4000 YEARS OF NUMBER

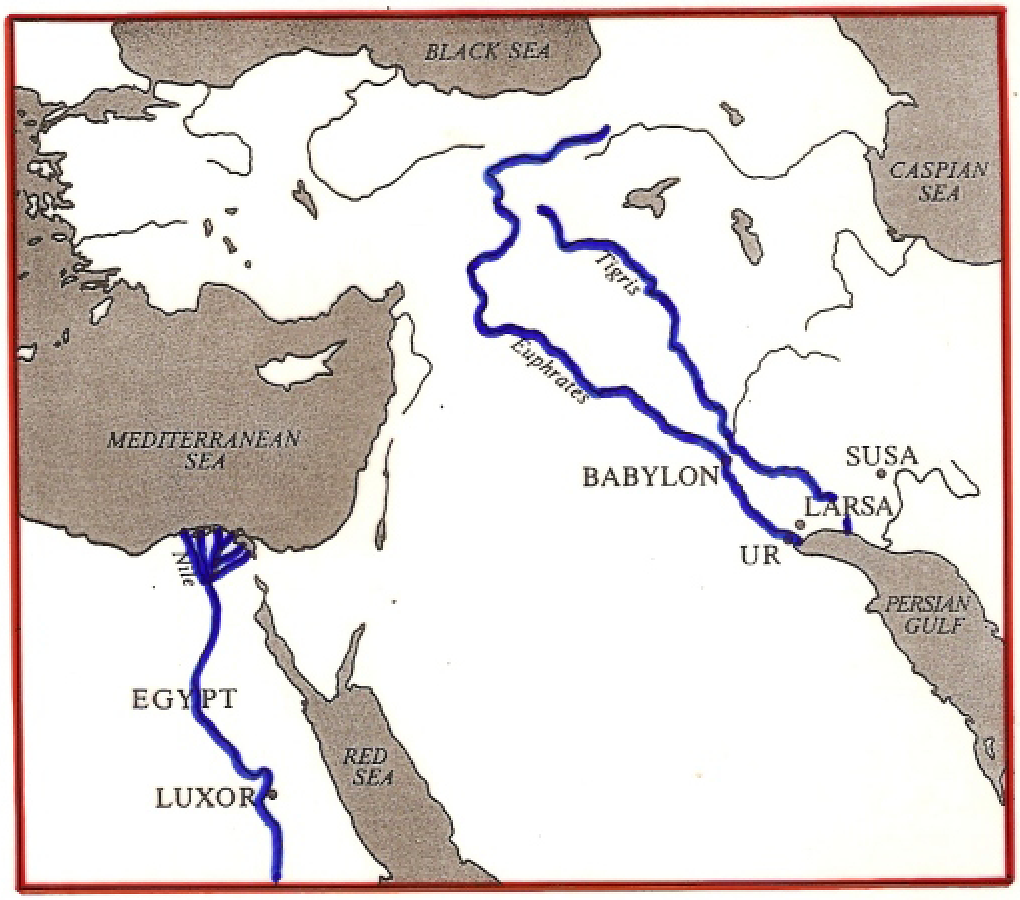
Robin Wilson

Number Words

English	Gothic	Latin	Greek	French
one	ains	unus	εἱς	un
two	twai	duo	δυο	deux
three	threis	tres	τρεις	trois
four	fidwor	quattuor	τετταρες	quatre
five	fimf	quinque	πεντε	cinq
six	saihs	sex	ἑξ	six
seven	sibun	septem	ἑπτα	sept
eight	ahtau	octo	ὀκτω	huit
nine	niun	novem	ἐννεα	neuf
ten	taihun	decem	δεκα	dix
eleven	ainlif	undecim	ἑνδεκα	onze
twelve	twalif	duodecim	δωδεκα	douze
thirteen	...	...	...	treize
...	...	...	...	...
twenty	twaitigjus	viginti	εἰκοσι	vingt

87 = quatre-vingt sept

Egypt and Mesopotamia

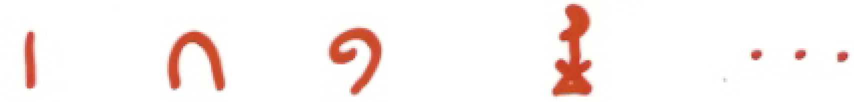


papyrus

clay tablet

Egyptian Counting and Calculation

Decimal system:	1	10	100	1000	. . .



	rod	heel bone	coiled rope	lotus flower	. . .

