Mathematics of Currency and Exchange:

Arithmetic at the end of the Thirteenth Century

Norman Biggs

London School of Economics

British Society for the History of Mathematics

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1. Money and trade

Part of the Cuerdale Hoard, early tenth century

Money can take many forms. Jevons (1872) distinguished between

- a medium of exchange (that is *money-objects*, such as coins, hacksilver, gold dust, cowrie shells); and

- a measure of value (that is *accounting units*, such as pounds, shillings and pence).

Jevons argued that money was needed because barter was inconvenient.

From an eleventh century ms, published by Salzman
English Trade in the Middle Ages (1931) with the title
A SIMPLE BARGAIN

Coins, like most other money-objects in medieval times, were valued by the amount of precious metal that they contained.

Two measurements were needed:

the mass, which could be found by weighing;

the fineness which could be found by assaying, or by reference to an authority.

Weighing was normal in trade.

A twelfth century Islamic story [Bib. Nat., Paris]

By 1300 there were many gold coins circulating in Europe, all with different mass and fineness.

1: Dinar from Alexandria (1196-1218).

2: Morabetino of Alfonso of Castile (1158-1214), imitating a dinar.

3: Augustale of Frederick II as King of Sicily (1197-1250)

4: Florin, minted in Florence (1252 onwards).

A Florentine banker, Pegolotti made a list giving the fineness. His notebook was compiled c1300, and printed in 1936 as *La Pratica della Mercatura*.

