THE STORY OF e

Robin Wilson, Open Univ.

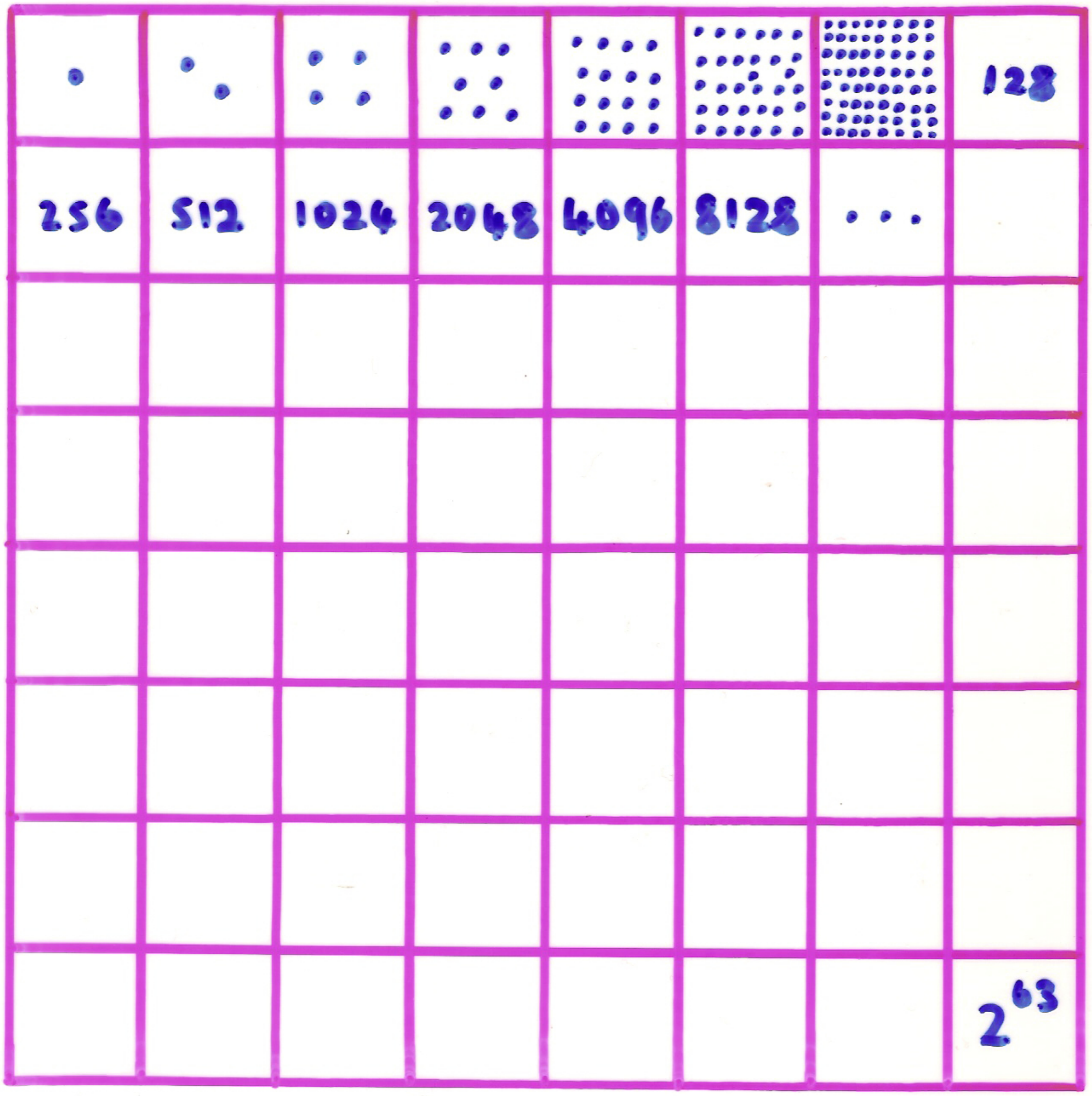
Old Macdonald had a farm

e, i, e, i, o,

And on that farm he had some pigs

e, i, e, i, o.

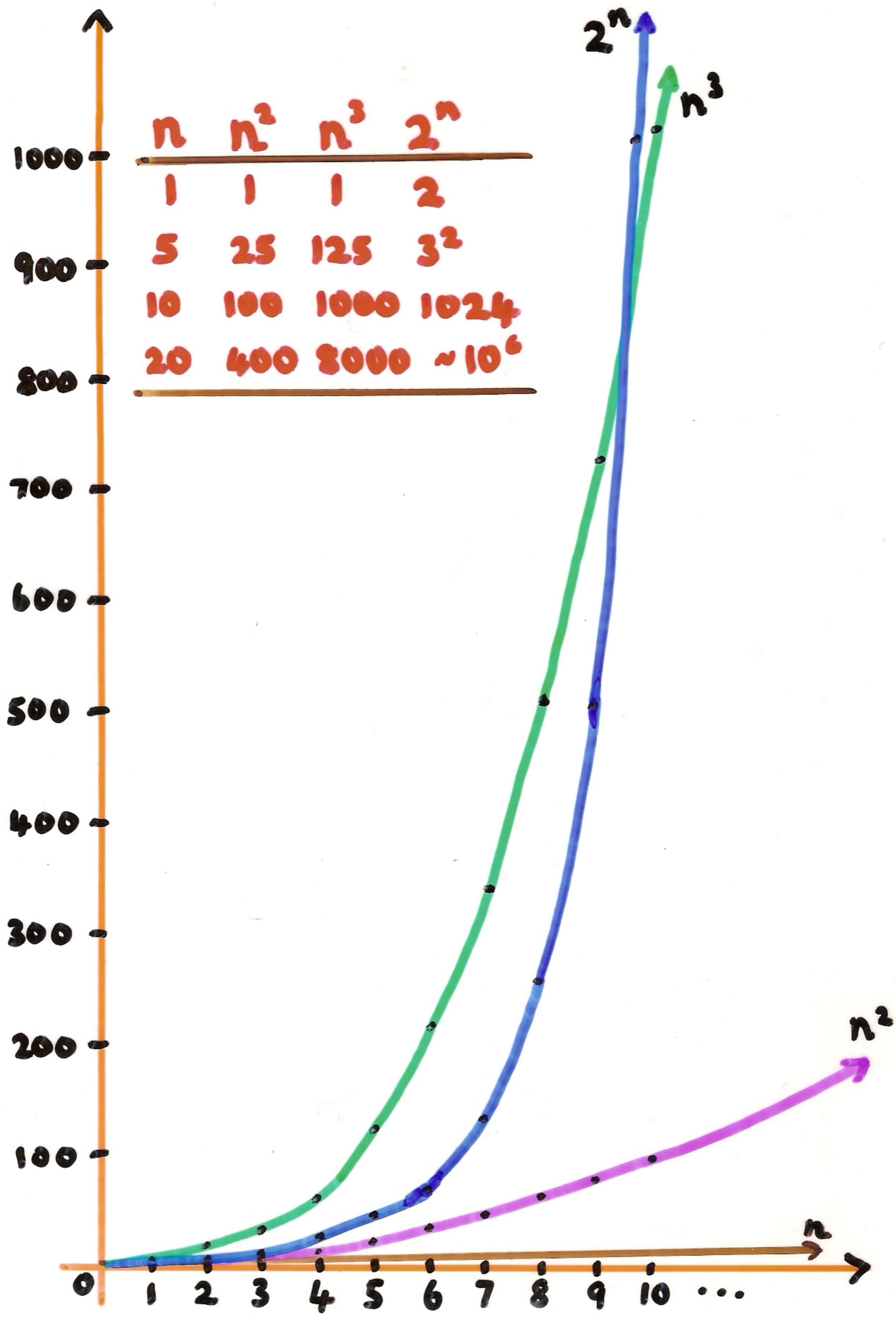
Grains of Wheat on a Chessboard



$$2^{63} = 9223372036854775808$$

How fast do things grow?

n	n^2	n^3	2^n
1	1	1	2
5	25	125	32
10	100	1000	1024
20	400	8000	$\sim 10^6$



running time*	input size $n = 10$	$n = 50$
n	0.00001 seconds	0.00005 seconds
n^2	0.0001 seconds	0.0025 seconds
n^3	0.001 seconds	0.125 seconds
n^5	0.1 seconds	5.2 minutes
2^n	0.001 seconds	35.7 years
3^n	0.059 seconds	2.3×10^{10} years

* based on one million computer operations per second

Compound Interest Tables, 1613

The Breuiat of the Table *of* 10. *per Cent.*

Yeares.		Yeares.	
1	11000000	16	45949729
2	12100000	17	50544702
3	13310000	18	55599173
4	14641000	19	61159090
5	16105100	20	67274999
6	17715610	21	74002499
7	19487171	22	81402749
8	21435888	23	89543024
9	23579476	24	98497326
10	25937424	25	108347059
11	28531167	26	119181765
12	31384283	27	131099941
13	34522712	28	144209936
14	37974983	29	158630929
15	41772481	30	174494022

Theſe Numbers in this Breuiat muſt alſo be eſteemed Numerators, each of them hauing for Denominator 10000000.

