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Mathematical Structure in Fiction

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Introduction

Mathematical ideas have often been used to create structural forms in writing. We explore some of these in today's lecture, playing with the various layers of meaning in the phrase "the structure of a story" to create three main threads of discussion.

Structural forms and constraint

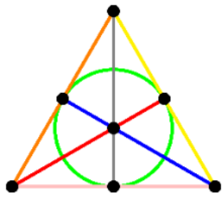
Many kinds of creative writing involve formal constraints. The most obvious examples are poetic forms like sonnets (14 lines each with ten syllables in iambic pentameter, with a specific rhyme scheme). Structures, such as a specified length and rhyme scheme, can be a spur to creativity, and mathematics is a fantastic source of structures to play with. Of course, what one does with them is the choice of the individual writer. I'm not a poet/If I try to write haiku/They won't be that great. (That last sentence was a haiku with absolutely no artistic merit, for example.) We begin our exploration with an introduction to a group that has contributed a wide variety of new structural constraints to literature.

The Oulipo

On November 24th, 1960, a group of writers and mathematicians formed what become known as the *Ouvroir de Littérature Potentielle*, or "*Oulipo*". This translates roughly as "workshop of potential literature". Instigated by Raymond Queneau and François Le Lionnais, the aim of the group was to explore the possibilities for new formal constraints in literary texts, particularly those arising from mathematical concepts and structures. Well-known members of the group have included Italo Calvino, Marcel Duchamp, Georges Perec and of course Queneau himself, and the group has been influential beyond its immediate membership. Work produced in an Oulipian spirit that happened to have been made before the foundation of the Oulipo is playfully termed "anticipatory plagiarism". This playfulness characterises much of the Oulipian output, and we'll see some examples as we go. The Oulipo can be seen as a kind of counterpoint to some of the work of the Surrealists, with their automatic writing and other methods of acquiring raw material from the human subconscious for new composition. It was also influenced by the work of "Bourbaki", a group of mostly French mathematicians writing anonymously under the name Bourbaki, with the ambitious aim of producing texts covering the entire architecture of mathematics in a uniform manner, proceeding from first principles. The books explained different mathematical structures in a very formal fashion, setting out axioms and definitions, then deriving the whole edifice of a particular abstract structure. This approach has a noble lineage, going back via Hilbert to Euclid. We will look here at a selection of constraints invented by the Oulipo. To set the scene, we mention a humorous potential axiomatisation of literature put forward in a 1976 article by Raymond Queneau, *Foundations for Literature (after David Hilbert)*. You can read it in [1], or see the discussion in [2]. Hilbert had given a set of axioms that allow us to consider several different kinds of geometry by changing the meaning of the words involved. For example, two of his axioms state the following.

1. Given two distinct points, there is always a line containing those points.
2. Given a line, any two points on the line uniquely determine that line.

In other words, any two points lie on one, and only one, line. This is true for standard Euclidean geometry, but the key insight is that by changing what we mean by words like *point* and *line*, we can see that the same axioms (and the consequences that derive from them) hold in other kinds of geometry. One example is the non-Euclidean geometry on the surface of a sphere. Here, our "points" are actually pairs of antipodal points, and "lines" are so-called "great circles" – such as the equator. With this interpretation, both the above axioms hold. But we can define much more esoteric geometries. One example is the so-called Fano Plane, shown. It consists of 7 points and 7 "lines" (the six straight lines, plus the circle, shown in the diagram).



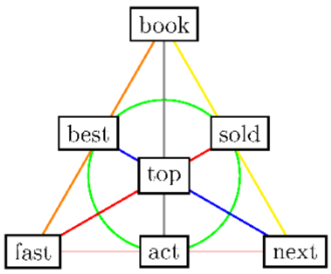
Every pair of points lies on exactly one line; every pair of lines intersects in exactly one point; every line contains exactly three points; every point lies on exactly three lines. It is a beautifully symmetrical object, and it obeys the two axioms above (and many others). In general, we can define a particular brand of geometry by saying it is a system of points, lines, and so on, that satisfies a particular set of axioms.

Queneau suggested that literary texts could be created in a similar way. Our "points" could be "words". Our "lines" could be "sentences". A particular literary structure could then be a text that satisfies a given set of axioms. For instance, the two axioms above would translate as

1. Given two distinct words in the text, there is always a sentence in the text containing those words.
2. Given a sentence in the text, any two words in the text uniquely determine that sentence.

Queneau points out that the text describing the axioms certainly does not satisfy the axioms. There are shades of Gödel here, in that one suspects that any axioms sufficiently sophisticated to describe a system cannot actually be stated within the system. This is not paradoxical, however, as we are not required to state the axioms governing a text within that text. In tribute to the Oulipo I have devised a piece of "Fano fiction" that satisfies the axioms of the Fano plane. (This may be an entirely new genre of literature, so I hope you are suitably grateful.) A work of Fano fiction uses a vocabulary of precisely seven words, and consists of seven sentences, each containing exactly three of the seven words. Each pair of words appear in exactly one sentence, and any pair of sentences have exactly one word in common. I have also stuck to the grammatical rule that a sentence must have at least one verb. The work tells the story of how you, a talent agency employee, were advised that it's best to get hold of the next big talent, and book her fast. A t-shirt line she endorsed flew off the shelves, and there was a bidding war for her autobiography. You encouraged her to write the next volume without delay, and to top her best previous achievements. She did so well that you could sell your share of the proceeds and retire a millionaire. Or, as the Fano version has it:

"Book top act! Best book fast!" Top sold fast. Next, book sold. "Act fast – next! Next: top best!" Best act: sold.



I never said it was great literature! Readers are encouraged to send me their (no doubt better) works of Fano fiction.