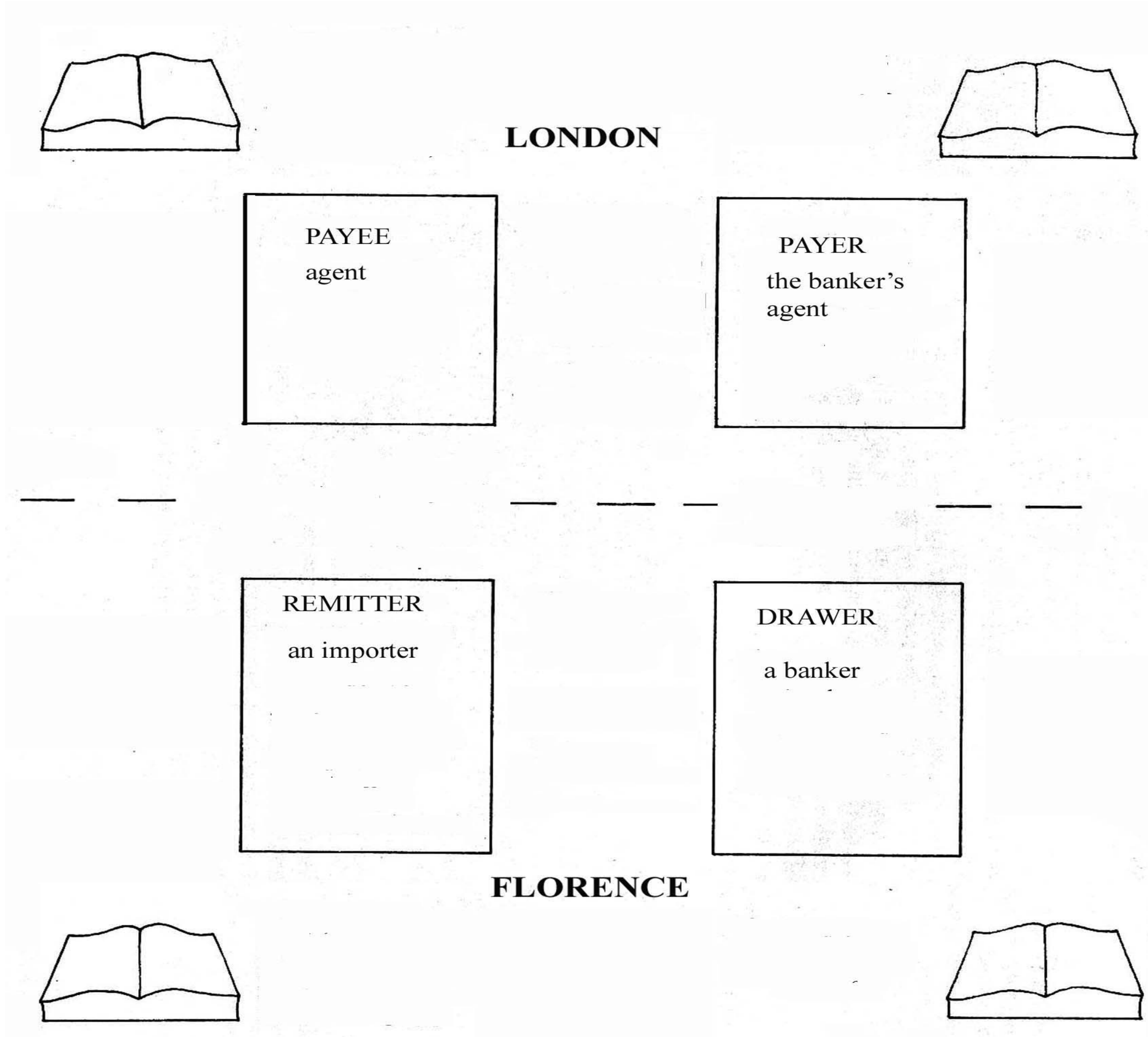
LEGHE DI MONETE D'ORO

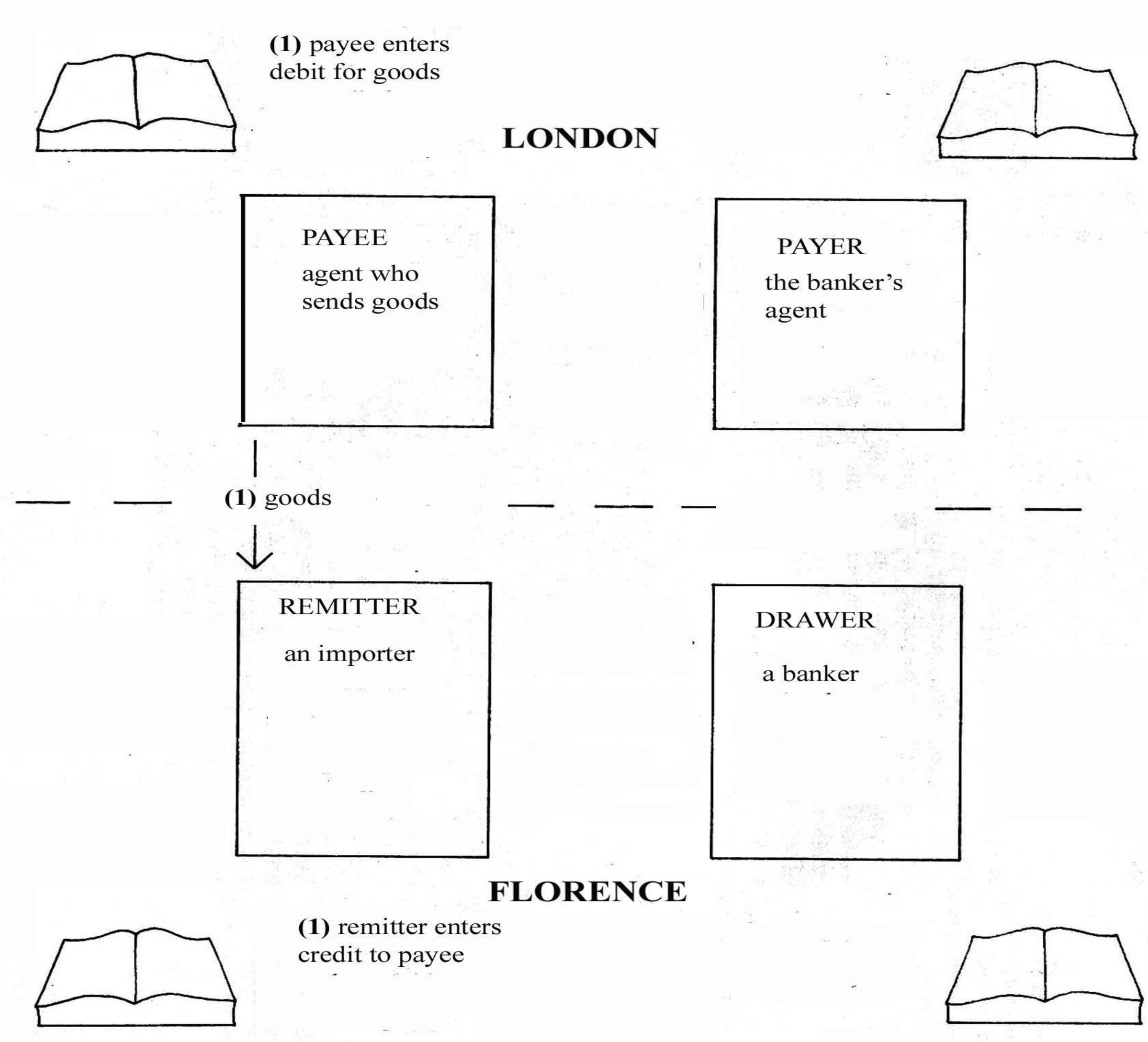
Fiorini d'oro sono a carati 24 d'oro fine per oncia.
Ducati d'oro a carati 24.
Gienovini d'oro vecchi a carati 23 1/4.
Gienovini d'oro nuovi a carati 24.
Lucchesi d'oro a cavallo, a carati 23 7/8.
Lucchesi d'oro a piede, a carati 23 3/4.
Carlini d'oro a carati 23 e 7/8.
Ragonesi d'oro di Cicilia a carati 23 e 7/8.
Romanini d'oro a carati 23 e 3/4.
Parigini d'oro a carati 23 e 3/4.
Dobbre da Rimirra d'oro a carati 23 3/4.
Dobbre d'oro di Morrocco a carati 23 3/4.
Castellani d'oro a carati 23 3/4.
Anfusini[3] d'oro vecchi a carati 20 1/2.
Anfusini[3] d'oro nuovi a carati 20 1/2.
Bisanti vecchi d'oro d'Allessandria piue nuovi a carati 23.
Bisanti vecchi d'oro d'Allessandria meno nuovi a carati 23 1/2.
Bisanti saracinati d'oro a carati 15.
Pezzi di bisanti d'oro a carati 12.
Pezzi di Tripoli d'oro a carati 11.
Oro di terì[1] a carati 16 e 2/3.
Agostantini d'oro a carati 20 e 1/2.
Casanini d'oro a carati 23 1/8.
Tanghi d'oro a carati 23 7/8.