Now follow the Queſtions: In working whereof, we will vſe this Breuiat.

Simple and Compound Interest

Invest £P at r per cent for k years.

Simple interest : P = 100, r = 5 :

£ 100 → 105 → 110 → 115 → 120 → ...

Compound interest : P = 100, r = 5 :

£ 100 → 105 → 110·25 → 115·76 → 121·55 → ...

Simple : £100 + 5n   Compound : $100 \times (1{\cdot}05)^k$

Compound interest several times / year :

Annually	n = 1	r/n = 0·05	£105·00
Every 6 months	n = 2	r/n = 0·025	£105·06
Every month	n = 12	r/n = 0·00417	£105·12
Every day	n = 365	r/n = 0·00014	£105·13

After 1 year, n times / year : $£100\left(1+\frac{0{\cdot}05}{n}\right)^n \longrightarrow £100 \times e^{0{\cdot}05}$

e as a limit

$$\left(1+\frac{1}{n}\right)^n, \quad n = 1, 2, 3, \ldots$$

n	$(1+1/n)^n$	
1	2^1	$= 2$
2	$(1\frac{1}{2})^2$	$= 2{\cdot}25$
3	$(1\frac{1}{3})^3$	$= 2{\cdot}37037$
4	$(1\frac{1}{4})^4$	$= 2{\cdot}44141$
5	$(1\frac{1}{5})^5$	$= 2{\cdot}48832$
10	$(1\frac{1}{10})^{10}$	$= 2{\cdot}59374$
50	$(1\frac{1}{50})^{50}$	$= 2{\cdot}69159$
100	$(1\frac{1}{100})^{100}$	$= 2{\cdot}70481$
1000	$(1\frac{1}{1000})^{1000}$	$= 2{\cdot}71692$
10000	$(1\frac{1}{10000})^{10000}$	$= 2{\cdot}71815$

$\cdots$

How to remember e

$e = 2{\cdot}718281828459\ldots$

By omnibus I traveled to Brooklyn

or

In Gresham I lectured on geometry.

To disrupt a playroom is commonly a practice of children.

In showing a painting to probably a critical or venomous lady, anger dominates – O take guard, or she raves and shouts.

Chuquet and Stifel: Powers of 2

n	2^n
1	2
2	4
3	8
4	16
5	32
6	64
7	128
8	256
9	512
10	1024
11	2048

$$2^3 \times 2^5 = 2^{3+5} = 2^8$$

$$8 \times 32 = 256$$

$$2^4 \times 2^7 = 2^{4+7} = 2^{11}$$

$$16 \times 128 = 2048$$

To multiply the powers, add the exponents.

John Napier,
Eighth Laird of
Merchiston
(1550 - 1617)

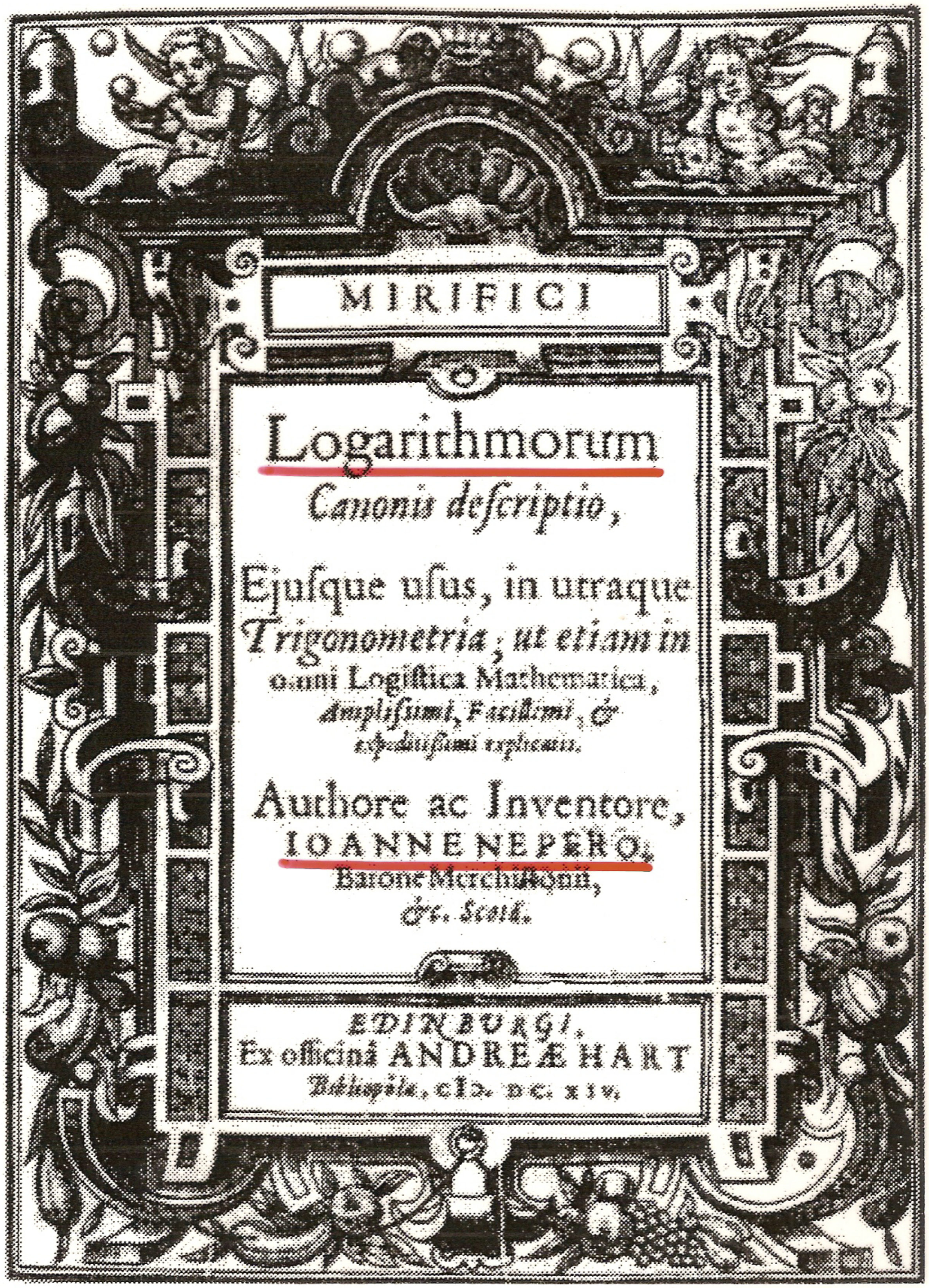
manure

coal-mining

Revelation of St John

calculating aids
(Napier's rods / bones)

Napier's Logarithms



MIRIFICI

Logarithmorum

Canonis descriptio,

Ejusque usus, in utraque
Trigonometria; ut etiam in
omni Logistica Mathematica,
Amplissimi, Facillimi, &
expeditissimi explicatio.

Authore ac Inventore,
IOANNE NEPERO,
Barone Merchistonii,
&c. Scoto.

EDINBURGI,
Ex officinâ ANDREÆ HART
Bibliopolæ, CIƆ. DC. XIV.

