Will Computers Outsmart Mathematicians?

K. Buzzard

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Computers v humans

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Of course, we will have to say what we *mean* by "mathematics" here.

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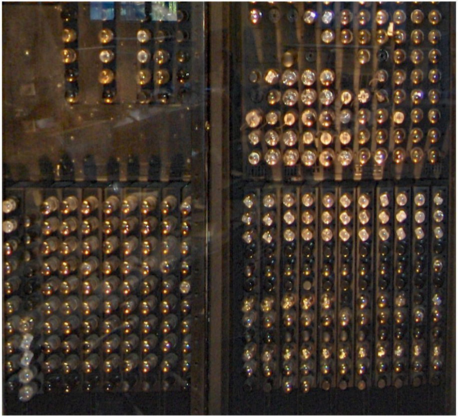

[Image of ENIAC uploaded by user `TexasDex` to Wikipedia]

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Nowadays, a computer can switch many millions of switches in one second.

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Computer programs can hence analyse systems which can be described with a finite amount of data.

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Nowadays, humans use computers to help them to learn about chess.

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The board game *go* is played on a much bigger board.

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After DeepMind's success, some computer scientists declared that the domain of board games was "solved".

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A *theorem* is an abstract mathematical statement, like a logic puzzle. Proving the theorem is like solving the puzzle.

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It is hence a natural target for AI, after board games like chess or go.

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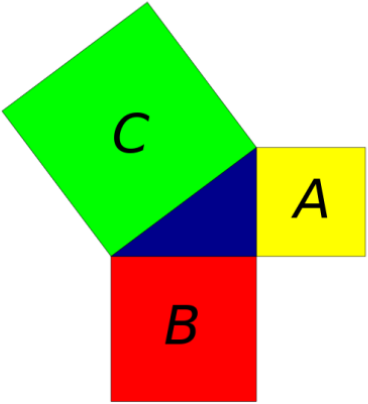
Is there such a thing as “2”?

Pythagoras' theorem



Pythagoras' theorem $A + B = C$ is also an abstraction. It is about an "abstract square".

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By the end of book 1 he has proved Pythagoras' Theorem.

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What is a *finite* list of rules which uniquely characterise these numbers?

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Most humans do not usually consider mathematics in this primitive state.

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Let's teach Peano's axioms to Lean.

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For example, let's try and define how to add 2 to a number.

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So $2+2=4$. QED!

$2 + 2 = 4$

Let’s see if Lean can do it.

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An AI then solved the “$2 + 2 = 4$” level of this maths game.

How much further does this go?

So what if we teach the computer an entire undergraduate pure mathematics degree?

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Other research groups are trying the same with other systems.

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Thank you for coming!