367 + 756 = 1123

Rhind papyrus, Problem 69 : 80 × 14 = 1120

80	1
800	10 /
160	2
320	4 /
1120	

[doubling and halving]

Egyptian fractions

Unit fractions: (reciprocals) $\frac{1}{n}$ (and $\frac{2}{3}$)

$$\frac{2}{11} = \frac{1}{6}\ \frac{1}{66}$$

$$\frac{2}{13} = \frac{1}{8}\ \frac{1}{52}\ \frac{1}{104}$$

Rhind papyrus, Problem 31

A quantity, its $\frac{2}{3}$, its $\frac{1}{2}$, and its $\frac{1}{7}$, added together, become 33.

What is the quantity?

[Solve: $x + \frac{2}{3}x + \frac{1}{2}x + \frac{1}{7}x = 33$]

Solution: The total is

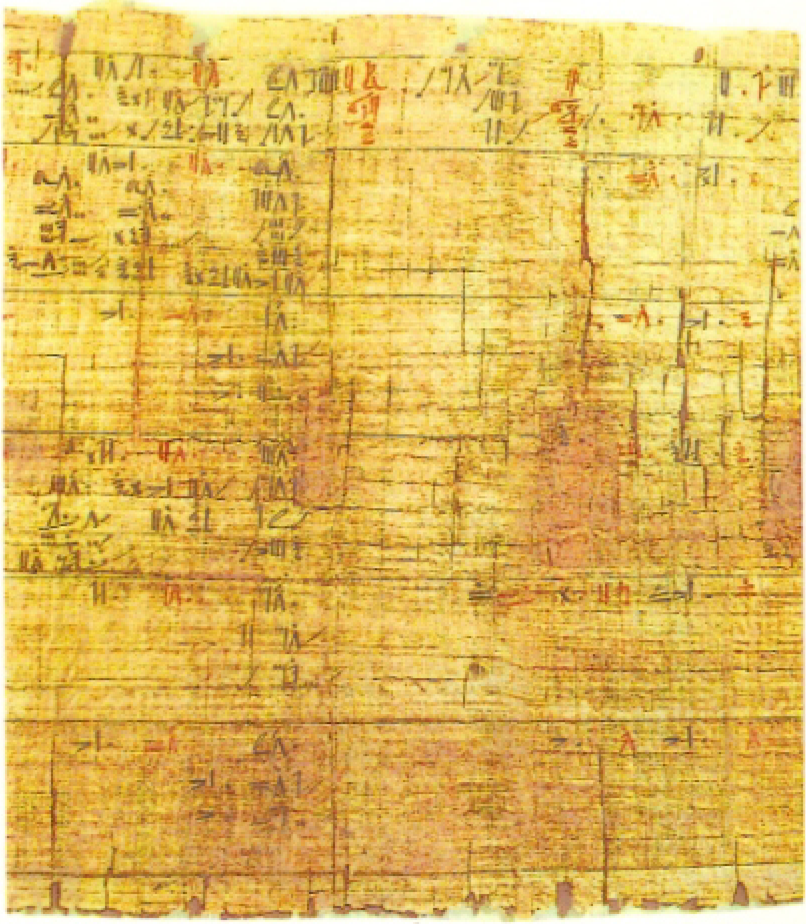
$$14\ \frac{1}{4}\ \frac{1}{56}\ \frac{1}{97}\ \frac{1}{194}\ \frac{1}{388}\ \frac{1}{679}\ \frac{1}{776},$$

$14\frac{28}{97}$

which multiplied by $1\ \frac{2}{3}\ \frac{1}{2}\ \frac{1}{7}$ makes 33.