A page of Napier's logarithms

Gr. 2

+ | −

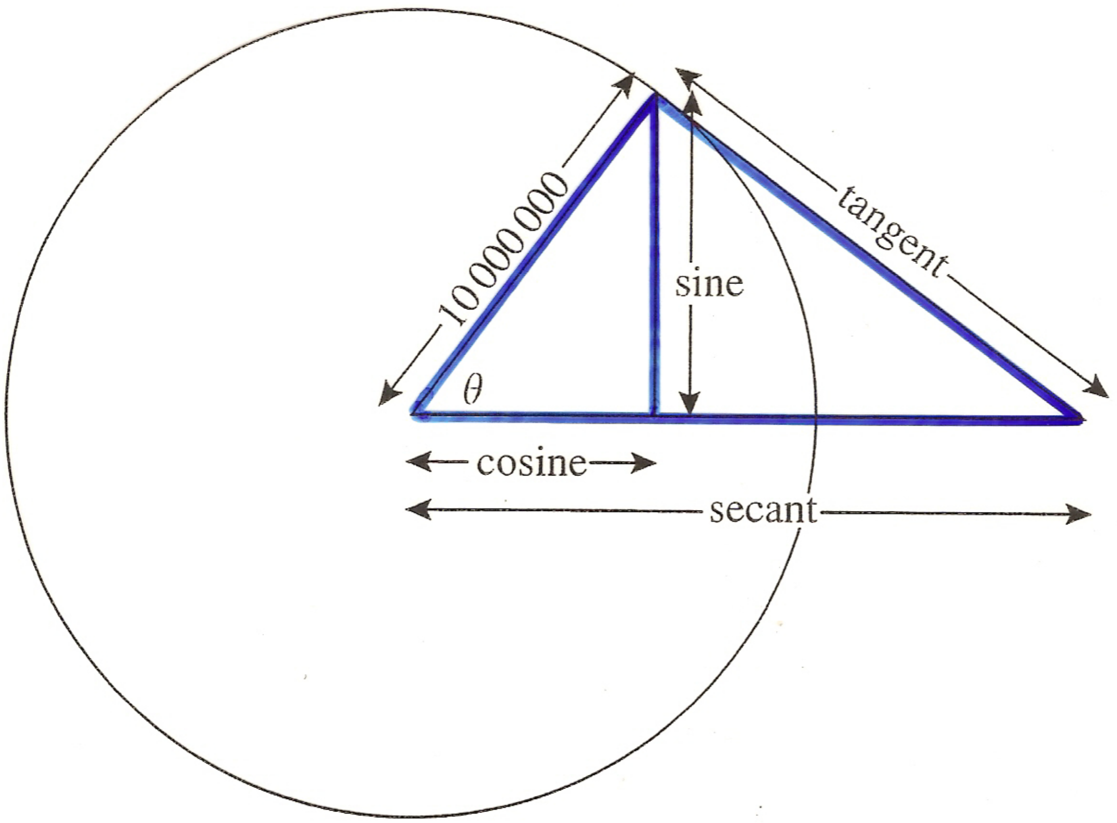
2 min	Sinus	Logarithmi	Differentiæ	logarithmi	Sinus	
0	348995	33552817	33546723	6094	9993908	60
1	351902	33469860	33463664	6196	9993806	59
2	354809	33387588	33381289	6299	9993703	58
3	357716	33305993	33299590	6403	9993599	57
4	360623	33225056	33218549	6507	9993495	56
5	363530	33144770	33138158	6612	9993390	55
6	366437	33065128	33058410	6718	9993284	54
7	369344	32986107	32979282	6825	9993177	53
8	372251	32907712	32900779	6933	9993069	52
9	375158	32829923	32822881	7042	9992960	51
10	378064	32752740	32745588	7152	9992850	50
11	380971	32676149	32668887	7262	9992740	49
12	383878	32600139	32592866	7373	9992629	48
13	386785	32524706	32517221	7485	9992517	47
14	389692	32449837	32442239	7598	9992404	46
15	392598	32375526	32367814	7712	9992290	45
16	395505	32301761	32293934	7827	9992175	44
17	398412	32228539	32220596	7943	9992060	43
18	401318	32155852	32147793	8059	9991944	42
19	404225	32083692	32075516	8176	9991827	41
20	407131	32012045	32003751	8294	9991709	40
21	410038	31940909	31932496	8413	9991590	39
22	412944	31870276	31861743	8533	9991470	38
23	415851	31800141	31791487	8654	9991349	37
24	418757	31730492	31721716	8776	9991228	36
25	421663	31661332	31652434	8898	9991106	35
26	424570	31592644	31583623	9021	9990983	34
27	427476	31524424	31515279	9145	9990859	33
28	430382	31456671	31447402	9270	9990734	32
29	433288	31389371	21379975	9396	9990608	31
30	436194	31322524	31313001	9523	9990482	30

87

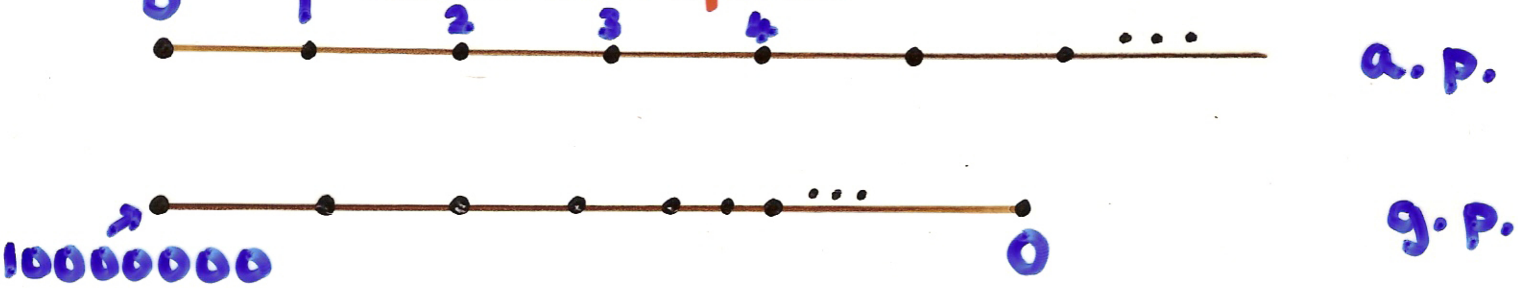
2° – 2° 30'

87° 30' – 88°

Napier's logarithms



constant speed



velocity ∝ distance still to travel

$$N\log(10000000) = 0$$

$$N\log(ab) = N\log(a) + N\log(b) - N\log(1)$$

Napier's 'Constructio' (1619)

THE

CONSTRUCTION OF THE

WONDERFUL CANON OF

LOGARITHMS;

And their relations to their own natural numbers;

WITH

An Appendix as to the making of another and better kind of Logarithms.

TO WHICH ARE ADDED

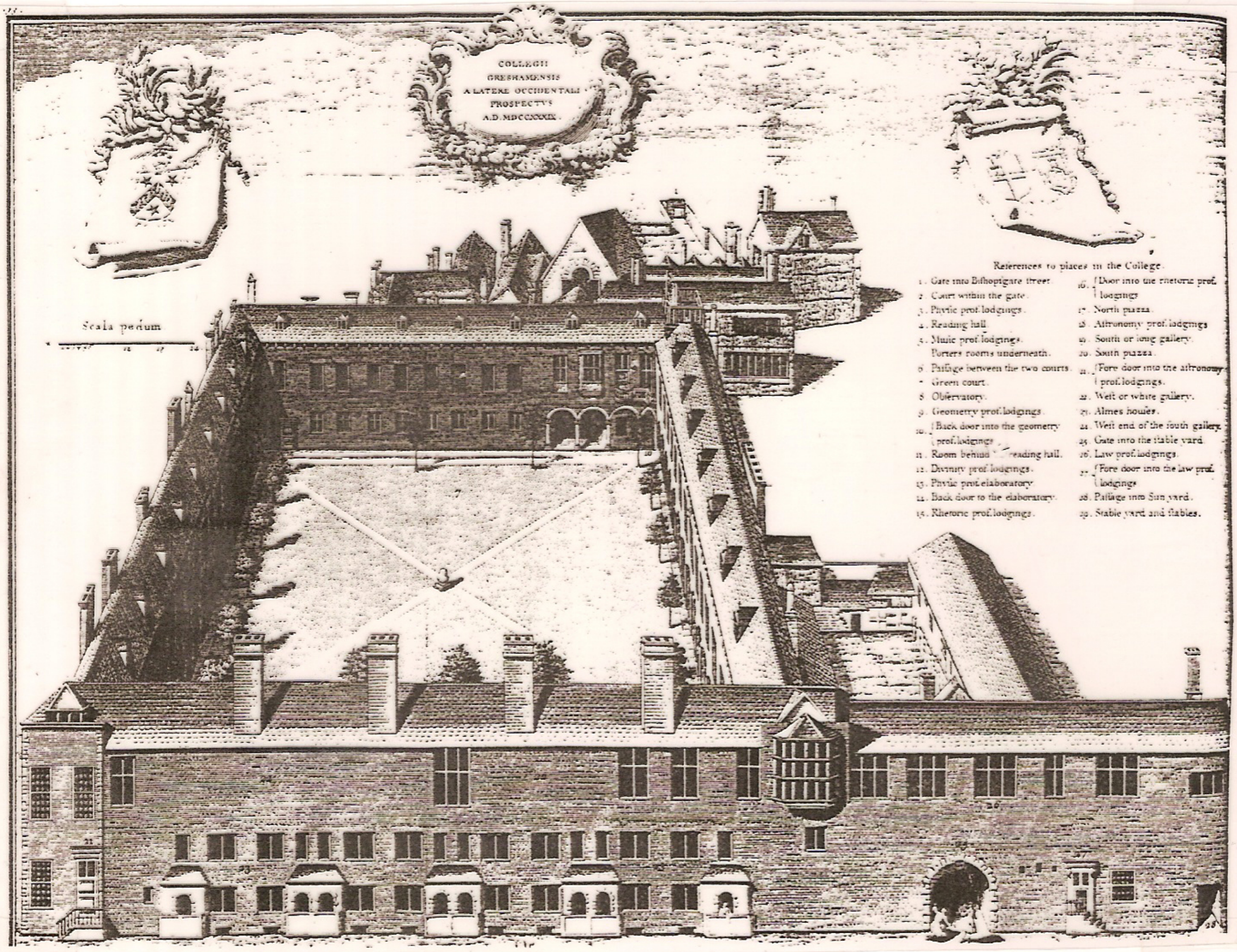
Propositions for the solution of Spherical Triangles by an *easier method: with Notes on them and on the above-mentioned Appendix by the learned* HENRY BRIGGS.

By the Author and Inventor, *John Napier*, Baron *of Merchiston*, &c., in Scotland.

Printed by ANDREW HART,
OF EDINBURGH;
IN THE YEAR OF OUR LORD, 1619.

Translated from Latin into English by William Rae Macdonald, 1888.

The original Gresham College



Henry Briggs on Logarithms

[John Napier] set my Head and Hands a Work with his new and remarkable logarithms. I never saw a Book which pleased me better or made me more wonder.