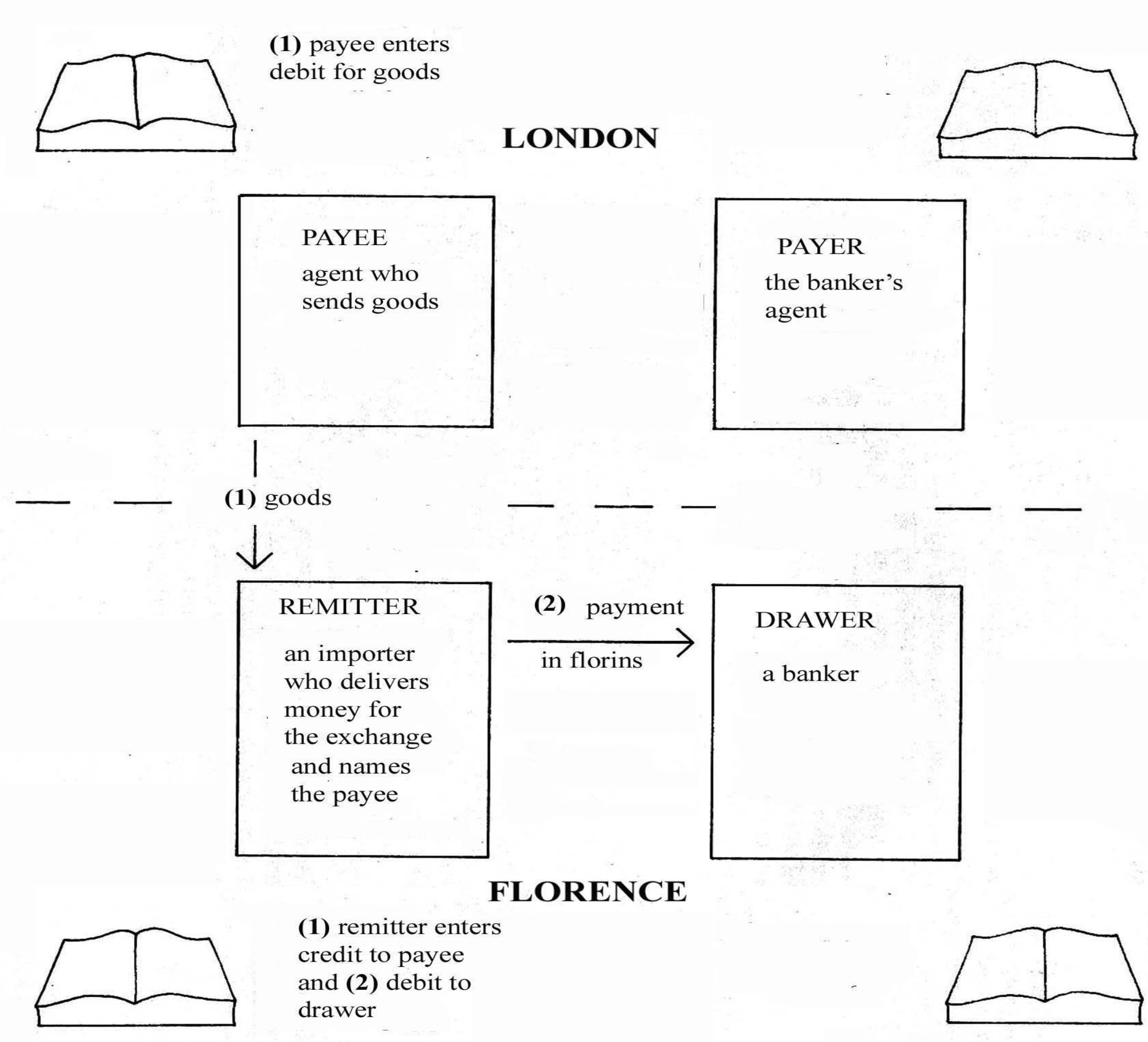
Even when the mass and fineness of the coins were known, that did not solve the practical problems of international trade.

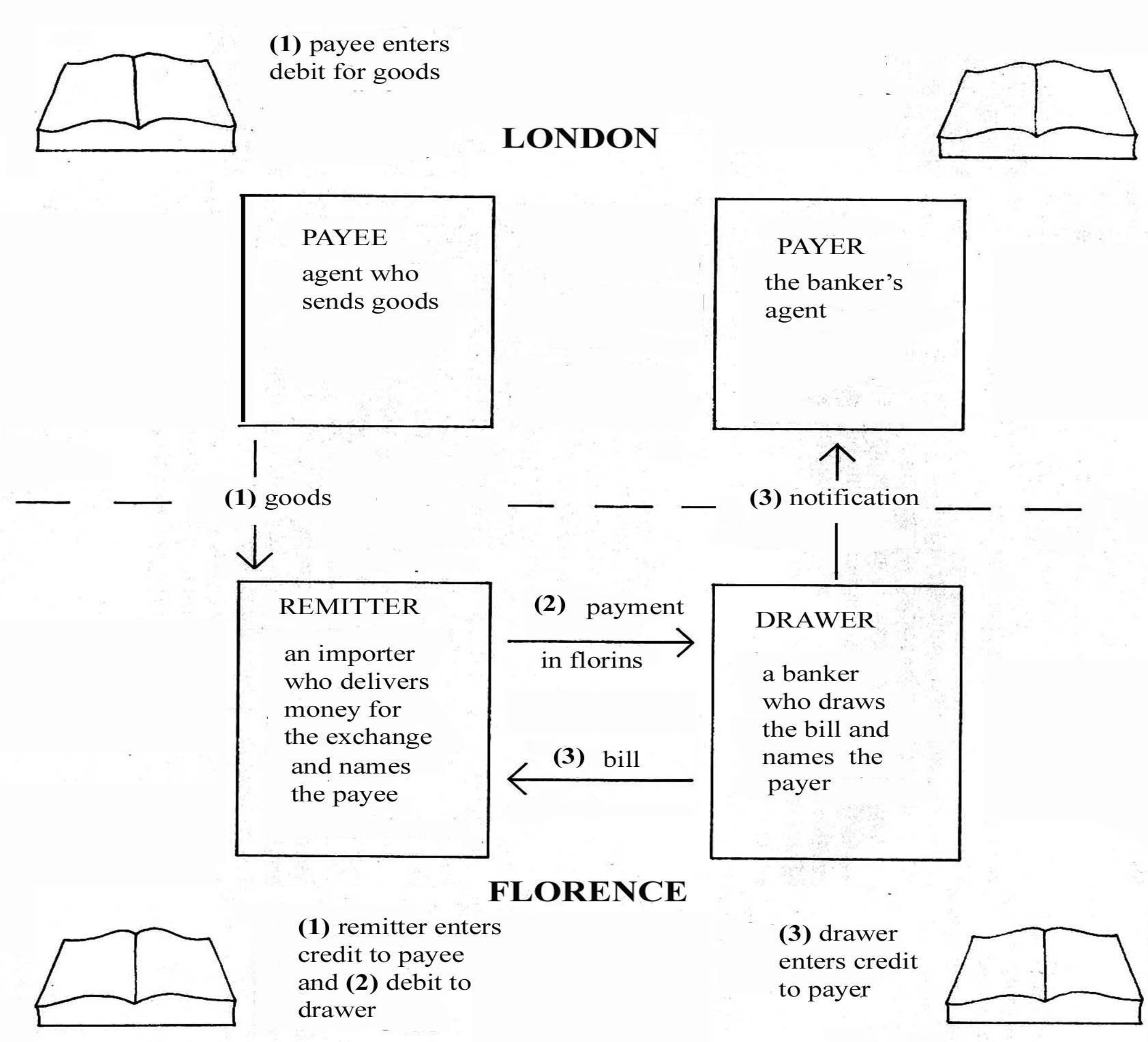
Sending coins on a long journey was a risky business, so the mechanism known as the bill of exchange was developed in the thirteenth century.