Table of Fractions

$^2/n$, for $n = 5, 7, 9, 11, \cdots, 99, 101$



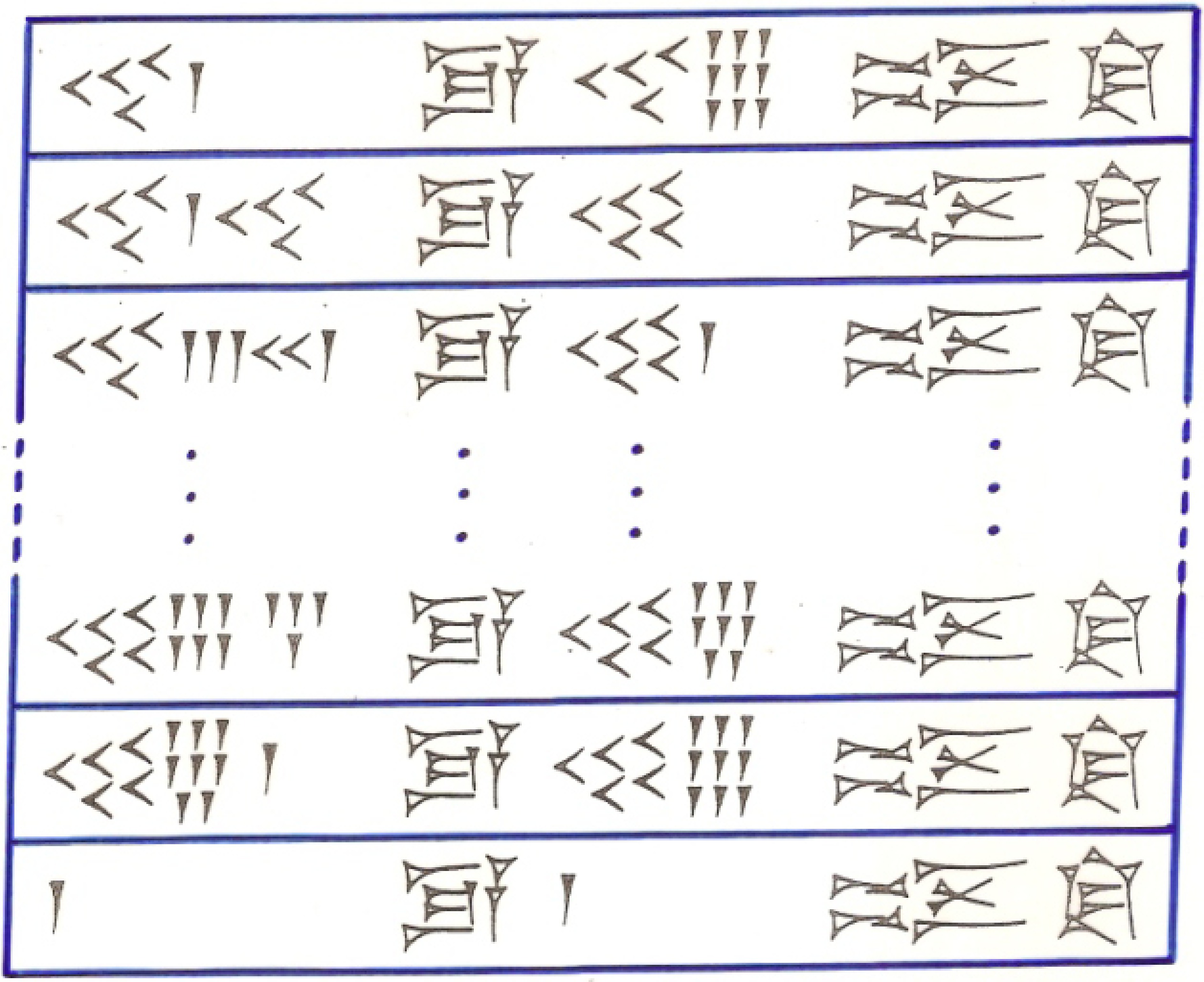
Mesopotamian Mathematics

clay tablets – cuneiform writing

place-value system based on 60: < , |

<<<<| <<<< = 41(60)+40, or $41\frac{40}{60}$, or …

Larsa table text



2401 equals 49 squared

2500 equals 50 squared

. . .

3600 equals 60 squared

Multiplication Tables

1	9
2	18
3	27
4	36
5	45
6	54
7	63
8	72
9	81
10	90
11	99
12	108
13	117
14	126

9 times table

1	5
2	10
3	15
4	20
5	25
6	30
7	35
8	40
9	45
10	50
11	55

5 times table

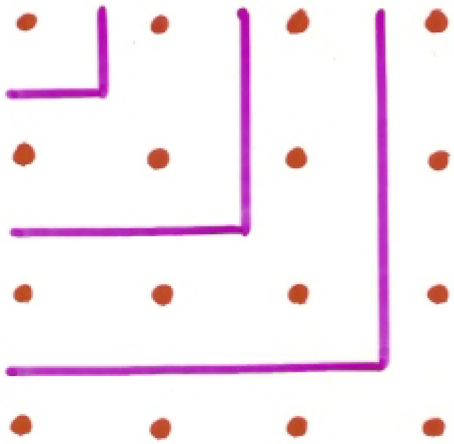
Table of Reciprocals

Two thirds of 1 is 0;40.
Its half is 0;30.
The reciprocal of 2 is 0;30.
The reciprocal of 3 is 0;20.
The reciprocal of 4 is 0;15.
The reciprocal of 5 is 0;12.
The reciprocal of 6 is 0;10.
The reciprocal of 8 is 0;07 30.