I myself, when expounding this doctrine publicly in London to my auditors in Gresham College, remarked that it would be much more convenient that 0 should be kept for the logarithm of the whole sine.

Briggs's 'Logarithmorum Chilias Prima' (1617)

2		Logarithmi.			Logarithmi.
1	0	0000,00000,00000	34	1	5314,78917,04226
2	0	3010,29995,66398	35	1	5440,68044,35028
3	0	4771,21254,71966	36	1	5563,02500,76729
4	0	6020,59991,32796	37	1	5682,01724,06700
5	0	6989,70004,33602	38	1	5797,83596,61681
6	0	7781,51250,38364	39	1	5910,64607,02650
7	0	8450,98040,01426	40	1	6020,59991,32796
8	0	9030,89986,99194	41	1	6127,83856,71974
9	0	9542,42509,43932	2	1	6232,49290,39790
10	1	0000,00000,00000	43	1	6334,68455,57959
11	1	0413,92685,15823	4	1	6434,52676,48619
12	1	0791,81246,04762	45	1	6532,12513,77534
13	1	1139,43352,30684	6	1	6627,57831,68157
14	1	1461,28035,67824	47	1	6720,97857,93572
15	1	1760,91259,05568	8	1	6812,41237,37559
16	1	2041,19982,65592	49	1	6901,96080,,02851
17	1	2304,48921,37827	50	1	6989,70004,33602
18	1	2552,72505,10331	51	1	7075,70176,09794
19	1	2787,53600,95283	2	1	7160,03343,63480
20	1	3010,29995,66398	53	1	7242,75869,60079
21	1	3222,19294,73392	4	1	7323,93759,82297
22	1	3424,22680,82221	55	1	7403,62689,49424
23	1	3617,27836,01759	6	1	7481,88027,00620
24	1	3802,11241,71161	57	1	7558,74855,67249
25	1	3979,40008,67204	8	1	7634,27993,56294
26	1	4149,73347,97082	59	1	7708,52011,64214
27	1	4313,63764,15899	60	1	7781,51250,38364
28	1	4471,58031,34222	61	1	7853,29835,01077
29	1	4623,97997,89896	2	1	7923,91689,49825
30	1	4771,21254,71966	63	1	7993,40549,45358
31	1	4913,61693,83427	4	1	8061,79973,98389
32	1	5051,49978,31991	65	1	8129,13356,64286
33	1	5185,13939,87789	6	1	8195,43935,54187
34	1	5314,78917,04226	67	1	8260,74802,70083

The first printed table of base-10 logs:

log 1 = 0, log 10 = 1, ...

MIRIFICI

Logarithmorum
Canonis descriptio,

Ejusque usus, in utraque
Trigonometria, ut etiam in
omni Logistica Mathematica,
Amplissimi, Facillimi, &
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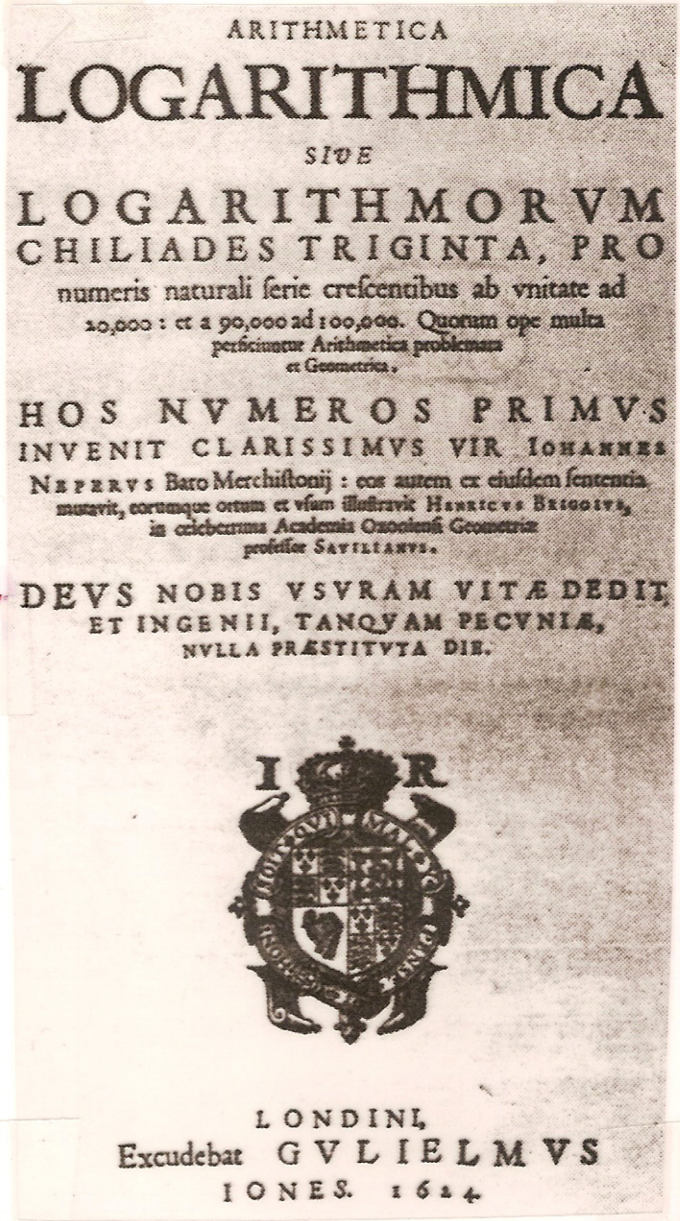
Napier's
Logarithms
(1614)

Briggs'
Logarithms
(1624)



ARITHMETICA
LOGARITHMICA
SIVE
LOGARITHMORVM
CHILIADES TRIGINTA, PRO
numeris naturali serie crescentibus ab vnitate ad
20,000: et a 90,000 ad 100,000. Quorum ope multa
perficiuntur Arithmetica problemata
et Geometrica.

HOS NVMEROS PRIMVS
INVENIT CLARISSIMVS VIR IOHANNES
NEPERVS Baro Merchistonij: eos autem ex eiusdem sententia
mutavit, eorumque ortum et vsum illustravit HENRICVS BRIGGIVS,
in celeberrima Academia Oxoniensi Geometriæ
professor SAVILIANVS.

DEVS NOBIS VSVRAM VITÆ DEDIT,
ET INGENII, TANQVAM PECVNIÆ,
NVLLA PRÆSTITVTA DIE.

I R

LONDINI,
Excudebat GVLIELMVS
IONES. 1624.

Briggs's logarithms (1624)

Num. absolut.	Logarithmi.	
17501	4,24306,28648,0481	2,48147,0057
17502	4,24308,76795,0538	2,48132,8279
17503	4,24311,24927,8817	2,48118,6517
17504	4,24313,73046,5334	2,48104,4771
17505	4,24316,21151,0105	2,48090,3042
17506	4,24318,69241,3147	2,48076,1329
17507	4,24321,17317,4476	2,48061,9631
17508	4,24323,65379,4107	2,48047,7951
17509	4,24326,23427,2058	2,48033,6286
17510	4,24328,61460,8344	2,48019,4638
17511	4,24331,09480,2982	2,48005,3005
17512	4,24333,57485,5987	2,47991,1389
17513	4,24336,05476,7376	2,47976,9789
17514	4,24338,53453,7165	2,47962,8206
17515	4,24341,01416,5371	2,47948,6638
17516	4,24343,49365,2009	2,47934,5086
17517	4,24345,97299,7095	2,47920,3551
17518	4,24348,45220,0646	2,47906,2032
17519	4,24350,93126,2678	2,47892,0528
17520	4,24353,41018,3206	2,47877,9042
17521	4,24355,88896,2248	2,47863,7571
17522	4,24358,35759,9819	2,47849,6116
17523	4,24360,84609,5935	2,47835,4678
17524	4,24363,32445,0613	2,47821,3256
17525	4,24365,80266,3869	2,47807,1850
17526	4,24368,28073,5719	2,47793,0460
17527	4,24370,75866,6179	2,47778,9086
17528	4,24373,23645,5265	2,47764,7728
17529	4,24375,71410,2993	2,47750,6386
17530	4,24378,19160,9379	2,47736,5061
17531	4,24380,66897,4440	2,47722,3752
17532	4,24383,14619,8192	2,47708,2459
17533	4,24385,62328,0651	2,47694,1181
17534	4,24388,10022,1832	2,47679,9920