Suppose a a shipload of wool is being exported from London to Florence. The following diagrams are based on the definitive account given by Peter Spufford, *Handbook of the Medieval Exchange* (1986).

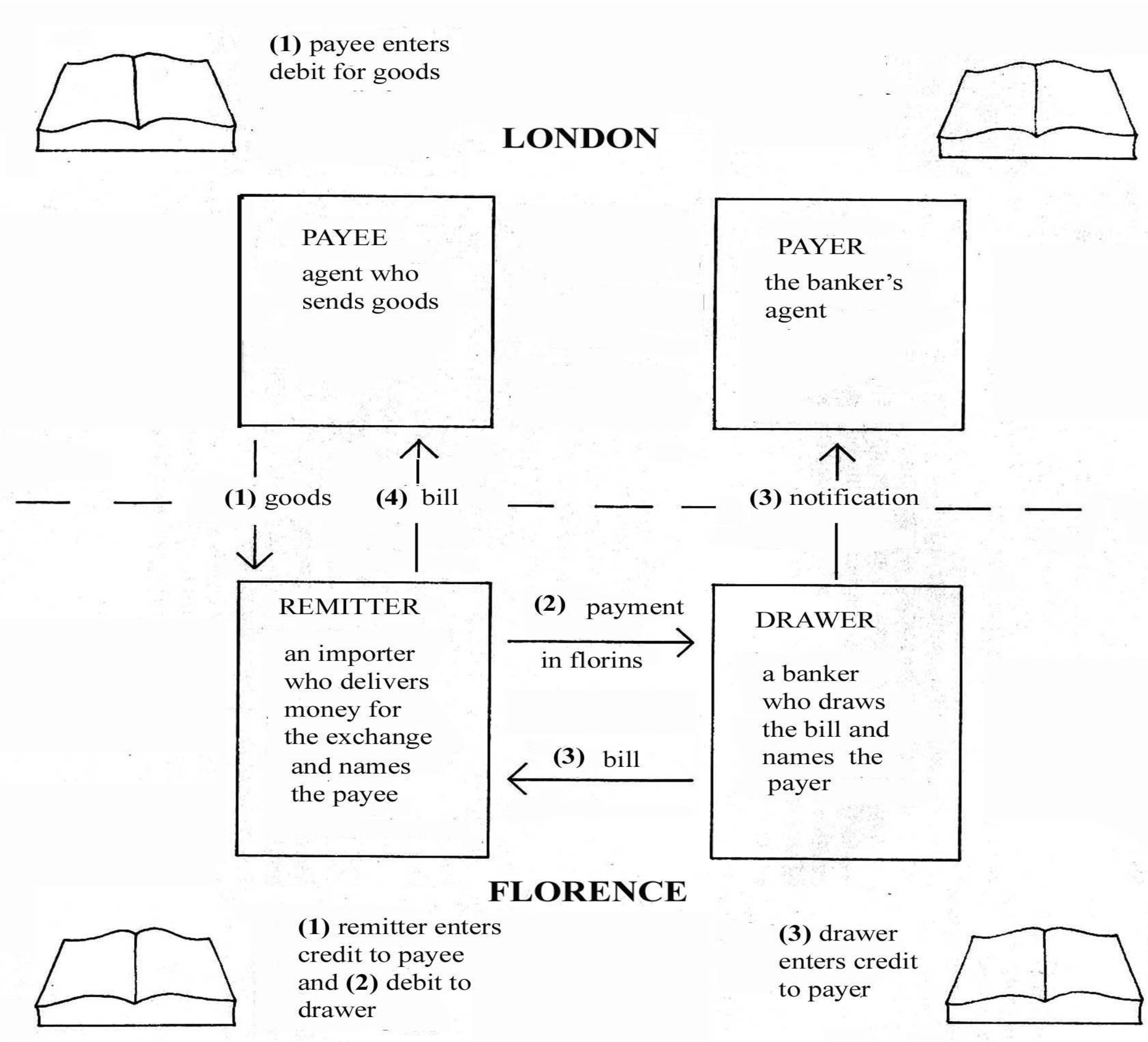

LONDON
PAYEE
agent
PAYER
the banker's
agent
REMITTER
an importer
DRAWER
a banker
FLORENCE

