for which the corresponding number of swaps is possible. In other words, you wouldn't usually do Triples on six bells, because the tenor would be played last in every row. The even numbered bells are given the stage names Minimus (4 bells), Minor (6), Major (8), Royal (10) and Maximus (12). Beyond that it's just Fourteen, Sixteen and so on. This leads to some wonderful names like *Primrose Surprise Major* and *Avon Delight Maximus*. The example we saw above for four bells has the proper name *Plain Bob Minimus.*

Once a method is started, it will run its course because the ringers simply follow the set pattern that they have learnt. Between methods, a 'call' tells the ringers what to do to switch to the next method (or the same one again but from a different starting point). For example, on five bells in positions $A$, $B$, $C$, $D$, $E$, you could perform Plain Bob Minimus on the bells in positions $A$, $B$, $C$ and $D$. Then the conductor could call $(DE)$, and Plain Bob Minimus could be repeated. This would result in 48 rows: 12345, …, 12435, followed by 12453, …, 12543.

We can check that Plain Bob Minimus is a true extent on four bells (in other words, that there are no repeated rows) just by looking. But an extent on five bells has 120 rows (plus the final repeated 12345 rounds). There is more room for error, and the problem gets worse as we move to six bells (720 rows) and seven bells (5,040 rows).

## The Search for Truth

A 'true' performance must not contain any repeated rows (except the final row of rounds). This can be very challenging for the composer, who is joining methods with calls that must ensure repetition is avoided. In trying to construct an extent of five bells, for example, we could try Plain Bob on bells $ABCD$, followed by $(DE)$, repeatedly. This gives 12345 … 12435 (which contains, exactly once, each row ending in 5), then $(DE)$ to get 12453, followed by another Plain Bob on $ABCD$, ending in 12543. This run-through of Plain Bob will obtain all the rows of five ending in 3. Another $(DE)$ will give us 12534, and Plain Bob now cycles through all the rows ending in 4. But it ends with 12354, so $(DE)$ would get us back to a row ending in 5. Maybe other calls than $(DE)$, such as $(AB)(DE)$ or $(BC)(DE)$ would work. We will know we have a true extent if we can keep track of what bell is in the fifth position. However, this is not a very satisfactory solution, even if we can achieve it, because it is boring for a bell to stay in the same place for 24 rows (especially if we are actually ringing six bells with the tenor bell already always playing last). So: we have some more questions. Can we ring an extent at all? Can we compose a 'non-boring' extent? Can we compose a non-boring extent that has sufficient structure that it can be learnt and rung in practice? And finally, how do we *prove* that we have rung a true extent?

Huge strides were made in the 17th century, as bellringing became very popular. There were clubs and societies across the land, and neighbouring parishes entered into fierce competition. In '*The School of Recreation*', published in 1684 (full title – *The School of Recreation, or the Gentleman's*

*Tutor: to those most ingenious exercises of hunting, racing, hawking, riding, cock-fighting, fowling, fishing, shooting, bowling, tennis, ringing and billiards*), bell ringing is described as 'highly esteemed for its excellent Harmony of Musick it affords the ear, for its Mathematical Invention delighting the Mind, and for the Violence of its Exercise bringing Health to the Body, causing it to Transpire plentifully'. Churches would have 'pealboards', recording dates and ringers of particularly memorable peals.[2] There was something of a hiatus in bell ringing during Cromwell's time, but after the restoration secular bell ringing started back up with a vengeance. It was thirsty work, and beer at that time was safer to drink that water. Bell ringers were actually paid in beer in many cases, and would also often keep a large jug, known as a 'gotch', of beer on hand to drink from throughout the day. The gotch at Beccles reputedly contained 33 pints! In the lecture we show a Victorian picture of 'ringing out the old year in the belfry of Cripplegate Church' – the church in question being St. Giles Cripplegate, London. Bell ringing societies sprang up, such as the Scholars of Cheapside (1603), the Society of Colledg Youths (1637) – later renamed, with better spelling, the Ancient Society of College Youths, the Society of Royal Cumberland Youths (1747) and so on. Careful records of ringing feats were kept, and if a peal was found not to be 'true' subsequently, it would be stricken from the record. Extents on four, five and six bells were all achieved in the 17th century. The first proven peal (more than 5000 different rows) was completed by the Norwich Scholars on 2 May 1715, ringing what we would now call Plain Bob Triples. The 'triples' indicates seven bells, but it was rung in the standard way on eight bells with the tenor last each time. A seven bell extent is 5,040 rows, but to go up to an eight bell extent would be 40,320 rows. This must have seemed an impossible thing, and in fact was not achieved by a single team of eight ringers until 1963, though a relay team managed it in 27 hours in 1761. Peals (not extents of course!) for up to twelve bells were all achieved by 1725. For example, the Union Scholars rang a nine-bell peal (Grandsire Caters) in January 1717. How were ringers designing these peals? A large part of the credit must go to Fabian Stedman, of whose contribution we now give a brief indication.