Num. absolut.	Logarithmi.	
17534	4,24388,10022,1832	2,47679,9920
17535	4,24390,57702,1752	2,47665,8675
17536	4,24393,05368,0427	2,47651,7447
17537	4,24395,53019,7874	2,47637,6234
17538	4,24398,00657,4108	2,47623,5037
17539	4,24400,48280,9145	2,47609,3857
17540	4,24402,95890,3002	2,47595,2692
17541	4,24405,43485,5694	2,47581,1544
17542	4,24407,91066,7238	2,47567,0412
17543	4,24410,38633,7650	2,47552,9296
17544	4,24412,86186,6946	2,47538,8196
17545	4,24415,33725,5142	2,47524,7112
17546	4,24417,81250,2254	2,47510,6044
17547	4,24420,28760,8298	2,47496,4992
17548	4,24422,76257,3290	2,47482,3957
17549	4,24425,23739,7247	2,47468,2937
17550	4,24427,71208,0184	2,47454,1933
17551	4,24430,18662,2117	2,47440,0946
17552	4,24432,66102,3063	2,47425,9974
17553	4,24435,13528,3037	2,47411,9019
17554	4,24437,60940,2056	2,47397,8080
17555	4,24440,08338,0136	2,47383,7157
17556	4,24442,55721,7293	2,47369,6249
17557	4,24445,03091,3542	2,47355,5358
17558	4,24447,50446,8900	2,47341,4484
17559	4,24449,97788,3384	2,47327,3624
17560	4,24452,45115,7008	2,47313,2781
17561	4,24454,92428,9789	2,47299,1954
17562	4,24457,39728,1743	2,47285,1143
17563	4,24459,87013,2886	2,47271,0349
17564	4,24462,34284,3235	2,47256,9570
17565	4,24464,81541,2805	2,47242,8807
17566	4,24467,28784,1612	2,47228,8060
17567	4,24469,76012,9672	2,47214,7329

Num. absolut.	Logarithmi.	
17567	4,24469,76012,9672	2,47214,7329
17568	4,24472,23227,7001	2,47200,6614
17569	4,24474,70428,3615	2,47186,5916
17570	4,24477,17614,9531	2,47172,5233
17571	4,24479,64787,4764	2,47158,4566
17572	4,24482,11945,9330	2,47144,3915
17573	4,24484,59090,3245	2,47130,3280
17574	4,24487,06220,6525	2,47116,2662
17575	4,24489,53336,9187	2,47102,2059
17576	4,24492,00439,1246	2,47088,1473
17577	4,24494,47527,2719	2,47074,0902
17578	4,24496,94601,3621	2,47060,0347
17579	4,24499,41661,3968	2,47045,9808
17580	4,24501,88707,3776	2,47031,9287
17581	4,24504,35739,3063	2,47017,8780
17582	4,24506,82757,1843	2,47003,8288
17583	4,24509,29761,0131	2,46989,7812
17584	4,24511,76750,7943	2,46975,7354
17585	4,24514,23726,5297	2,46961,6912
17586	4,24516,70688,2209	2,46947,6485
17587	4,24519,17635,8694	2,46933,6074
17588	4,24521,64569,4768	2,46919,5679
17589	4,24524,11489,0447	2,46905,5300
17590	4,24526,58394,5747	2,46891,4937
17591	4,24529,05286,0684	2,46877,4590
17592	4,24531,52163,5274	2,46863,4259
17593	4,24533,99026,9533	2,46849,3944
17594	4,24536,45876,3477	2,46835,3645
17595	4,24538,92711,7122	2,46821,3361
17596	4,24541,39533,0483	2,46807,3094
17597	4,24543,86340,3577	2,46793,2843
17598	4,24546,33133,6420	2,46779,2607
17599	4,24548,79912,9027	2,46765,2388
17600	4,24551,26678,1415	2,46751,2185

ARITHMETICA
LOGARITHMICA,

SIVE

LOGARITHMORVM
CHILIADES CENTVM, PRO
Numeris naturali ſerie creſcentibus
ab Vnitate ad 100000.

VNA CVM

CANONE TRIANGVLORVM,
SEV TABVLA ARTIFICIALIVM
Sinuum, Tangentium & Secantium,
Ad Radium 10,00000,00000. & ad ſingula
Scrupula Prima Quadrantis.

QVIBVS NOVVM TRADITVR COMPENDIVM, QVO NVLlum nec admirabilius, nec utilius ſolvendi pleraque Problemata Arithmetica & Geometrica.

HOS NVMEROS PRIMVS INVENIT
Clariſſimus Vir IOHANNES NEPERVS Baro
Merchiſtonij: eos autem ex ejuſdem ſententiâ mutavit, eorumque ortum & uſum illuſtravit HENRICVS BRIGGIVS,
in celeberrimâ Academiâ Oxonienſi Geometriæ Profeſſor Savilianus.

Editio Secunda aucta per ADRIANVM VLACQ *Goudanum.*

DEVS NOBIS VSVRAM VITAE DEDIT ET INGENII,
TANQVAM PECVNIAE, NVLLA
PRAESTITVTA DIE.

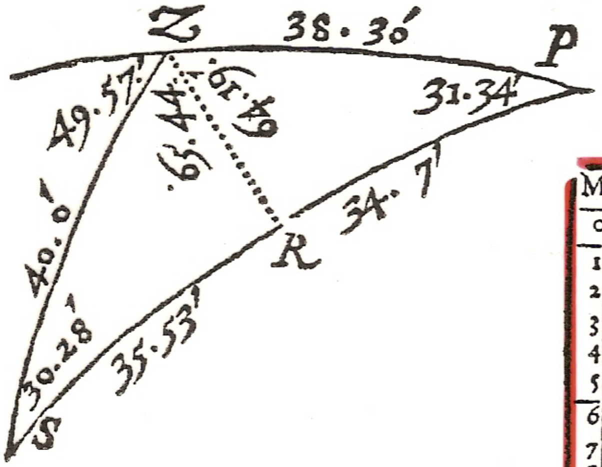
GOVDAE,
Excudebat Petrus Rammaſenius.

M. DC. XXVIII.

Cum Privilegio Illuſt. Ord. Generalium.

CANON
TRIANGVLORVM,
SIVE
Tabulæ Sinuum et Tangen
tium artificialium ad Radium
10000. 0000. & ad scru
pula prima quadrantis.

Per EDM: GVNTER, Professorem
Astronomiæ in Collegio
Greshamensi.



LONDINI,
Excudebat *Gulielmus Iones*
MDCXX.

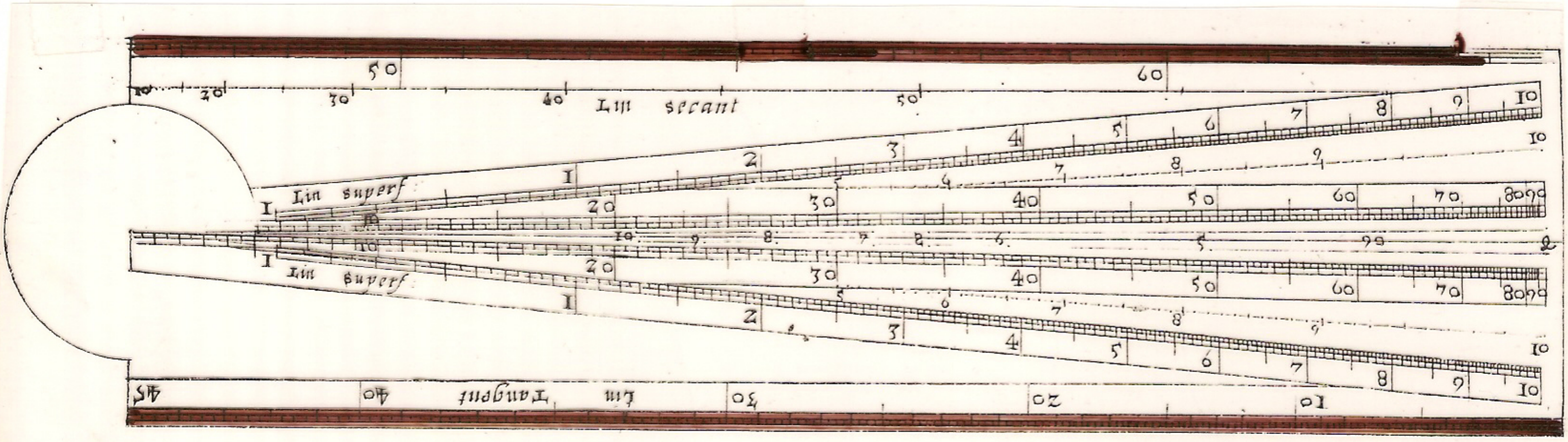
Edmund
Gunter's
'Canon
triangulorum'
(1620)

M	Sin. 38				Tan. 38				
0	9789	3419	9896	5321	9892	8098	10107	1901	60
1	9	5036		4334	3	0702	6	9297	59
2	9	6651		3346	3	3305	6	6694	58
3	9	8266		2357	3	5908	6	4091	57
4	9	9880		1368	3	8511	6	1488	56
5	9790	1492		0379	4	1113	5	8886	55
6	0	3104	9895	9388	4	3715	5	6284	54
7	0	4714		8398	4	6317	5	3683	53
8	0	6324		7406	4	8918	5	1081	52
9	0	7933		6414	5	1519	4	8481	51
10	0	9541		5421	5	4119	4	5880	50
11	1	1148		4428	5	6719	4	3280	49
12	1	2753		3434	5	9319	4	0680	48
13	1	4358		2440	6	1918	3	8081	47
14	1	5962		1445	6	4517	3	5482	46
15	1	7565		0449	6	7116	3	2884	45
16	1	9167	9894	9453	6	9714	3	0285	44
17	2	0768		8456	7	2312	2	7687	43
18	2	2368		7459	7	4909	2	5090	42
19	2	3968		6461	7	7507	2	2493	41
20	2	5566		5462	8	0103	1	9896	40
21	2	7163		4463	8	2700	1	7300	39
22	2	8759		3463	8	5296	1	4704	38
23	3	0355		2463	8	7891	1	2108	37
24	3	1949		1462	9	0487	0	9512	36
25	3	3542		0460	9	3082	0	6917	35
26	3	5135	9893	9458	9	5677	0	4323	34
27	3	6726		8455	9	8271	0	1728	33
28	3	8317		7452	9900	0865	10099	9134	32
29	3	9907		6448	0	3458	9	6541	31
30	4	[illegible]		[illegible]	0	[illegible]	9	3948	30