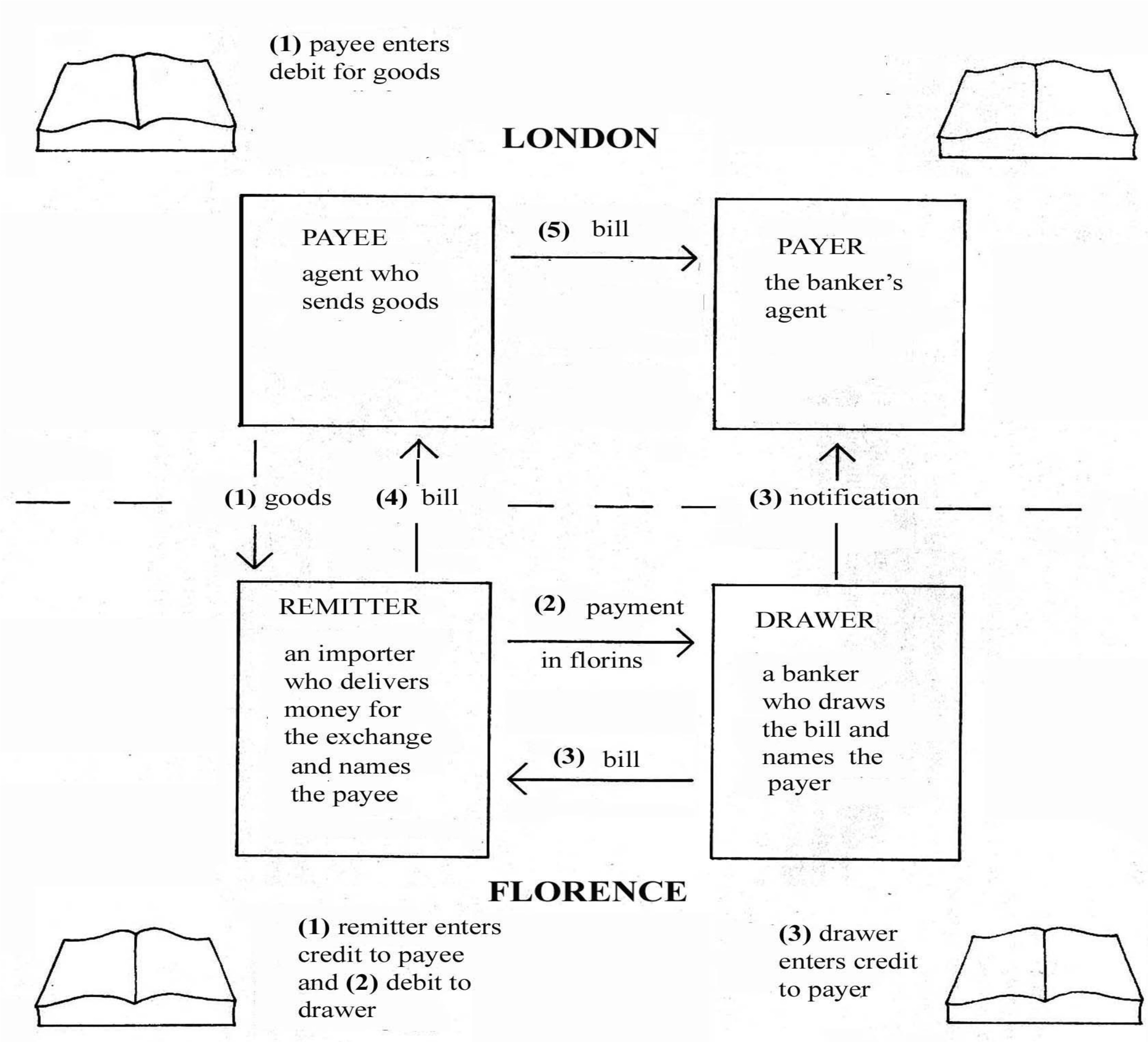

(1) payee enters debit for goods
LONDON
PAYEE
agent who sends goods
PAYER
the banker's agent
(1) goods
REMITTER
an importer
DRAWER
a banker
FLORENCE
(1) remitter enters credit to payee