Lipograms

There are other potential axioms one could adopt to create constraints. Lipograms, for example, are texts in which one or more letters are forbidden. We could frame the constraint as an axiom or axioms stating which letters are forbidden. The best-known example of a lipogram is George Perec's 1969 *La Disparition* [3], which is a text obeying a single axiom: *the letter E is forbidden.* In fact, it was not the first E-less novel. That honour goes to Ernest Vincent Wright's 1927 book *Gadsby*, an act of anticipatory plagiarism if ever there was one. What raises Perec's work to a higher level is something remarked on by fellow Oulipian Jacques Roubaud, in a discussion of two often-followed Oulipian precepts: firstly, that a text written according to a given constraint must speak of this constraint in some way, and secondly, that if the constraint is mathematical in nature, then the text must make use of some mathematical consequences of the constraint. *La Disparition* observes the first of these rules. It is, says Roubaud, "a novel about a disappearance, the disappearance of the E; it is thus both the story of what it recounts and the story of the constraint that creates that which is recounted". Raymond Queneau suggests a measure of the "lipogrammatic difficulty" of a given text, based on the frequency distribution of letters in the language of composition. In English, for example, out of every 100 letters in a typical piece of text, there would be 2 W's and 13 E's. Thus, the frequency of W is 0.02, and the frequency of E is 0.13. (We can be more accurate if we wish: to five decimal places the frequency of W is 0.02360 and E is 0.12702.) The lipogrammatic difficulty of a text of n words which omits a letter of frequency f is defined to be $n \times f$. (It might be more precise to count the number of letters, rather than words, but one loses ease of calculation that way.) If all letters are allowed, then the difficulty is of course zero. The difficulty of composing a 100-word text without the letter W is just 2. The difficulty of composing a 300-word text in English without the letter E is much higher: 38 (to the nearest whole number).

We can use this idea to see how much of a challenge *La Disparition* would have been to write. The frequency of E in French (including variations like É, Ê and so on) is 0.16716. An 80,000-word novel (about the length of *La Disparition*) would therefore have difficulty 13,373! Although a work of the same length in English would theoretically have a difficulty of "just" 10,162, I have a profound respect for the translators of such works, who must meet the lipogrammatic challenge while simultaneously providing a faithful translation of the original text. *La Disparition* was brilliantly translated as *A Void* by Gilbert Adair [4]. Perec's sequel to *La Disparition* was *Les Revenentes* [5]– the only vowel it allowed was E. It's less than half the length of *La Disparition*, at an estimated 36,000 words. With our mathematical measure of lipogrammatic difficulty, we can ascertain which is harder: 80,000 words in French with no E, or 36,000 words with only E. Combining the frequencies (in French) of all variants of A, I, O and U gives a total frequency of 0.28018. The difficulty of *Les Revenentes* is therefore 10,086, slightly lower than *La Disparition*, but there's not much in it (especially as we don't have the exact word counts to work with). We can instantly see why *Les Revenentes* is shorter. Meanwhile, the English translation by Ian Monk, as The *Exeter Texts* (contained in [6]), has a lipogrammatic difficulty of $0.25398 \times 36{,}000 = 9{,}143$, though again this does not reflect the additional challenge of translation.

There are many other lipogrammatic texts. Another well-known example is *Eunoia* (2001), by Canadian author Christian Bök [7]. The main part of the book consists of five chapters, each of which use only one vowel (A, E, I, O and U in that order). The letter Y is omitted throughout. You may like to speculate about which of the chapters has the highest lipogrammatic difficulty. 'Eunoia' is the shortest word in the English language which contains all the vowels – it is a medical term for being in a state of good mental health. In French, there is a much nicer shortest word with all the vowels: *oiseau*.

Poetry

Poetry, of course, already had a long tradition of structural constraints before the Oulipo came along – as well as sonnets and haiku, there are sestinas, alexandrines, even limericks. What I have chosen to discuss here is a 1979 collection of poems by Paul Braffort: *Mes Hypertropes: Vingt-et-un moins un poèmes à programme* [8] (*My hypertropes: 21 minus 1 programmed poems*) I'm not aware that an English translation has been published, but excerpts of one, and a discussion of the construction of the text, are given in [2]. The reason for my choice is that these poems respect Roubaud's second dictum on structure, that any text making use of something mathematical should incorporate a mathematical consequence of that something. The something in question here is the Fibonacci sequence, and the consequence is a 1972 theorem by the mathematician Eduard Zeckendorf. Remember that the Fibonacci sequence begins

$$1, 2, 3, 5, 8, 13, 21, \ldots$$

(Some people prefer to begin it with two 1's.) Each term in the sequence is the sum of the two previous terms. For example, $13 = 5 + 8$. The next term in the sequence after 21 is therefore $13 + 21 = 34$. The Fibonacci sequence has many interesting properties. One is that the sequence $\frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \frac{21}{13}, \ldots$ of ratios of consecutive terms converges to the golden ratio $\frac{1+\sqrt{5}}{2} \approx 1.618$.

Zeckendorf's result is about ways to break down numbers (specifically positive whole numbers) as sums of Fibonacci numbers. Trying to express numbers in terms of special things such as squares or primes is a long-standing area of research for number theorists. For example, Joseph Louis Lagrange proved in 1770 that every positive integer can be expressed as the sum of at most four perfect squares ($23 = 3^2 + 3^2 + 2^2 + 1^2$ for example). The Goldbach conjecture, that every even number greater than 2 is the sum of two primes, is still unsolved. Such expressions are not necessarily unique – for instance we have $34 = 17 + 17 = 23 + 11$. There are instances of unique expressions in terms of particular sequences. We can write any positive integer in exactly one way as a sum of powers of ten, for example – we do this every day without thinking in the way we write numbers: $21 = 2 \times 10 + 1 = 10^1 + 10^1 + 10^0$. In binary it's even better: every positive integer can be expressed in exactly one way as a sum of *distinct* powers of 2. That is, every power of 2 is used at most once in the sum, with no repetitions. What Zeckendorf managed to prove is that every positive integer can be expressed in exactly one way as a sum of *distinct, non-consecutive* Fibonacci numbers. We need the non-consecutive bit to get uniqueness, because otherwise any Fibonacci number in the sum could be replaced by the two immediately preceding it, to get things like 35 = 1 + 34 = 1 + 21 + 13. It was only while researching this article that I learned that in fact Zeckendorf's Theorem appears to have been independently proved some two decades earlier by the Dutch mathematician Gerrit Lekkerkerker. However, though he wrote about it, he didn't publish a formal paper on it, so this may be the reason why he was not credited. Zeckendorf published his paper on Fibonacci numbers in French, while Lekkerkerker wrote in Dutch, which is another reason why Braffort was more likely to have encountered Zeckendorf's proof. Neither of them seems to have been aware of Lekkerkerker's work, at any rate.