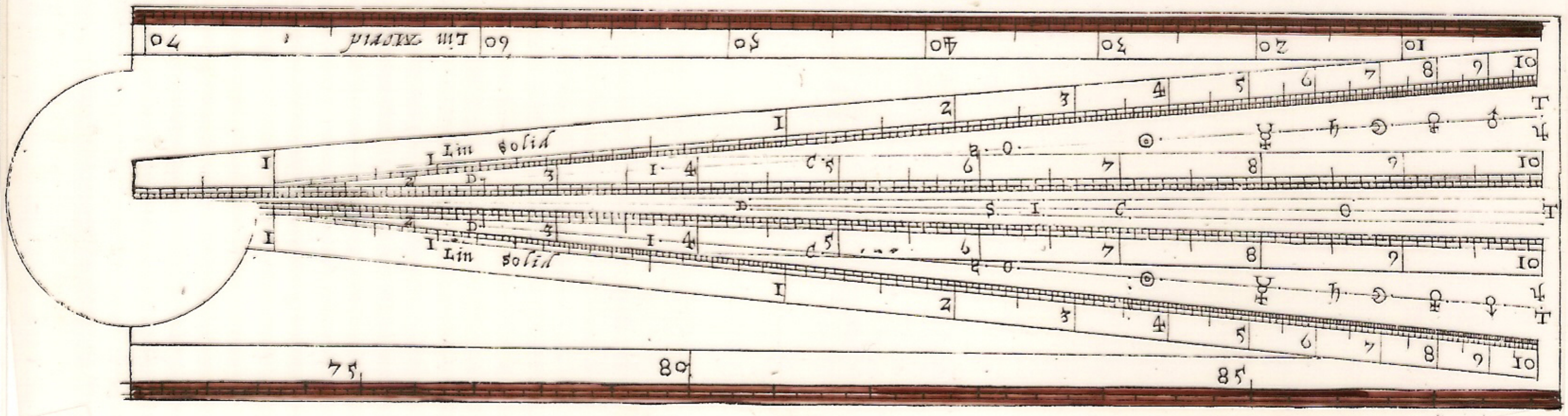
51° ↑

base-10 logarithms
of trigonometrical
functions

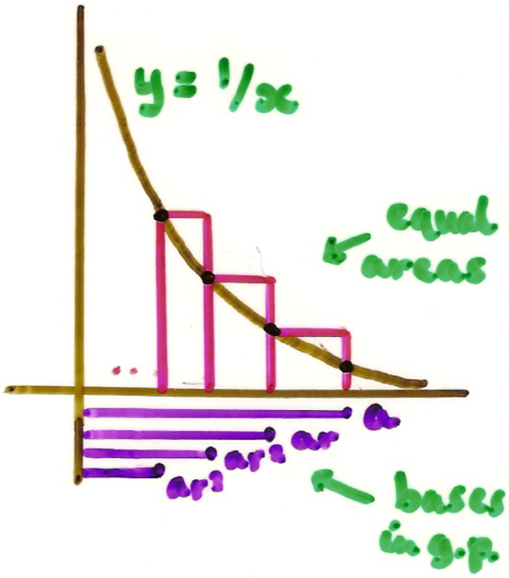
The Gunter sector (1636)



These and all other Mathematicall Instruments are made in Brass by Elias Allen dweling with out Tempel barr a gainst St Clements Church



Results on logarithms



1647: G. de Saint-Vincent

$$\int_1^n \frac{1}{x}\,dx = \log n$$

(Fermat knew $\int_1^n x^k\,dx$ for all $k \neq -1$.)

1660s: Newton (also Brouncker 1668)

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$$

so (taking areas) $\log(1+t) = t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots$

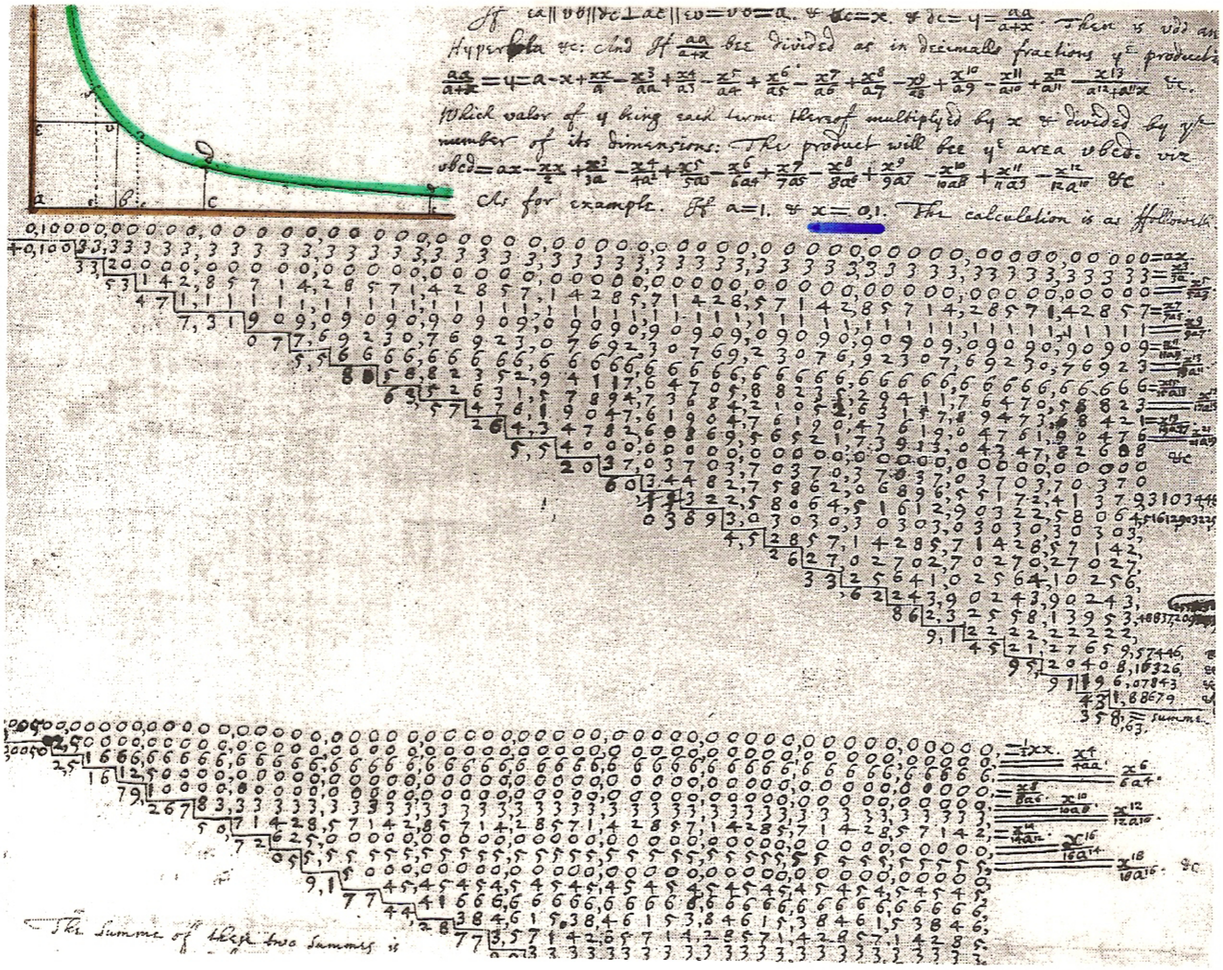
$$\int_0^t \frac{1}{x+1}\,dx = \log(1+t)$$

$$\int_0^t (1 - x + x^2 - \cdots)\,dx = t - \frac{t^2}{2} + \frac{t^3}{3} - \cdots$$

Newton and the hyperbola

x

$x^3/3$



If $ea \parallel vb \parallel dc \perp ae \parallel ev = vb = a$. & $bc = x$. & $dc = y = \frac{aa}{a+x}$. Then is vod an Hyperbola &c: And If $\frac{aa}{a+x}$ bee divided as in decimall fractions ye product is

$$\frac{aa}{a+x} = y = a - x + \frac{xx}{a} - \frac{x^3}{aa} + \frac{x^4}{a^3} - \frac{x^5}{a^4} + \frac{x^6}{a^5} - \frac{x^7}{a^6} + \frac{x^8}{a^7} - \frac{x^9}{a^8} + \frac{x^{10}}{a^9} - \frac{x^{11}}{a^{10}} + \frac{x^{12}}{a^{11}} - \frac{x^{13}}{a^{12} + a^{11}x} \text{ \&c.}$$

Which valor of y being each terme thereof multiplyed by x & divided by ye number of its dimensions: The product will bee ye area vbcd. viz

$$vbcd = ax - \frac{xx}{2} + \frac{x^3}{3a} - \frac{x^4}{4a^2} + \frac{x^5}{5a^3} - \frac{x^6}{6a^4} + \frac{x^7}{7a^5} - \frac{x^8}{8a^6} + \frac{x^9}{9a^7} - \frac{x^{10}}{10a^8} + \frac{x^{11}}{11a^9} - \frac{x^{12}}{12a^{10}} \text{ \&c}$$

As for example. If $a=1$. & $x=0{,}1$. The calculation is as followeth.