Fabian Stedman (1640-1713) was a printer by trade, living and working in London for most of his life. He was a member, and later Treasurer, of the Scholars of Cheapside, then in 1664 joined the Society of Colledg Youths. He would eventually become Master of the society. He is known today for his contributions to the first two books on change ringing, both highly influential. *Tintinnalogia – or, the Art of Change Ringing* (1668) was written by Richard Duckworth, a fellow of Brasenose College, Oxford, possibly with contributions from Stedman, but Stedman arranged the printing. Beginning with plain changes (one swap at a time), the book then introduces the relatively recent idea of cross-peals, where multiple swaps happen at the same time. Descriptions are given of Plain Bob Minimus, as well as peals on five and six bells. Some of these are written out accurately in full. The second book, *Campanalogia – or the Art of Ringing Improved* (1677), was written by Stedman himself, and contains many new methods, on up to eight bells. There are fifty-three 'London Peals' of Stedman's own invention, including 'Stedman's Principle', which we now call 'Stedman Doubles'. At the start of the book, Stedman writes, 'Although the Practick part of *Ringing* is chiefly the subject of this Discourse, yet first I will speak something of the Art of *Changes*, its Invention being Mathematical, and produceth incredible effects, as hereafter will appear'.

What is particularly striking in *Campanalogia* is that, although of course Stedman does not use modern mathematical language, he puts considerable effort into explaining how we can be sure that the peals described are true extents, in terms that, though the words used are different, seem very similar logically to how we would proceed today. Stedman's importance as a contributor to change ringing is reflected in the memorial plaque to him at the church of St. Andrew Undershaft in the City of London, where he was buried on November 16, 1713. The plaque was erected in 1983 by the

[2] Nowadays there are national databases of ringing performances. The Felstead database, started in the 1950s by Canon K.W.H. Felstead, lists the date, place and method of all known peals. PealBase contains more comprehensive details such as ringers' names, of peals rung since 1950. Both are available online.

Society of College Youths. A lot of what Stedman was doing can nowadays be best described using the mathematics of groups, and it is to those wonderful structures that we turn next.

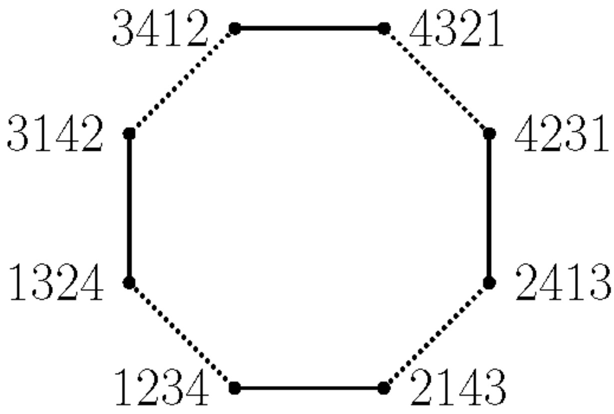
The Mathematical Structure of Bellringing