So, having established that any positive integer has a unique Fibonacci decomposition in terms of non-consecutive Fibonacci numbers, how did Braffort use this decomposition in his *Hypertropes*? There are 20 poems (which is 21 minus 1). Each non-Fibonacci numbered poem references characters, words, or imagery from the poems corresponding to the Fibonacci numbers in its Zeckendorf decomposition. Thus, Poem 7 involves allusions to Poems 2 and 5, while Poem 17 echoes themes from Poems 1, 3 and 13. Poems whose numbers are already Fibonacci numbers use the property that the Fibonacci number is the sum of the two previous Fibonacci numbers. Thus, Poem 13 refers to Poems 5 and 8. Only Poem 1 does not have any "programming". This is all rather subtle, but Braffort leaves several clues – the title of the work uses two Fibonacci numbers, 1 and 21, instead of just saying there are 20 poems, and there are other hints in the text itself, including line endings that make the sounds Fi, Bo, Nach, Chi.

Geometric Progression and The Luminaries

One of the criticisms that is sometimes levelled at the Oulipo is that the constraints and structures imposed only serve to produce clever intellectual puzzles rather than great writing. There are many ripostes to this. Among them is that the Oulipo is a workshop of *potential* literature – its purpose is to provide possible structures, that is, not to provide the literature. Another is to remark that there's nothing intrinsically wrong with clever intellectual puzzles, which can be most enjoyable. We can also say that there are an awful lot of dreadful sonnets; it does not mean that the sonnet form is in itself bad. As Queneau said, "[w]e place ourselves beyond aesthetic value, which does not mean that we despise it". Oulipians such as Queneau, Georges Perec, and Italo Calvino have certainly produced art that has stood the test of time. However, I wanted to give an example of the successful use of a mathematical constraint in a recent mainstream work of literary fiction. *The Luminaries*, by New Zealand author Eleanor Catton, won the Booker Prize in 2013. Set in 1860s New Zealand, the novel begins with a prospector, Walter Moody, arriving in the gold-mining town of Hokitika. He becomes embroiled in a mystery involving murder, opium, love affairs, and £4,096 worth of stolen gold. Catton made use of several structural constraints in the novel. It contains twelve chapters, each of which takes place on a single day over the course of 1865-66. Twelve characters are each associated with a specific sign of the zodiac; their actions in a given chapter are in part determined by the sign's astronomical position on the day the chapter's events unfold. Seven further characters are associated with astronomical bodies (Walter Moody is Mercury, for example). Each chapter is divided into parts such that the chapter number plus the number of parts equals 13. Thus, Chapter 1 has twelve parts, Chapter 2 has 11 parts, and so on to Chapter 12, which has just one part. The most mathematical, and most impressive, of the constraints is that each chapter is half the length of the last.

Let us explore the implications of this length constraint. If the first chapter has length L, say (measured in words, or lines, or pages, whatever we prefer), then, if there are twelve chapters, the total length of the book is

$$L\left(1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\cdots+\frac{1}{2048}\right).$$

This sequence $1,\frac{1}{2},\frac{1}{4},\frac{1}{8},\cdots$ is an example of what's called a geometric progression, where each term is a fixed multiple of the last. In this instance that multiple is $\frac{1}{2}$. There's a really nice way to find the sum of a geometric progression. Suppose the fixed multiple (called the common ratio) is r. Then the sequence is $1, r, r^2, r^3, \ldots$. Now, suppose we want to add up the first n terms. The n^{th} term is r^{n-1}, as we start with $r^0 = 1$. But

$$(1-r)(1+r+r^2+\cdots+r^{n-1}) = (1+r+r^2+\cdots+r^{n-1}) - (r+r^2+\cdots+r^{n-1}+r^n)$$

$$(1-r)(1+r+r^2+\cdots+r^{n-1}) = 1-r^n$$

$$1+r+r^2+\cdots+r^{n-1} = \frac{1-r^n}{1-r} \quad \text{(as long as } r \neq 1\text{)}.$$

So, with $n = 12$ chapters and $r = \frac{1}{2}$, we have that the book length is $L\left(\frac{1-(1/2)^{12}}{1-(1/2)}\right) = 2L\left(1-\frac{1}{4096}\right)$. There's that £4096, by the way! The interesting thing is that when $0 < r < 1$, as we have here, r^n tends to zero as n tends to infinity, so the sum of the progression is limited above by $\frac{1}{1-r}$. This means that the book length is bounded above by $L\left(\frac{1}{1-(1/2)}\right) = 2L$. However many chapters the book has, more than half the book consists of the first chapter! Now let's look at things from the other direction. Write S for the length of the shortest chapter. If the book has n chapters, then $L = 2^{n-1}S$. So, the book length is $2^{n-1}S\left(\frac{1-(1/2)^n}{1-(1/2)}\right) = (2^n - 1)S$. Clearly, the length of chapters cannot be being measured in pages, because if $S = 1$ and $n = 12$, we would get $4{,}096 - 1 = 4{,}095$ pages! *The Luminaries* is long, at 832 pages, but not that long. If we estimate that a standard book should not exceed 1,000 pages (it's difficult to bind anything much longer), and each page of standard text has about 400 words, then that gives an upper bound of 400,000 words. Most novels are considerably shorter than this! If our shortest chapter has only 100 words, then a twelve-chapter book would already be 409,500 words. The shortest chapter of *The Luminaries* has 95 words, which means that there cannot be more than twelve chapters. For an absolute upper bound, even if the shortest chapter contained just one word, then the maximum value for the number n of chapters is obtained by solving $2^n - 1 \leq 400{,}000$. We get $n \leq 18$. Those extra six final chapters would be extremely short, having only 63 words in total between them.