$\frac{x^2}{2}$

The Summe of these two Summes is

calculation of log (1 + 0·1)


LEONHD. EULER
NAT. BASILEÆ MDCCVII.
MORT. PETROP MDCCLXXXIII.

Euler's 'Introductio' (1748)

INTRODUCTIO
IN ANALYSIN
INFINITORUM.

AUCTORE

LEONHARDO EULERO,

Professore Regio BEROLINENSI, & *Academiæ Imperialis Scientiarum* PETROPOLITANÆ *Socio.*

TOMUS PRIMUS.

LAUSANNÆ,

Apud MARCUM-MICHAELEM BOUSQUET & Socios.

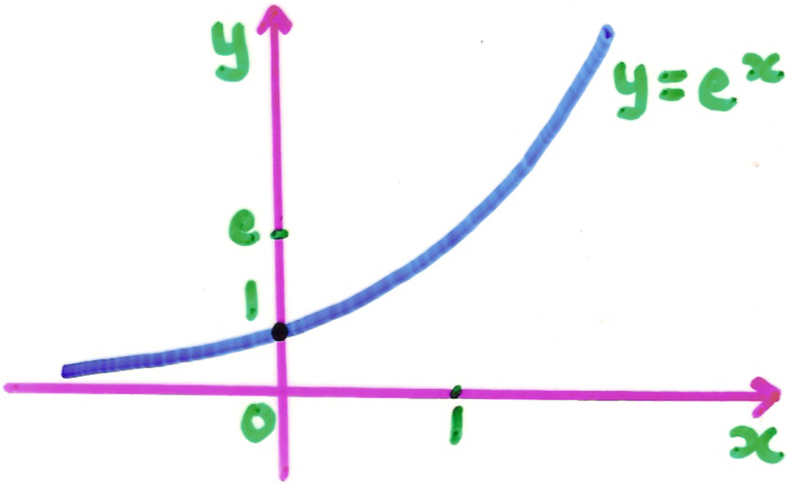
MDCCXLVIII. (1748)

The exponential function

e = limit of $\left(1+\frac{1}{n}\right)^n$ for large n

For any x:

$$\boxed{e^x = \text{limit of } \left(1+\frac{x}{n}\right)^n \text{ for large } n}$$

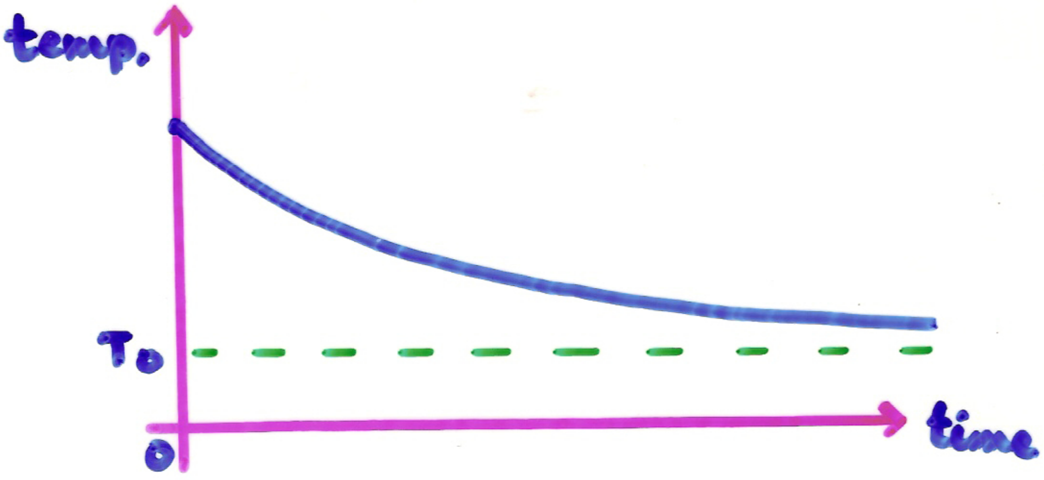


$$\boxed{\text{at any point, the slope } \frac{dy}{dx} = y}$$

Other exponential functions:

$2^x = e^{x \log 2}$; $b^x = e^{x \log b}$.

Cooling of a cup of tea



Newton's law of cooling:

The rate at which the tea cools

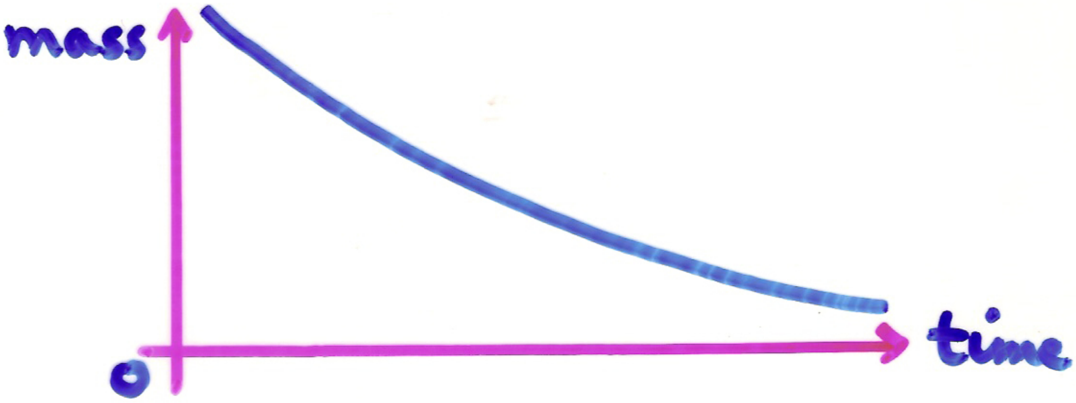
$\propto$ the difference in temperature between

the tea and the surrounding room.

$$\frac{dT}{dt} = -K(T - T_0)$$

$$\boxed{T = T_0 + Ce^{-Kt}}$$

Radioactive decay



Rate of decay $\propto$ mass

$$\frac{dm}{dt} = -Km \text{, for } K \text{ constant}$$

$$\boxed{m = Ce^{-Kt}}$$

Continued fractions

$$e = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{2}{3 + \cfrac{3}{4 + \cfrac{4}{5 + \cdots}}}}}$$

$$\frac{e+1}{e-1} = 2 + \cfrac{1}{6 + \cfrac{1}{10 + \cfrac{1}{14 + \cdots}}}$$

The series for e^x

$$\left(1+\frac{1}{n}\right)^n = 1 + n\left(\frac{1}{n}\right) + \frac{n(n-1)}{2!}\left(\frac{1}{n}\right)^2$$

$$+ \frac{n(n-1)(n-2)}{3!} + \cdots + \left(\frac{1}{n}\right)^n$$

$$= 1 + 1 + \frac{(1-1/n)}{2!} + \frac{(1-\frac{1}{n})(1-\frac{2}{n})}{3!} + \cdots + \frac{1}{n^n}.$$

Let $n \to \infty$: $\boxed{e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots}$

For any x:

$$\boxed{e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots}$$

So: $1/e = (1-1) + \frac{1}{2!} - \frac{1}{3!} + \cdots$

The Derangement Problem

Given any series of n letters a, b, c, d, e, etc., to find how many ways they can be rearranged so that none is in its original position.

n:	1	2	3	4	5	6	7	8	...
D_n:	0	1	2	9	44	265	1854	14833	...