The mathematical idea of a group is one that we encountered in Lecture 1 of this series, looking at the different ways of transforming musical motifs. It is an indication of the versatility of the concept that we meet it again here in a completely different musical context. A *group* is a collection of objects that can be combined using some operation to produce further objects that still lie in the collection, satisfying four basic properties, which we shall illustrate with some examples. In our case, the objects are permutations of the bells. For example, suppose we have four bells, in positions $A, B, C$ and $D$. The notation $(ABC)$, for example, means 'send the bell in position $A$ to position $B$, the bell in position $B$ to position $C$, and the bell in position $C$ to position $A$, and fix all other bells'. Permutations can be combined by simply following one with the other. Thus, for instance $(AB) \cdot (BC) = (ACB)$. The composition of two permutations is another permutation (this is called the *closure* property). Every permutation $\alpha$ has a corresponding *inverse* permutation $\alpha^{-1}$: you simply reverse what you just did. There is also an *identity* permutation, which just leaves all bells where they are. When you combine the identity permutation with any other permutation $\alpha$, you just get $\alpha$ back again. Finally, there is a technical property called *associativity,* which says that if we are working out $\alpha\beta\gamma$, we get the same answer whether we do $(\alpha\beta)\gamma$ or $\alpha(\beta\gamma)$. Lots of situations produce groups. For example, the integers form a group, with the operation being addition. We can check that the four required properties of closure, existence of identity, existence of inverses, and associativity, do hold. If $a$ and $b$ are integers, then $a + b$ is an integer (closure). Meanwhile the identity element is $0$, because $a + 0 = 0 + a = a$. Every integer has an additive inverse: the inverse of $a$ is $-a$, because $a + (-a) = 0$. Finally, we have associativity, because it is always true that $a + (b + c) = (a + b) + c$, for any integers $a, b, c$. Another example of a group is the set of symmetries of a shape – known as the 'symmetry group' of that shape. These are the reflections, rotations and so on which leave the shape it looking the same, such as rotating a square through 90$^\circ$ about its centre. The operation is just composition – do one symmetry followed by the next.

The set of all permutations on $n$ bells is a group, and in mathematics it is known as $S_n$. It has $n!$ elements. The elements of the group are the permutations, not the rows themselves. They are the transitions between rows. In a bell ringing performance, we aim to ring a sequence of rows, all different, starting from rounds, $123 \cdots n$. The only allowed transitions from one row to the next are combinations of swaps of adjacent bells. Question: Are all possible rows obtainable using legal moves? Answer: Yes! To get any particular row, first decide what bell will be in position $n$. If it is already in position $n$, then there's nothing to do. Suppose that the bell is in position $k$. Then do $(P_k P_{k+1}), \cdots, (P_{n-1} P_n)$. Repeat this process with the bell that is going to be in position $n - 1$, and so on. This shows that any individual row can be produced, but it does not tell us how to create the full extent – each row appearing exactly once. To approach that question we need a little more group theory.

A subset of a group that happens also to be a group (with the same operation) is called a *subgroup*. For example, the set of even numbers is a subgroup of the group of integers. The set of rotational symmetries of a shape is a subgroup of its symmetry group. Importantly for us, the set of permutations that can be made with $r$ and $s$ (the two permutations we use for Plain Hunting) is a subgroup of the group of permutations, called the Hunting Group. Associativity is inherited from the bigger permutation group; closure follows from the fact that any two permutations made using only $r$ and $s$ combine to produce another such permutation; the inverse of any $r, s$-permutation is simply the original permutation written in reverse order; and the identity permutation is just (for example) $rr$. Any expression involving $r$ and $s$ can be simplified by cancelling any $rr$ and $ss$ instances, because $rr$ and $ss$ are both the identity permutation. Therefore, every expression is just $rsrsr \cdots$ or

$srsrs\cdots$ (or the identity, which we often write as $e$). Finally, we have observed that $(rs)^n = e = (sr)^n$. This means we can cancel these expressions from any longer expression. But, actually, any expression longer than $n$ terms can be shortened too. For example, when $n = 4$, we get $rsrsrsrs = e$, which means (post-multiplying by $s$) that $rsrsrsrss = s$. Now cancel the $ss$ to get $rsrsrsr = s$. Similarly, $rsrsrs = sr$. We also have $rsrsr = srs$ and $rsrs = srsr$. We can represent this in a diagram – here, the solid lines indicate the change $r$ (going in either direction), while the dotted lines indicate $s$.