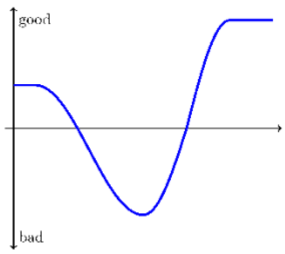
One reason why the use of this geometric progression is successful is because it is not a random choice, as for instance requiring every twelfth word to contain exactly three vowels would be. The successive halving of the chapter lengths echoes the waning moon, so tying in with the astronomical and astrological themes of the book; it adds impetus to the narrative as we feel the inexorable tightening of the focus. The other characters gradually fall away, and by the final, one paragraph, chapter we are left only with an intimate conversation between the two lovers, who represent the sun and the moon. This is at the heart of the entire novel.

Plot and narrative

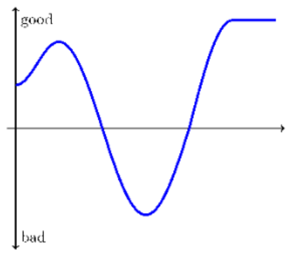
The second sense in which mathematics can contribute to the structure of a story is in terms of its plot, and we will meet several examples in this section.

Graphs, choice, and randomness

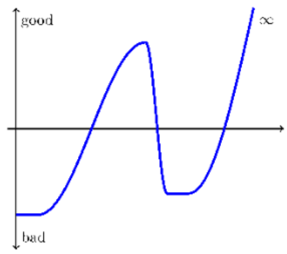
Many authors will produce diagrammatic representations of the plots of their novels, and these can be mathematical to varying degrees. An amusing illustration of the graphs of some possible stories was given by Kurt Vonnegut [9] in a public lecture in 2004. The last one, "Metamorphosis", refers to the Franz Kafka story of that name.



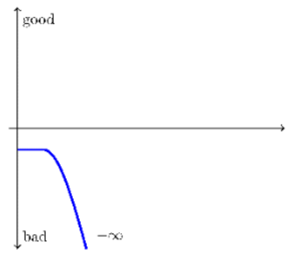
"Man in a hole"



"Boy meets girl"

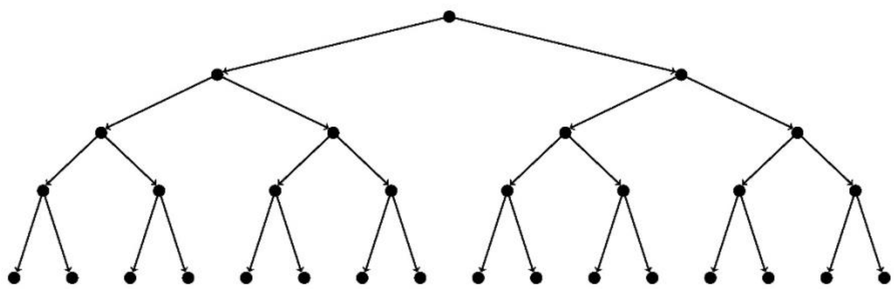


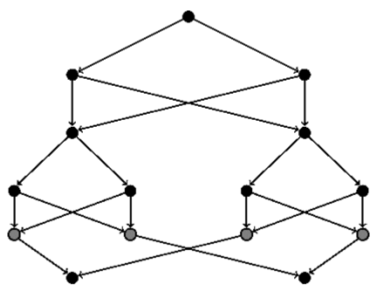
"Cinderella"



"Metamorphosis"

(Recall also the diagram in *Tristram Shandy* of the narrative path to that point in the novel!). Real graphs can, though, be used in plots, in particular where writers choose to make available several paths through a story or poem, either through randomisation or the reader's choice. The graphs we are talking about are networks with vertices, or nodes, joined by edges. One example is the "theatre tree" devised by Paul Fournel and Jean-Pierre Énard. The idea is that a play is performed, and at the end of each scene the actors ask the audience to choose between two possible developments. For example, a masked man dramatically enters the stage. The audience is asked to decide whether he is the king's son, or the queen's lover. A different scene is performed based on each outcome. This is all very well, but the number of scenes the poor actors have to learn soon becomes unreasonable, never mind all the different props, costumes and sets required. If we wish the audience to make four choices, then there will be five scenes. There have to be two different versions of Scene 2, four Scene 3's, 8 Scene 4's, and 16 Scene 5's, which is 31 scenes in all. Fournel explained that their suggested alternative, which is indistinguishable from a model with many more scenes, in fact only requires 15 scenes to be learnt (as long as the audience members only attend one performance of the play). The full tree (requiring 31 scenes) is shown on the left below; the theatre tree is on the right. Note that although there are four choices still made and five scenes, the two possible final scenes (happy ending or sad ending) each have two variant first halves depending on the route to the scene. These are shown as four grey circles in the diagram.


Full Tree


Theatre Tree

The theatre tree contains different 15 scenes (or half-scenes), and four choices are made, thus there are 16 possible plays. Fournel says that to perform 16 five-scene plays normally you'd have to write 80 scenes. Therefore, graph theory has created a saving of 65 scenes!

This has not perhaps caught on in theatres, but precisely this kind of mixture of free and constrained choice is used very effectively in things like choose-your-own-adventure books, where there are hundreds of choices to be made (to ask the wood sprite's help, turn to page 74; to steal the magic potion instead, turn to page 112), or in interactive TV shows, where there is necessarily a small number of genuinely different paths through the programme, as actors cannot be asked to shoot 500 variants of a story – they tend to have better union representation than wood sprites.

Another way to direct readers through a narrative is by giving instructions on how it is to be read. I'll just give two examples, one mathematical and one not. Reverse poems are poems that are read the "right way" first, and are followed by the instruction "now read this poem backwards". Often, the first, top to bottom, reading of the poem is pessimistic, but then the reverse version is uplifting. Well-known examples include *Refugees*, by Brian Bilston, and *Lost Generation*, by Jonathon Reed. The latter begins as follows.

> I am part of a lost generation
> And I refuse to believe that
> I can change the world

When you read the poem backwards, these lines transform into an optimistic ending.