Euler: $D_n = (n-1)D_{n-2} + (n-1)D_{n-1}$

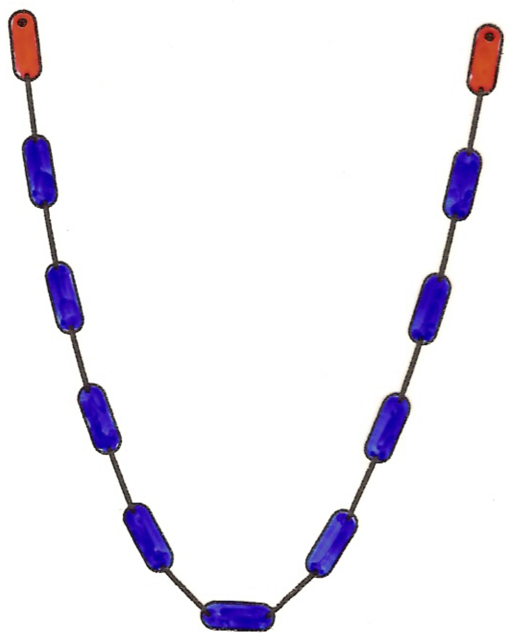
$$D_n = nD_{n-1} + (-1)^n$$

$$\Rightarrow \quad D_n = n!\left\{1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots \pm \frac{1}{n!}\right\}$$

$$\approx n!/e$$

$n = 8$, $D_n = 14833$, $n!/e = 14832 \cdot 9$

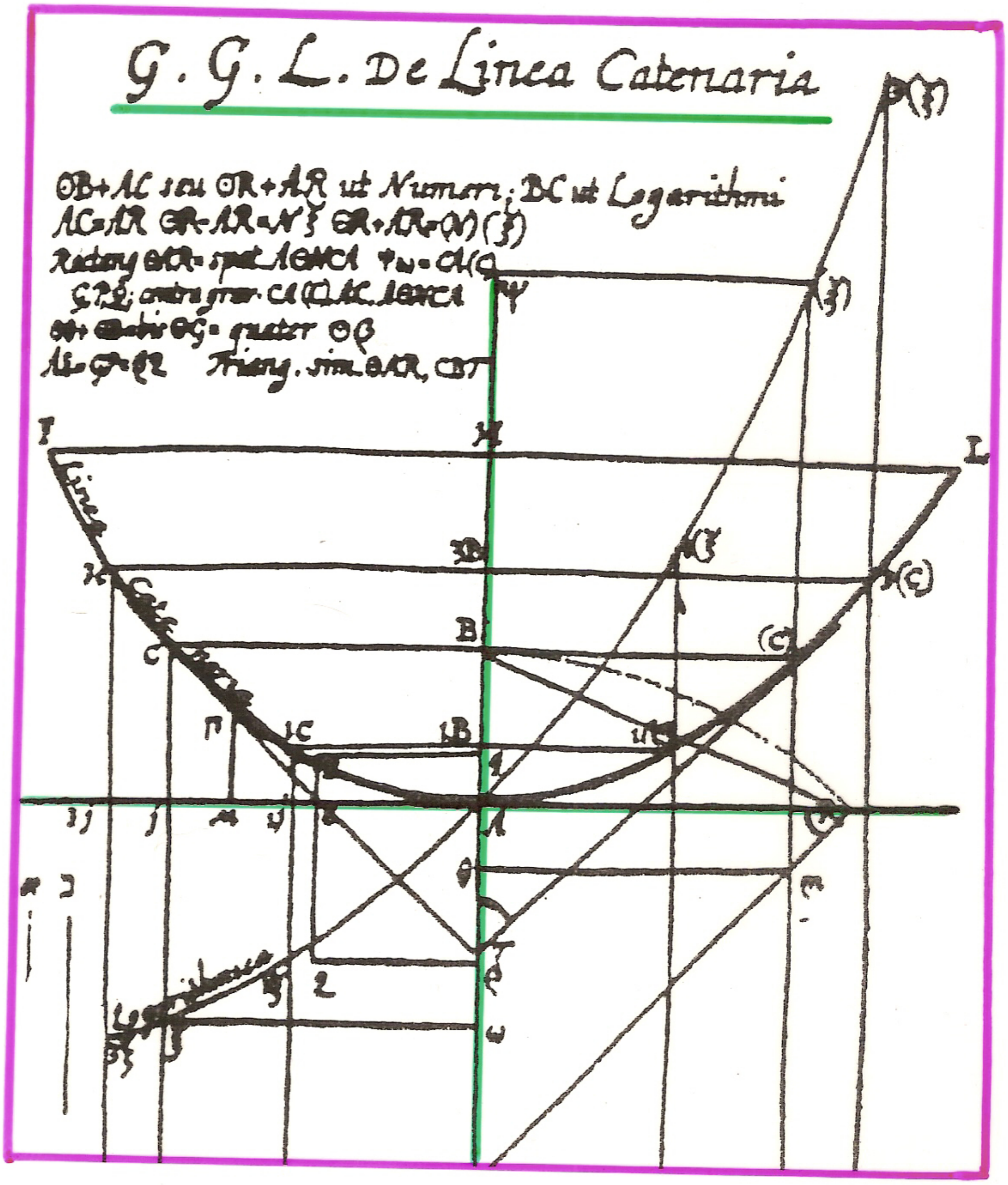
The Catenary



$$y = \frac{1}{2}\left(e^{x} + e^{-x}\right)$$

$$= \cosh x$$

catena (Latin) = chain



Gateway Arch, St Louis

cosh x and sinh x

$$\cosh x = \frac{1}{2}(e^{x} + e^{-x})$$

$$\sinh x = \frac{1}{2}(e^{x} - e^{-x})$$

$$\left\{\begin{aligned} &\cos^2 x + \sin^2 x = 1 \\ &\qquad\qquad \cosh^2 x - \sinh^2 x = 1 \end{aligned}\right.$$

$$\left\{\begin{aligned} &\sin(x+y) = \sin x \cos y + \cos x \sin y \\ &\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y \end{aligned}\right.$$

$$\left\{\begin{aligned} &\frac{d}{dx}(\sin x) = \cos x, \quad \frac{d}{dx}(\sinh x) = \cosh x \\ &\frac{d}{dx}(\cos x) = -\sin x, \quad \frac{d}{dx}(\cosh x) = \sinh x \end{aligned}\right.$$

$$\left\{\begin{aligned} \cosh ix &= \cos x \\ \sinh ix &= i \sin x \end{aligned}\right.$$

$$\cos ix = \cosh x$$

$$\sin ix = i \sinh x$$

Power Series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

So: $$e^{ix} = 1 + (ix) + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \cdots$$

$i^2 = -1$

$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)$$

$$+ i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)$$

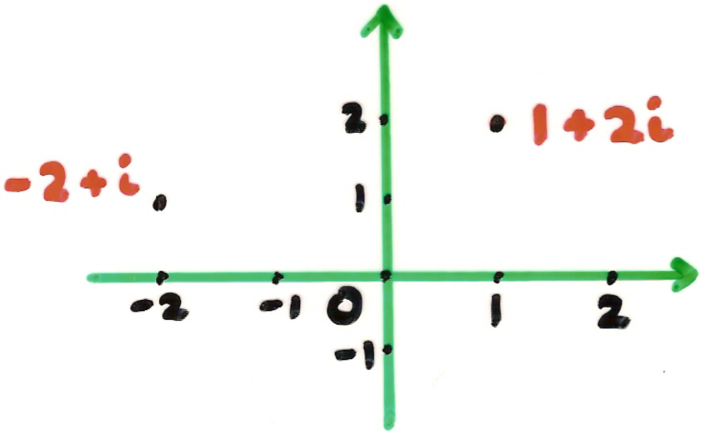
$$\boxed{e^{ix} = \cos x + i \sin x}$$

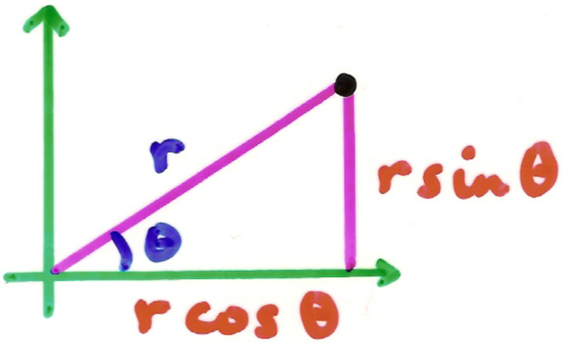
$e^{i\pi} = -1$, so $e^{i\pi} + 1 = 0$

Complex numbers

Two forms: $a + ib$, where $i^2 = -1$

$3+4i$, $7-12i$, ...





$$r(\cos\theta + i\sin\theta) = re^{i\theta}$$

Complex logarithms

$$2\log(-a) = \log(-a)^2 = \log a^2 = 2\log a,$$

so $\log(-a) = \log(a)$ (!)

If $z = re^{i\theta}$,

then $\log z = \log r + i\theta$

so $\log i = \log 1 + i\pi/2 = i\pi/2$.

and $i^i = e^{i\log i} = e^{-\pi/2}$.

Euler: There is always an infinity of logarithms which belong equally to each given number:
or, if y denotes the logarithm of the number x, I say that y contains an infinity of different values.

Transcendental numbers

$\sqrt{2}$, $\sqrt[3]{7}$, $\sqrt{2}+\sqrt{3}$ are algebraic:

they are solutions of polynomial equations

with integer coefficients:

$$x^2 - 2 = 0, \quad x^3 - 7 = 0, \quad x^4 - 10x^2 + 1 = 0$$

All other numbers are transcendental.

1873 (Hermite) e is transcendental

1882 (Lindemann) π is transcendental

Let $r, s, t, \dots z$ be distinct numbers (real or complex) which are algebraic

– similarly, $a, b, c, \dots, n$ (not all 0).

Then $ae^r + be^s + ce^t + \dots + ne^z \neq 0$.

But $e^{\pi i} + 1 = 0$, so πi is not algebraic

$a = b = 1$

$c = \dots = 0$

– so $\boxed{\pi \text{ is transcendental}}$