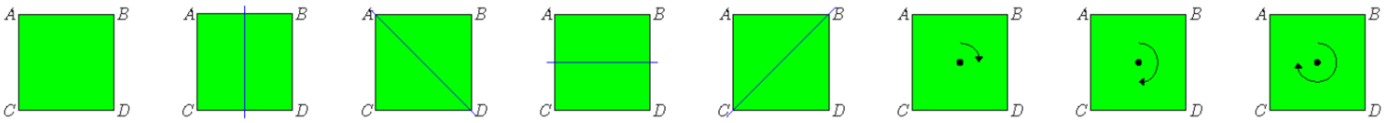



We can see that, for example, we can get from 1234 to 3142 by either $rsrsrs$ or $sr$. Looking back to our plain hunting, we can now see that the rows of the plain hunt correspond to a subgroup of order 8. The final term, 1324, is the permutation $s$ (or $(BC)$ in this case) on 1234. If we now define $t$ to be $(CD)$, then performing $t$ on 1324 gives 1342, which corresponds to the permutation $st$. Repeat plain hunting with this new starting point, and what we will get is the permutations from the plain hunt, but composed with $st$ at the beginning.

Suppose you have a group $G$ and a subgroup $H$. We can define things called *cosets* of $H$ in the following way. For some element of $G$, we let $xH$ be the set $\{xh: h \in H\}$. In other words, it is the set obtained by taking $x$ composed in turn with each element of $H$. For instance, in the case of the permutations of four bells, $S_4$, we can let $H$ be the hunting group, and $x = st$. In the table below, we show the rows of plain bob along with the permutations that produce them (in each case starting from 1234). The first column is just the eight rows of the plain hunt. The second column corresponds to the second set of eight rows in plain bob. This is precisely the coset $stH$. The final element of this column must be $sts$, because if $sts$ were followed by $s$ we would get $stss$, which equals $st$, returning to the top of the column. If we instead do $t$, we get the row resulting the permutation $stst = x^2$. This is another coset of $H$, and the three columns together form Plain Bob Minimus, an extent on four bells.

| | | | | | |
|---|---|---|---|---|---|
| 1234 | $e$ | 1342 | $st$ | 1423 | $stst$ |
| 2143 | $r$ | 3124 | $str$ | 4132 | $ststr$ |
| 2413 | $rs$ | 3214 | $strs$ | 4312 | $ststrs$ |
| 4231 | $rsr$ | 2341 | $strsr$ | 3421 | $ststrsr$ |
| 4321 | $rsrs$ | 2431 | $strsrs$ | 3241 | $ststrsrs$ |
| 3412 | $rsrsr$ ($= srs$) | 4213 | $stsrs$ | 2314 | $ststsrs$ |
| 3142 | $rsrsrs$ ($= sr$) | 4123 | $stsr$ | 2134 | $ststsr$ |
| 1324 | $rsrsrsr$ ($= s$) | 1432 | $sts$ | 1243 | $ststs$ |
| | | | | and finish with 1234 | $ststst$ ($= e$) |

Interestingly, these hunting groups crop up in a quite different context – symmetry groups. We have mentioned that the set of symmetries of a shape (in other words, things you can do to it that leave it looking the same), forms a group. The group operation is just composition of maps – do one symmetry, then do the next. So, for instance, there are eight symmetries of a square. We can rotate about the centre by 0° (the identity map $e$), 90°, 180° or 270°. We can also reflect in any of the four mirror lines. If *A, B, C* and *D* indicate the positions of vertices, then these symmetries give rise to the permutations shown.