> I can change the world
> And I refuse to believe that
> I am part of a lost generation

Both are easily found online for those interested to read them in full. A more mathematical example is *Frame-Tale* (1963) by John Barth. The entire text is a single page, printed on both sides, with the directions "cut on dotted line, twist end once and fasten AB to ab, CD to cd". On the first side of the part to be cut out is the text "ONCE UPON A TIME THERE", and on the other side is "WAS A STORY THAT BEGAN". But when the story is assembled, one can follow the loop with one's finger seamlessly, forever, so the story is an infinite regress of stories: Once upon a time there was a story that began, 'Once upon a time there was a story that began, 'Once upon a time…". A frame-tale, by the way, is a story within a story. The shape created when you follow the instructions is the well-known Möbius Strip, which famously only has one side. Other writers, such as Gabriel Josipovici, have made more sophisticated use of this shape. In Josipovici's story *Mobius* [sic] *the Stripper*, the text is divided throughout into a top half and a bottom half. One can either read the top half first followed by the bottom half, or vice versa. The clever part is that events in one half leak through into the other half. The halves are related, mimicking the fact that in a physical Möbius strip there is at any point a matching point on the reverse side.

The reader can choose from two possible paths through *Mobius the Stripper*. But there have been poems and books where much more choice is allowed. The most famous Oulipian example is Raymond Queneau's 1961 *Cent Mille milliards de poèmes* (a hundred thousand billion poems). It consists of ten fourteen-line sonnets, printed on ten consecutive sheets. All the first lines rhyme with each other, all the second lines rhyme with each other, and so on, in a sort of three-dimensional poem. The sheets can be cut horizontally after each line, such that new sonnets can be constructed by selecting any of the ten possible lines at each stage. The total number of possibilities is 10^{14}. The upshot is that what appear to be ten poems are actually 100 thousand billion (or as we would now say, 100 trillion). It is an interesting philosophical question, the extent to which each of these poems exists or was written by Queneau. If you read a different sonnet every minute, non-stop, it would take 190,128,527 years to read the whole ten-page book. (According to [10], Queneau calculated 190,258,751 years, which worried me at first, but then I realised that this is the answer you get if you assume one sonnet per minute, but forget about leap years. Perhaps he was allowing a day off every February 29th.) It is certainly an excellent example of "potential literature". Of course, 10^{14} is tiny. In 1969, B. S. Johnson published *The Unfortunates*. It is an unbound "book in a box". There are 27 chapters: a first and last chapter, along with 25 intervening chapters that can be read in any order. This means there are 25 choices for your second chapter, 24 for your third, 23 for your fourth and so on, until there is just one possible 26th chapter left. The total number of ways to read the book is therefore

$$25 \times 24 \times 23 \times \cdots \times 2 \times 1 \approx 1.5511210043330985984 \times 10^{25} \approx 15.5 \text{ septillion.}$$

There is another stage at which choice can be ceded by the author, and that is in the construction of the text. An example of this is what's called the $N + 7$ rule. The process is to take a text, but then replace every noun by the seventh noun following it in the dictionary. This is not a random process, because words close in the dictionary often have a common root. Starting at the word "college" in my Chambers dictionary, the next seven nouns are colleger, collegialism, collegiality, collegian, collegianer, college pudding, collegiate church. "The college is near Chancery Lane," under $N + 7$ would become "The collegiate church is near Chandlery langouste". It certainly makes for a more interesting mindscape – if you don't mind a spiky lobster in your candlemaker's shop!

As well as graphs, other mathematical structures have been deployed in the design of texts. Claude Berge, Jacques Roubaud and Georges Perec proposed the use of what Berge refers to as bi-Latin squares, and what we now call orthogonal Latin squares, or sometimes Graeco-Latin squares, to construct stories. A Latin square is an $n \times n$ grid each of whose entries is an element from some set of size n (usually, for convenience, the numbers 1 to n), such that every element appears exactly once in each row and column. Sudoku grids are an example. Bi-Latin squares are a more complex design where two Latin squares are superimposed in such a way that each possible combination of pairs from the two sets occurs exactly once. For example, we could try to arrange the four highest cards (Jack, Queen, King, Ace) of each suit (Hearts, Diamonds, Clubs, Spades) in a deck of playing cards in a four by four square array, such that each row and column contains exactly one Jack, one Queen, one King, and one Ace, and exactly one representative of each suit. (The fact that we only have one of each different card will ensure the final requirement of each pair occurring exactly once.) One solution is shown.

A ♠	K ♥	Q ♣	J ♦
K ♦	A ♣	J ♥	Q ♠
J ♣	Q ♦	K ♠	A ♥
Q ♥	J ♠	A ♦	K ♣

There are 144 different solutions to this problem (different means here that the solutions cannot be transformed into each other by one of the eight reflections or rotations of the square; if you view each of those as different, then there would be $144 \times 8 = 1152$ solutions). The first person to correctly classify the solutions was the British mathematician Katherine Ollerenshaw.

Euler tried to determine for which values of n these orthogonal Latin squares exist. He proved that there are no examples when $n = 2$ and $n = 6$, and he conjectured in 1782 that there are also no orthogonal Latin squares of order 10, 14, 18 and so on. This was only disproved in 1959, when E. T. Parker, R. C. Bose, and S. S. Shrikhande showed that orthogonal Latin squares exist for all orders $n > 2$ except $n = 6$. This result was covered by Martin Gardner in the November 1959 edition of Scientific American, whose front cover features a 10×10 counterexample to Euler's conjecture. Thus, this mathematics was pretty recent when Perec's *La Vie Mode d'Emploi* [11] was published in 1978 (translated by David Bellos as *Life, a User's Manual* [12]). The narrative takes place in a building of ten floors with ten rooms on each floor. The stories of the 100 rooms of the building each feature unique combinations of different characteristics taken from lists of ten – for instance there was a list of ten fabrics. To make it even better, the stories are told consecutively by following an order given by a knight's tour of a 10×10 chessboard. When Perec was describing the construction of the book, he remarked, rather cryptically, that it should be noticed that the book has not 100 chapters, but 99. "For this the little girl on pages 295 and 394 is solely responsible [pages 231 and 318 in the English translation]."