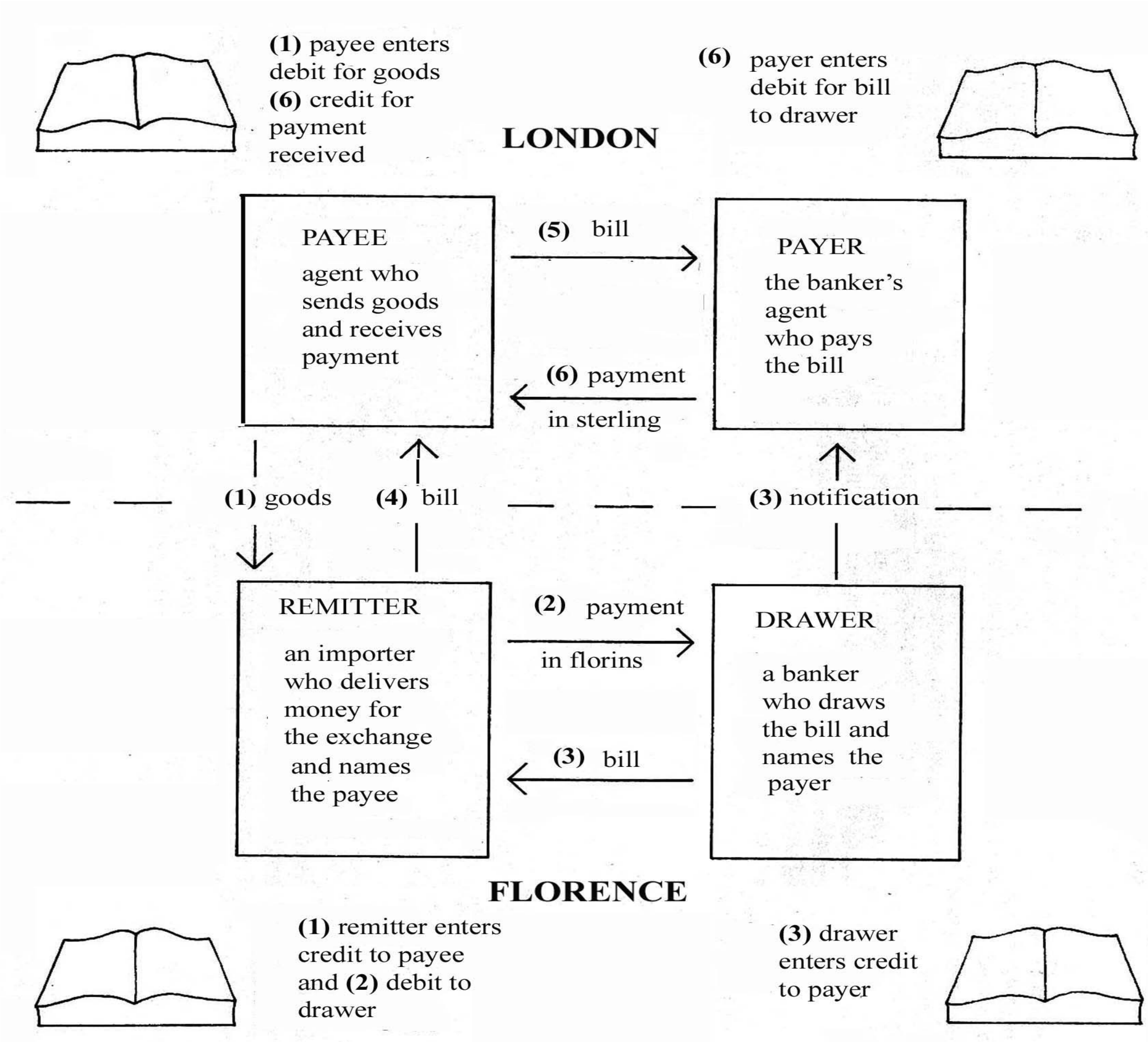

(1) payee enters debit for goods
LONDON
PAYEE
agent who sends goods
PAYER
the banker's agent
(1) goods
REMITTER
an importer who delivers money for the exchange and names the payee
(2) payment
in florins
DRAWER
a banker
FLORENCE
(1) remitter enters credit to payee and (2) debit to drawer



(1) payee enters debit for goods
LONDON
PAYEE
agent who sends goods
PAYER
the banker's agent
(1) goods
(3) notification
REMITTER
an importer who delivers money for the exchange and names the payee
(2) payment in florins
(3) bill
DRAWER
a banker who draws the bill and names the payer
FLORENCE
(1) remitter enters credit to payee and (2) debit to drawer
(3) drawer enters credit to payer


(1) payee enters
debit for goods
LONDON
PAYEE
agent who
sends goods
PAYER
the banker's
agent
(1) goods
(4) bill
(3) notification
REMITTER
an importer
who delivers
money for
the exchange
and names
the payee
(2) payment
in florins
(3) bill
DRAWER
a banker
who draws
the bill and
names the
payer
FLORENCE
(1) remitter enters
credit to payee
and (2) debit to
drawer
(3) drawer
enters credit
to payer


(1) payee enters
debit for goods
LONDON
PAYEE
agent who
sends goods
(5) bill
PAYER
the banker's
agent
(1) goods
(4) bill
(3) notification
REMITTER
an importer
who delivers
money for
the exchange
and names
the payee
(2) payment
in florins
(3) bill
DRAWER
a banker
who draws
the bill and
names the
payer
FLORENCE
(1) remitter enters
credit to payee
and (2) debit to
drawer
(3) drawer
enters credit
to payer