$e$ $(AB)(CD)$ $(BC)$ $(AC)(BD)$ $(AD)$ $(ABDC)$ $(AD)(BC)$ $(ACDB)$

This is precisely the hunting group in disguise! We already know that all elements of the hunting group can be generated with combinations of $r = (AB)(CD)$ and $s = (BC)$. These correspond to reflections in the vertical and north-west to south-east diagonal. This immediately tells us that every symmetry of the square can be obtained as a product of reflections! Similarly, the hunting group on six bells is the same, mathematically speaking, as the symmetry group of the regular hexagon. This is one of the great things about groups – something you prove in one context can give you information in an area that appears to be completely unconnected.

Let us say a little more about cosets. Suppose $xg = xh$. Then $x^{-1}xg = x^{-1}xh$, and so $g = h$. Therefore, different elements of a subgroup $H$ of a group $G$ lead to different elements of $xH$. Hence, $|xH| = |H|$. But also, if we make a new coset by picking some $y$ that isn't in any of the cosets we have so far, then we can be sure all the elements of $yH$ are different from anything we have yet seen. This is because if the worst happens and $yh$ is in $xH$ for some $x$, then $yh = xg$, say, for some $g, h$ in $H$, but then $y = xgh^{-1}$. Now $H$ is a subgroup, so $gh^{-1}$ is in $H$. But this implies that $y \in xH$, which contradicts the fact that we chose $y$ to be an element we hadn't yet seen. To sum all this up: every coset has the same number of elements, namely $|H|$, and if cosets are different then they are completely disjoint sets. This means we can break up our group as a disjoint union of some number, say $m$, different cosets of $H$, all of the same size. Therefore, $|G| = m|H|$. This fact is known as Lagrange's Theorem.

## Back to Bells

The Plain Bob Minimus construction shows Lagrange's Theorem in action: we have broken up $S_4$, which has 24 elements, into three cosets of the hunting group $H$, which has 8 elements. (The cosets are $H$ (or $eH$ if you prefer), $xH$, and $x^2H$.) Our observations about cosets show that if the first row of any new coset is different from any previous row, then every row in that coset will be different from every previous row. This considerably reduces the amount of checking we have to do, especially for the hunting group case where the first row always begins with 1, and we can observe that only the first and last rows of each column/coset have that property. So we just need to check a handful of rows in order to definitively prove that what we have is a true extent. Can we do this more generally? Let us imagine trying to produce an extent on five bells. One approach is to start with the Plain Bob Minimus on the first four bells. This corresponds to an $S_4$ subgroup inside $S_5$ (because bell 5 remains where it is throughout). Since $S_5$ has 120 elements, $S_4$ has 24, and $120 \div 24 = 5$, we know there are five cosets. So, we could try and find legal moves to transition between these cosets. However, as we have mentioned, it is undesirable to have any one bell staying in the same place for so long.

Another approach was devised by Fabian Stedman (though of course without using the language of groups and cosets). It is called *Stedman Doubles*, because it uses the three possible double swaps on five bells. If we choose $u = (AB)(DE)$ and $v = (BC)(DE)$, then $u$ and $v$ generate a subgroup of order 6. Combining carefully with the element $w = (AB)(CD)$, one can produce ten distinct cosets of the subgroup, for a 'plain course' of 60 rows. Specifically, we repeat a defined sequence of twelve changes (consisting of two cosets of the order six subgroup) five times, alternating the two different extents on the first three bells with an application of $w$. The sequence is $u, v, u, v, u$ (the net effect of which is equivalent to $v$). Then $w$, then $v, u, v, u, v$ (the net effect of which is equivalent to $u$). This works because the permutation $vwuw$ has order 5.