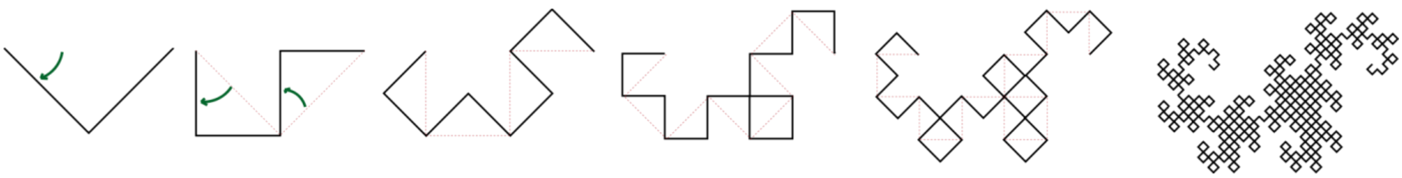
Mathematics as a plot device

There are a few novels where the construction of the plot itself is informed by a particular mathematical concept. From least to most esoteric, I refer you to *Jurassic Park* by Michael Crichton [13], the murder mysteries of Catherine

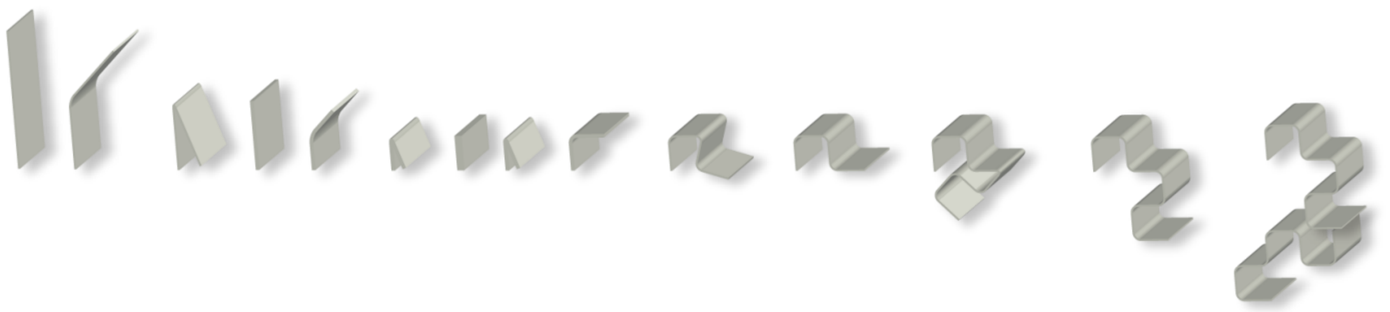
Shaw, and the detective story *Qui a tué le Duc de Densmore*, by Claude Berge (available in translation as *Who killed the Duke of Densmore* in [1]).

Jurassic Park is the story of an avaricious inventor who exploits the newly discovered techniques of genetic engineering to bring dinosaurs back to life – not for noble scientific reasons, but to make money by opening a theme park. One of the themes of the novel is that complex systems can grow in ways that we cannot predict, and that what may seem simple to begin with can end up much more complicated than we can understand or control. In the story, the park owners hubristically believe they have tamed nature, but of course nature cannot be controlled and I'm sorry to have to tell you that some people do get eaten by velociraptors.

As an illustration of complex things coming out of simple ones, Crichton divides his book into seven "iterations", and the first page of each section shows one iteration in the construction of the fractal curve known, appropriately enough, as the dragon curve. It is produced by starting with a simple image created using straight lines and right angles, and then the same procedure is followed at each step to produce, iteratively, a very complex design. The iterative step is at each stage to replace a given line segment by two shorter line segments, such that the original segment and the two new shorter segments form an isosceles right-angled triangle. The first few steps, and the ninth step, are as shown. (Note that you alternate whether the replacing happens on the right or the left of the original segment.)



Crichton's "first iteration" is actually the fourth step above, his second is the fifth, and so on. The dragon curve can also be thought of as the limiting process of repeatedly folding a strip of paper on itself to create a series of creases, then unfurling the strip, as shown overleaf.



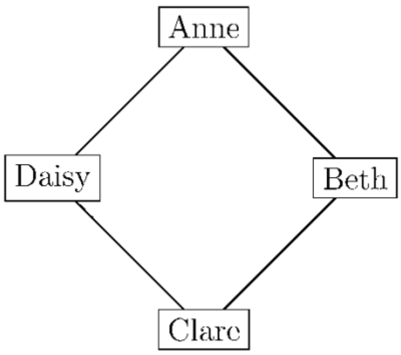
An extremely simple act like folding a piece of paper, which you would imagine to be completely predictable, actually looks totally chaotic, and of course this reflects the chaos that ensues in the novel.

Catherine Shaw is our next example. She is the author of a series of detective novels set in Victorian England and featuring a mathematically-minded amateur sleuth called Vanessa Duncan. Mathematicians and mathematical ideas feature in several ways in the plots. Since I want to mention one of these plot devices without spoiling the story within which it is contained, I shall not reference particular books, but they are easy to find online if you know that Catherine Shaw is the pseudonym of mathematician Leila Schneps (for example, the books are linked from her Wikipedia page).

The mathematical concept in question is the idea of non-transitivity. To explain what that is, we need to know about transitivity. Relationships between objects in a set, such as "being bigger than", "being equal to" for numbers, and "being congruent" for shapes, often have the property called transitivity. For "greater than", for instance, it's clear that if $a > b$ and $b > c$, then $a > c$. If $a = b$ and $b = c$, then $a = c$. If my husband is taller than my daughter, and my daughter is taller than me (sad but true), it follows that my husband is taller than me. Imagine now that we have a set of three slightly unusual dice. They have six sides like normal dice, and are "fair" in the sense that the probability of any one side coming up is $\frac{1}{6}$. But the numbers on the sides are different. Die A has sides 2, 2, 4, 4, 9, 9, Die B has 1, 1, 6, 6, 8, 8, and Die C has sides 3, 3, 5, 5, 7, 7. If we play a game where I roll Die A, you roll Die B, and the higher score wins, then on average who will win?

We can calculate the possible outcomes – there are nine (I roll 2, you roll 1, I roll 2, you roll 6, and so on). In 5 of the 9 cases, Die A wins. Therefore, Die A beats Die B (in the long run). Similarly, if we use Die B and Die C, then Die B beats Die C in 5 of every 9 games. So, A beats B and B beats C. It seems obvious that A must therefore beat C. But if we do the calculation, it turns out that actually C beats A, again $\frac{5}{9}$ths of the time! This means that these dice are *non-transitive*. Now, if we replace dice by people and the word "beats" by the word "murders", then things get interesting. How could A murder B, B murder C, and C murder A? Catherine Shaw finds an ingenious solution.