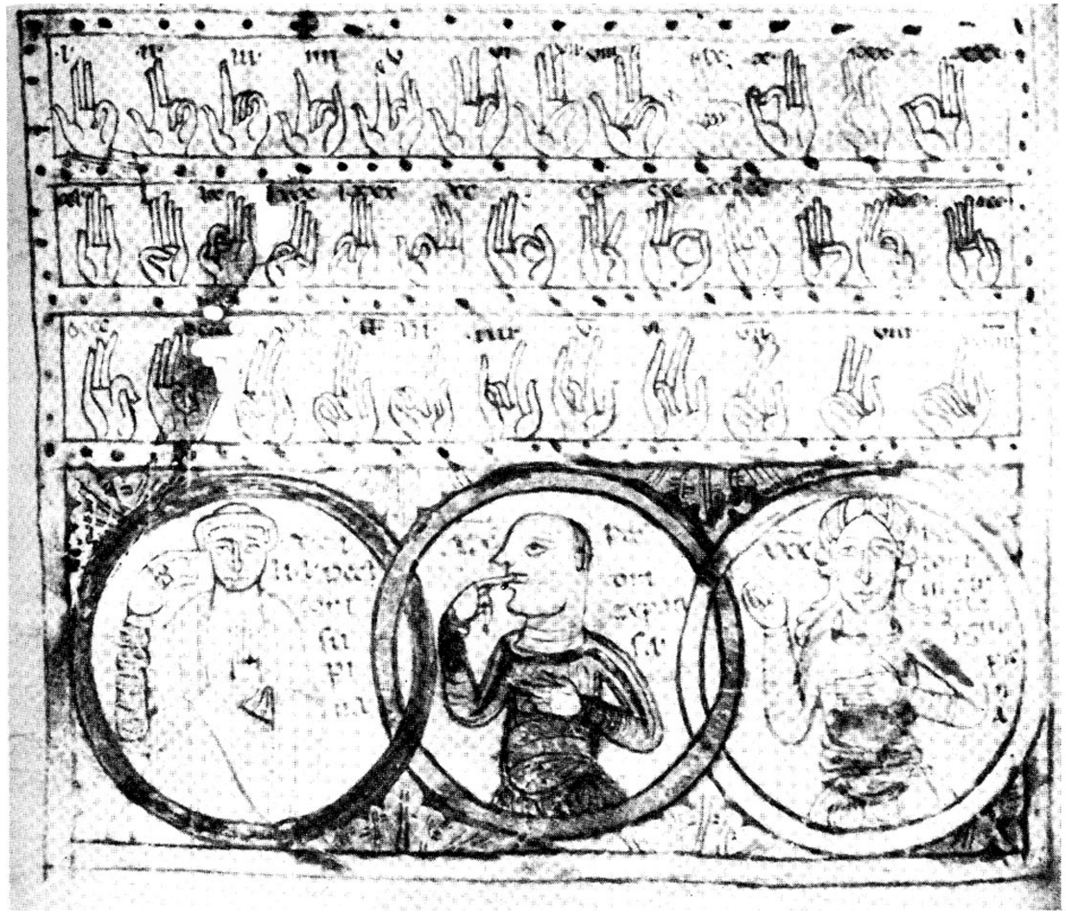

(1) payee enters debit for goods (6) credit for payment received
(6) payer enters debit for bill to drawer
LONDON
PAYEE agent who sends goods and receives payment
(5) bill
PAYER the banker's agent who pays the bill
(6) payment in sterling
(1) goods
(4) bill
(3) notification
REMITTER an importer who delivers money for the exchange and names the payee
(2) payment in florins
DRAWER a banker who draws the bill and names the payer
(3) bill
FLORENCE
(1) remitter enters credit to payee and (2) debit to drawer
(3) drawer enters credit to payer

2. The tools of arithmetic

It is a truism that the Roman numerals, I, V, X, C, and so on, are not useful for calculations.

The Romans used them only to *record* numbers. For calculations they used *calculi*, small stones that were placed on a grid known as an *abacus*.

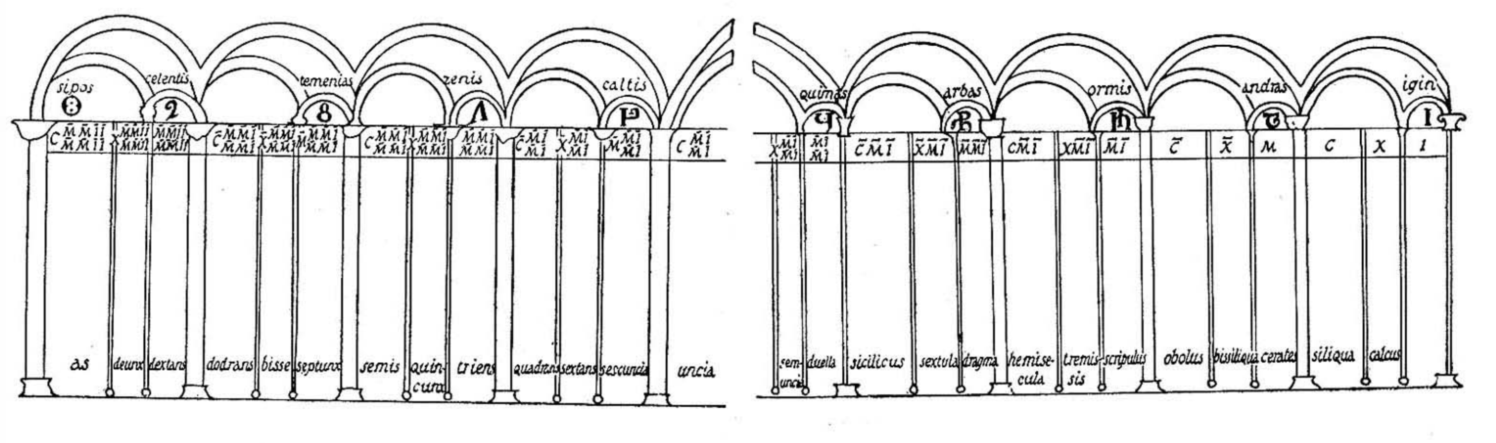
Not much evidence of the abacus survives from the early medieval period, but there is some evidence of other aids to arithmetic.



From a tenth century copy of Bede, *De Tempore Ratione*
[BL ms Royal BA XI fo.33v]

Around the turn of the millenium, things began to improve.

'Gerbert's abacus', as described by Yeldham (1926)



From Ramsey Abbey, c1110, [St John's Oxford, ms 17]

A similar account was given by Turchill (c1130), a clerk in the royal household (Poole 1912).

Gerbert's abacus had two distinctive features.

The grid was arranged in *columns*.

The counters were labelled with *Hindu-Arabic numerals*.