At this point we need a fact about permutations. We say a permutation is *odd* if it can be produced by composing an odd number of swaps, and *even* if it can be produced by composing an even number of swaps. If we imagine flicking a light-switch every time a swap is done, then (starting with the light off) a permutation like $(AB)(DE)$ would leave the light off, because $(AB)$ turns it on and $(DE)$ turns it back off again. But that means no permutation we could ever produce by combining double swaps like $u$, $v$ and $w$ could ever leave the light on. In particular, no even permutation can also be odd; after all, a light cannot be on and off at the same time! This means $S_n$ can be split equally into even and odd permutations. Moreover, because the composition of any two even permutations is also even, the set of even permutations can be shown to be a subgroup of $S_n$. It's called $A_n$. It consists of half the permutations of $S_n$; the remaining half of the group is therefore a coset of $A_n$, and if we let $x$ be any odd permutation, then $xA_n$ is precisely the set of all odd permutations of $S_n$. In $S_n$, then, there are 120 permutations in total, of which 60 are even. The plain course above must contain only even permutations as it is produced using combinations of the three even permutations $u$, $v$, and $w$. So, in fact, the plain course corresponds to precisely the subgroup $A_5$. This means that if at the end of it we perform any single swap at all, we will move into the odd coset of $A_5$, and then repeating the plain course must give us the remaining 60 elements of $S_5$.

This idea can be extended to any odd number of bells. When you extend to seven bells, you get *Stedman Triples*. This time, we have $u = (AB)(DE)(FG)$ and $v = (BC)(DE)(FG)$, producing a subgroup of order 6, and now $w = (AB)(CD)(EF)$. It turns out that the plain course we derive has 84 rows. It is less clear how to find a set of calls that would produce the sixty different repetitions of the plain course (from different starting points each time) needed to produce a full extent (5,040 rows) on 7 bells. One method uses the triple swap $(AB)(CD)(FG)$, along with $(AB)(CD)$. This $(AB)(CD)$ leaves three bells fixed, so is less desirable. Ringers spent three centuries trying to answer the question of whether an extent using Stedman Triples was possible without using such changes (in other words, only using triple swaps). Stedman thought it should be possible but couldn't prove it. Because triple swaps are odd permutations, we at least don't have to worry about being trapped inside $A_7$. It wasn't until 1994 that the problem was solved – it is indeed possible to ring an extent of seven bells with Stedman Triples using only triple swaps. The problem was solved independently by Colin Wyld, and Andrew Johnson and Philip Saddleton. The Johnson-Saddleton composition was successfully rung early in 1995 by a Cambridge University Guild band.

## Where Next?

Computers have been used in composition almost from their inception. The Ferranti Mk 1 computer at Manchester University ran a program in the 1950s analysing a particular ringing problem. They can certainly test candidate peals or extents to see if they are true, but they are less good at judging their musical quality or other desirable factors such as how easy they are to learn. Humans will always need to be involved. There are still many unanswered questions about possible peals and extents, which leaves fertile territory for mathematicians and bell ringers alike. For further reading and listening, a selection of excellent books and online resources is given over the page. If you are interested in ringing bells, but don't have a belfry to hand, you could always consider acquiring a set of handbells. Ringing a bell in each hand, it only requires a team of four to ring a peal on eight bells. With three people you can ring six bells. But you will have to practice for some time to equal the 2007 feat of Philip Earis, Andrew Tibbetts and David Pipe (members of the Ancient Society of College Youths), who rang a sequence of 100 different extents on six bells. They rang continuously for 24 hours and nine minutes!

Further Reading


Budden, F. (1972). *The fascination of groups.* Cambridge: Cambridge University Press.

Fauvel, J., Flood, R., & Wilson, R. (2003). *Music and Mathematics.* Oxford: Oxford University Press.

Harkleroad, L. (2006). *The Math Behind the Music.* Cambridge: Cambridge University Press.

Harrison, J. (2016). *Bells and Bell-Ringing.* Shire Publications.

Starmer, W. (1901). Bells and Bell Tones. *Proceedings of the Musical Association, 28*(1), 25-44. doi:10.1093/jrma/28.1.25

White, A. (1996). Fabian Stedman: The First Group Theorist? *American Mathematical Monthly 103*, 771-778.