We stay with detective fiction for our final example. The Oulipo had several offshoots; a field X can have an Ou-X-po – a workshop for potential X. So, there is an Oubapo for Bandes-Dessinées (cartoons), an Oupeinpo for peinture (painting), and so on. Obviously the next step is an Ou-ouXpo-po, and then one would need an ou-ou-ouXpo-po-po…but I digress, as usual. The Oulipopo is the Ouvroir de Littérature Policière Potentielle: the workshop for potential detective fiction, and the detective story we will discuss is Claude Berge's *Qui a tué le Duc de Densmore?* (*Who killed the Duke of Densmore?).* This story, like Braffort's *Hypertropes* earlier, makes use not only of a mathematical idea, but of a theorem concerning that idea. In summary, a detective is trying to solve an old case – the murder of the Duke of Densmore. He interviews several suspects. Each says they paid a single visit of a few days to the Duke's house around the time of the murder, but given how much time has passed, they aren't able to remember exact dates. However, they can remember who they met there during their visit. If two people met, then of course their stays must have overlapped. What this gives us is a collection of intervals of time such that we don't know the intervals, but we do know which ones overlap. We can use this to construct what's called an *interval graph*. The vertices of the graph correspond to the intervals, and we draw an edge between two intervals if they overlap. Suppose for example that Anne, Beth, Clare and Daisy all visited; Anne says she met Beth and Daisy, Beth says she met Anne and Clare; Clare says she met Beth and Daisy, and Daisy says she met Clare and Anne. The corresponding graph is this.



But wait! There is a theorem in graph theory that says every interval graph is chordal – which means that every cycle of length greater than 3 must contain at least one chord. This means that at least one of these women is lying! With this toy example we can't tell who is lying, but in the Berge story, there are more suspects and the graph obtained has the property that there is a unique vertex which, if removed, results in a true interval graph. Therefore, the suspect corresponding to that rogue vertex was the one lying, which makes them the probable murderer. The detective in the story knows the relevant theorem on interval graphs, and uses it to crack the case.

The Structure of a Story

The final interpretation of "the structure of a story" gives me the chance to discuss in slightly more detail a story by a much loved and mathematically-minded writer – Jorge Luis Borges. The structure in question is the Library of Babel, from the story of the same name. My quotations are from James E. Irby's English translation in [14]. There are many mathematical allusions in Borges's writing. For two other examples one could pick *The Lottery of Babylon* with its discussion of chance and infinite series, and *Avatars of the Tortoise*, an essay on (among other things) infinity and Zeno's paradox. But *The Library of Babel* is probably Borges's most famous story. It begins as follows.

"*The Universe (which others call the Library) is composed of an indefinite and perhaps infinite number of hexagonal galleries, with vast air shafts between, surrounded by very low railings. From any of the hexagons one can see, interminably, the upper and lower floors. The distribution of the galleries is invariable. Twenty shelves, five long shelves per side, cover all the sides except two [..]. One of the free sides leads to a narrow hallway which opens on to another gallery, identical to the first and to all the rest. To the left and the right of the hallway there are two very small closets. [These are for sleeping and other physical requirements.] Also through here passes a spiral stairway, which sinks abysmally and soars upwards to remote distances.*"

Each hexagon, we are told, contains 20 bookshelves. Each bookshelf contains 32 books identical in format (the translation unfortunately says 35 but the original Spanish, when I checked it, says 32); each book contains 410 pages; each page, 40 lines; each line, 80 characters. The spine of the book also has characters. It is not specified how many characters, but since the pages have 40 lines, there is presumably room for 40 characters on the spine, as these would be written vertically, so we assume 40 characters are available (some may be blank spaces, of course). There are 25 characters: 22 alphabet letters, along with the comma, the full stop, and the space. The actual alphabet is not directly specified.

The inhabitants of this universe wander through the library and try and understand its structure. They postulate that the entire library has the structure described, and that therefore it is endless – but no two identical books have ever been discovered, and they believe it is a law of the universe that no two books are identical, and moreover that every possible book (with the specified dimensions) exists in the library. This bounds the size of the library, as we shall see, but therein lies a seeming paradox: if the structure stretches out endlessly in all directions, how can it also be finite?

For a much more in-depth discussion of these and other mathematical questions arising from *The Library of Babel*, I recommend the excellent book by William Bloch [15]. We will restrict ourselves here to calculating the number of books in the library, and one or two brief comments on its possible structure.

To calculate the number of books, recall that there are 25 possibilities for each character. The number of different sequences of n characters is therefore 25^n. To see this, a much smaller example is perhaps useful. Suppose we only have 3 characters, A, B and C. There are three sequences of 1 character: A, B, or C. For a sequence of two characters, there are nine possibilities, AA, AB, AC, BA, BB, BC, CA, CB, CC. For each of the three choices for the first character we have three choices for the second character. Similarly, there would be $3^3 = 27$ sequences of 3 characters, $3^4 = 81$ sequences of 4 characters, and in general 3^n sequences of n characters. Thus, with 25 options for each character, there are 25^n books with n characters. We are told that each book has 410 pages, each with 40 lines of 80 characters. Therefore, inside the book there are $410 \times 40 \times 80 = 1{,}312{,}000$ characters, and hence $25^{1{,}312{,}000}$ possibilities for the contents of a book. However, this ignores the spines of the books. Those supply, by assumption, another 40 characters, which means we have $25^{1{,}312{,}040}$ books in the library. This is a vast number. The nearest power of ten to it is $10^{1{,}839{,}153}$. Given that our universe contains an estimated 10^{80} atoms, the library certainly doesn't exist in our universe, and the universe where it does exist must be vastly larger than our own.