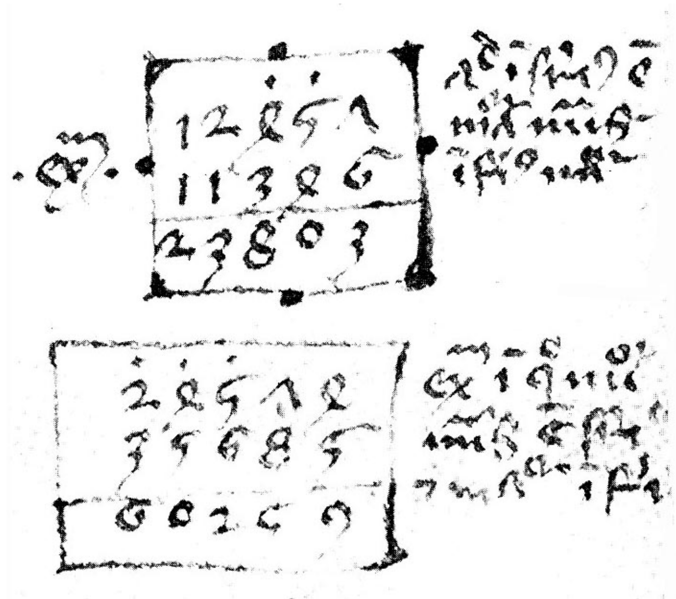
It seems likely that during the 12th century Gerbert's abacus was used in government and by the some of monastic orders that owned large estates.

Simpler versions of the abacus, with plain counters, were used by merchants and others. We know that this continued until the 17th century, if not later.

But Gerbert's abacus evolved into a system where the numerals themselves were 'moved', rather than the counters. We shall refer to this as *pen-reckoning*.

The methods used for Gerbert's abacus and for pen-reckoning were very similar.

Calculations were normally ephemeral (e.g. done on slates) but some survive.



Pen-reckoning c.1320 [Bodleian, ms Ashmole 1522 fo. 18r]

There is much confusion about how arithmetic was done at this time.

For example, many books refer to the fact that the Florentine bankers guild ‘banned’ the Hindu-Arabic numerals in 1299. In fact, the ban applied only to the use of the numerals in published records, not to the means of calculation, where they had already been in use for many years.

By 1345 Giovanni Villani reported that over a thousand children were learning learning *abaco* and *algorismo* in Florence.

Recommended: Alexander Murray, *Reason and Society in the Middle Ages* (1978).

Another source of confusion!

Reisch, *Margarita Philosophica* (1503)

3. Arithmetic of foreign exchange

Around 1300, in a typical place X trade would be carried on with a money-object M_X and accounts would be kept in X-units. The value of M_X was declared to be v_X X-units. Specifically:

In London (L), accounts were kept in sterling pence (L-units). M_L was an amount of silver, known as a mark, worth $v_L = 160$ pence.

In Florence (F), accounts were kept in denari piccoli (F-units). M_F was a gold florin worth $v_F = 348$ denari.

Trade between London and Florence was conducted in terms of an exchange rate:

$e = e_{LF} =$ the number of florins that equate to one mark.

Note that the corresponding quantity $e_{FL} = 1/e$.

The exchange rate varied because it was dependent on economic factors.

Pegolotti gave a table for the exchange between London and Florence:

A denari 33	sterl. il fior. viene il mar.	lire 7,	sol. —,	den. 7 e 2/11 [3/11] a fior.
A denari 33¼	starl. il fior. viene il mar.	lire 6,	sol. 19,	den. 6 e 78/133
A denari 33½	sterl. il fior. viene il mar.	lire 6,	sol. 18,	den. 6 e 6/67
A denari 33¾	sterl. il fior. viene il mar.	lire 6,	sol. 17,	den. 4 e 8/45 [d. 5 7/9]
A denari 34	sterl. il fior. viene il mar.	lire 6,	sol. 16,	den. 5 e 11/17
A denari 34¼	sterl. il fior. viene il mar.	lire 6,	sol. 15,	den. 5 e 95/137
A denari 34½	sterl. il fior. viene il mar.	lire 6,	sol. 14,	den. 5 e 21/23
A denari 34¾	sterl. il fior. viene il mar.	lire 6,	sol. 13,	den. 6 e 42/139
A denari 35	sterl. il fior. viene il mar.	lire 6,	sol. 12,	den. 6 e 6/7
A denari 35¼	sterl. il fior. viene il mar.	lire 6,	sol. 11,	den. 7 e 81/141
A denari 35½	sterl. il fior. viene il mar.	lire 6,	sol. 10,	den. 8 e 32/71
A denari 35¾	sterl. il fior. viene il mar.	lire 6,	sol. 9,	den. 9 e 69/143
A denari 36	sterl. il fior. viene il mar.	lire 6,	sol. 8,	den. 10 e 2/3
α		β		

Each line of the table is a statement of the form:

α L-units equates to M_F $\implies$ M_L equates to β F-units.

Note that:

β is expressed in multiples ($\ell\ s\ d$) of the accounting unit (denaro $= d$) – this is simply a matter of convenience;

two types of money occur, because money-objects were used for payments and the books were kept in accounting units – this requires some explanation.

If α sterling pence make a florin, then α/v_L marks make a florin. Hence

$$\alpha/v_L = 1/e.$$

Similarly $\beta = ev_F$, and so

$$\beta = \frac{v_L v_F}{\alpha}.$$

Pegolotti gave a table of values of β as a function of α.

A typical *user* of the table would have been a clerk in Florence, who needed to account for goods worth m marks, given that the ‘exchange on London’ was α. He would look up the corresponding β and multiply to get the answer, $m\beta$.

But the *compiler* of the table had to divide $v_L v_F$ by α in order to get β.

This implies two different levels of expertise: the first using a simple abacus, the second using Gerbert’s abacus or pen-reckoning.

4. Conclusions and Implications

• There were different levels of expertise in the practice of arithmetic. The higher levels are often obscured from the modern historian, partly by the imbalance of evidence, and partly by intentional secrecy.

• Numerate thinking was driven by commercial considerations as well as scientific ones. But progress in conceptual terms was slow – for example in framing the idea of number as a continuum. (Decimal notation does not appear until the sixteenth century.)