White, A., & Wilson, R. (1995). The Hunting Group. *The Mathematical Gazette, 79*(484), 5-16.


Further Listening

- Bill Hibberts completed a PhD in the acoustics of church bells and has collected recordings of thousands of bells. Explore them at his excellent website: http://www.hibberts.co.uk/belllist.htm.
- You can hear six ringers https://www.youtube.com/watch?v=4E9RWPn2nGU ringing the largest six bells at the Cathedral Church of Christ in Liverpool – the heaviest six bell ring in the world. The video also gives a good indication of the physical effort required in change ringing.
- The Grantham Bells ringing the start of a quarter peal of 'Bristol Surprise Maximus' in 2016. https://www.youtube.com/watch?v=gYXleRD8qNQ&t=75s.
- You can see and hear the Bell of Good Luck (the largest working bell in the world) ringing at https://www.youtube.com/watch?v=L83XAgsl_Gg
- To hear the gigantic carillon at the Rockefeller Chapel in Chicago, an instrument with 72 bells made with 100 tons of bronze, see Rob Scallon's video at https://www.youtube.com/watch?v=VkRIMgTU7cA.



Questions & Answers

1. Is the bell-ringing a bridge to the lectures on mathematics and fiction? I was thinking of Dorothy L Sayers. (Dr Gerry Leversha)

   As you saw, we did indeed use that quote!

2. Are all bell making calculations the same mathematically? (Bea)

I’m not sure exactly what is being asked, but if it’s about the making of the physical bell, then the Simpson Tuning has been in place for at least 150 years and is the standard approach; this gives the essential dimensions of the bell.

3. Why is the 2 always left out when there are not enough ringers? In every tower, it is invariably the 2. (Mrs Susan Ellis)

   (RESPONSE PROVIDED BY Simon Head) On 8 bells it sounds better to have the lowest and highest notes an octave apart. So although figuratively 2345678 might sound better than 1345678, once you get into changes (Start Ringing Combinations) the latter sounds better. A similar though less pronounced pattern exists on six bells

4. Until 4 years ago I owned the biggest and smallest handbells in the world (passed on from my father). Do handbells have any differences with tower bells in terms of sound or other features of performance? (Clive)

   In theory one has more freedom because you can be nimbler with handbells than with huge heavy church bells. But my understanding is that the same rules are followed nevertheless.

5. Is there any difference in construction of bells and bell ringing in the Western and Russian Orthodox church where bell ringing plays a significant role? (M Burrell)

   Yes, quite a lot. For example, the bells are fixed rather than the full circle ringing we have. The construction is a bit different too, with different tunings from our bells. I believe they add silver to their bells as well, whereas we have standard bronze.

6. Are bells tuned to a major chord? (Wrigley F)

   (RESPONSE PROVIDED BY Scott Allan Orr) No, they have a minor third predominantly. Some experimental bells have a major third but this has never caught on.

7. Maths grad (35 yrs ago) looking forward very much to this. Just watched video of Dorothy L Sayers' 'The Nine Taylors'. Is the book accurate? (Jean Tarry)

   As far as I can tell, yes it does a good job (apart from the likelihood of getting murdered which I hope is substantially less in real life!)

8. Except she demonstrated the full-circle ringing by saying it pauses at the top and then carries on but it doesn't, it goes back the way it came, full circle the other way - the stay stops it going any further than vertical. (Janet Betham)
   *Gresham College Flagging this up to Prof Hart in case this is an error that needs correcting*

   Yes, sorry you are right about this. I didn’t express myself very well!

9. Are all bell making calculations the same mathematically? (Bea)

   Same question as earlier.

10. Can you give us some links to actual ringings of the patterns you described? (Susan Cooper)

    (RESPONSE PROVIDED BY Antony Gay)
    https://www.ringingmethods.co.uk/methods/order/6

    Yes, there's loads online. I had to be a bit careful about (a) copyrighted performance and (b) technical challenges. Grateful to Antony Gay for providing that link.