Is this a whole number of hexagons? Each hexagon has 640 books. But the number of books is a power of 25. In particular, it is not an even number, so cannot be a whole number multiple of 640! Bloch suggests that one way to deal with this might be to change the number of characters allowed to 28, and make each shelf hold 49, not 32, books. But I prefer to keep the restrictions stated in the story, as far as possible. Is there anything we can do? The only wriggle room is the lettering on the spine. Changing the number of letters doesn't help, because we still get a power of 25, but changing the number of characters allowed just on the spine may help. Book titles don't typically have full stops, so my suggestion is that we suppose only 24, not 25, characters are available for the lettering on the spine. Noting that $640 = 5 \times 128 = 5 \times 2^7$, this would give the total number of books as

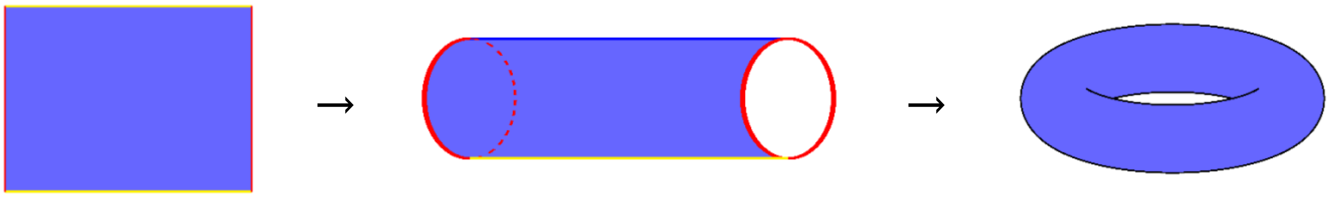
$$25^{1,312,000} \times 24^{40} = (25 \times 25^{1,311,999}) \times (2^{40} \times 12^{40}) = 640 \times (5 \times 25^{1,311,999} \times 2^{33} \times 12^{40}).$$

An alternative suggestion is that, since one of the example titles given in the story is *Axaxaxas mlö*, there may be more than 25 choices for the characters on the spine – just adding one more choice, for 26, will also give a factor of 2^{40} in the final book count. Either way, we end up with a whole number of hexagons.

Does the library contain all possible Earth books? Yes, if we allow longer ones to be split into different volumes, and devise conventions for representing non-standard symbols or pictures in some way (for example "pi" for "π"). You would need three volumes for *The Luminaries*, but just part of one volume for this transcript. You could fill the rest of the volume with the transcripts of my remaining Gresham lectures. Unfortunately, the library also contains a large number of volumes with things that look like the transcripts, but contain egregious mathematical errors and spelling mistakes. If you find any of these errors in what you thought was a real transcript, perhaps you have simply borrowed the wrong volume from the library. We can also amuse ourselves with questions about whether there is a book in the library that contains a list of all books that do not refer to themselves, and whether that book refers to itself.

But, back to the main point. The conclusion from these calculations of possible books is that the total number of hexagons, though enormous, is finite. How, then, can the library continue indefinitely in the way described? What is the structure of this library? We have a lot to go on. The presence of the ventilation shaft passing through the centre of each hexagon means that the vertical layers of the structure have the hexagons in exactly the same configurations – but what about the walls, and in particular the two walls that do not contain bookshelves? At least one of these has a spiral staircase, again going up and down as far as the eye can see. All hexagons are identical in design, so either both have the staircase, or exactly one wall of every hexagon has a staircase. The latter possibility seems unlikely because it would imply just a single pair of linked hexagons on each level, whereas the story refers to instances of travelling "a few miles to the right". The appropriate interpretation seems to be that both the free walls have a spiral staircase and a link to an adjacent hexagon (though perhaps only one of the walls has the vestibule with sleeping area and other facilities). The spiral staircases, along with the ventilation shafts, force the same floor plan to be replicated on all levels. There are infinitely many possible floor plans – if our free walls are opposite each other, we get essentially a straight line, but if not, then the possibilities are much more extensive; we also have to decide whether we require the library to be connected in that all hexagons are in theory reachable from all others.

Just as the circumference of a circle is a one-dimensional space that, while finite, can be traversed forever without reaching the end, and the surface of a 3-dimensional sphere is a two-dimensional space with the same property, one solution for the geometry of the library is that it is the 3-dimensional surface of a 4-dimensional sphere. Another solution, which I find more intuitively appealing, perhaps because of a mis-spent youth, is what you could call the "space invaders" solution. In some old video games where memory limits required a certain ingenuity on the part of the programmers, when your spaceship exited the screen from one edge, it would simply reappear at the corresponding point on the opposite edge. Topologists have no problem creating spaces like this – they simply decree that, for example, the top boundary of the screen is the same set of points as the bottom boundary and identify those two edges with each other. We can make a slightly distorted version of this in three dimensions by wrapping the screen around on itself top to bottom to create a cylinder (a bit like we did with the Möbius strip before, but omitting the twist). The vertical stairs and ventilation shafts do now continue indefinitely along the paths of the circles now created. But what about the horizontal levels? We can have the exact same solution horizontally too. In other words, when our spaceship exits the screen on the left, it reappears on the right. We do this by decreeing that the left-hand edge be identified with the right-hand edge. Again with some distortion, we can visualise this as bending our now-cylinder round horizontally, to create a so-called torus (or doughnut, but one of the ones with a hole in the middle rather than jam!). There are ways to create this space-invaders surface without distortions, but they require you to embed the surface in 4-dimensional space, so I will not go into details. The diagram shows, on the left, the rectangle made into a surface by identifying the top and bottom edges (shown in yellow in the colour picture) – this gives the cylinder in the centre – and then the left and right edges (shown in red). A visualisation of this surface as a torus is shown on the right (though it distorts some of the lengths).



There are infinitely many potential libraries whose geometries one could explore. Clearly, what we need is an Oubipo (Ouvroir des Bibliothèques Potentielles), to take the discussion further.

Further Reading

There are many more links between mathematics and fiction – some of which were explored in my previous Gresham lecture, *Mathematical Journeys into Fictional Worlds* [16]. For some other starting points, I suggest Tony Mann's excellent paper on the uses of mathematics in fiction [17], and Alex Kasman's extensive online mathematical fiction catalogue [18]. For collections of English translations of Oulipo texts, see [1], [10] and [19].

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