The reciprocal of 9 is 0;06 40.
The reciprocal of 10 is 0;06.
The reciprocal of 12 is 0;05.
The reciprocal of 15 is 0;04.
The reciprocal of 16 is 0;03 45.
The reciprocal of 18 is 0;03 20.
The reciprocal of 20 is 0;03.
The reciprocal of 24 is 0;02 30.
The reciprocal of 25 is 0;02 24.
The reciprocal of 27 is 0;02 13 20.
The reciprocal of 30 is 0;02.
The reciprocal of 32 is 0;01 52 30.
The reciprocal of 36 is 0;01 40.
The reciprocal of 40 is 0;01 30.
The reciprocal of 45 is 0;01 20.
The reciprocal of 48 is 0;01 15.
The reciprocal of 50 is 0;01 12.
The reciprocal of 54 is 0;01 06 40.
The reciprocal of 1 00 is 0;01.
The reciprocal of 1 04 is 0;00 56 15.
The reciprocal of 1 21 is 0;00 44 26 40.
<Its half>

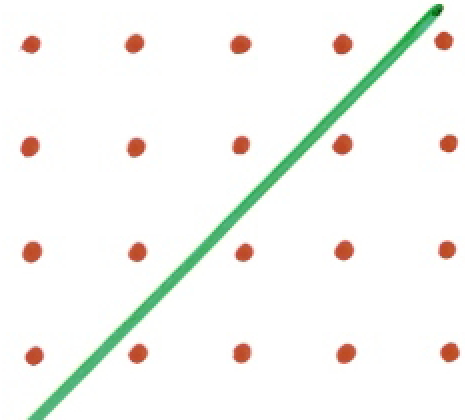
Pythagorean Figurate Numbers

Represent numbers geometrically...

$15 = 1 + 2 + 3 + 4 + 5$

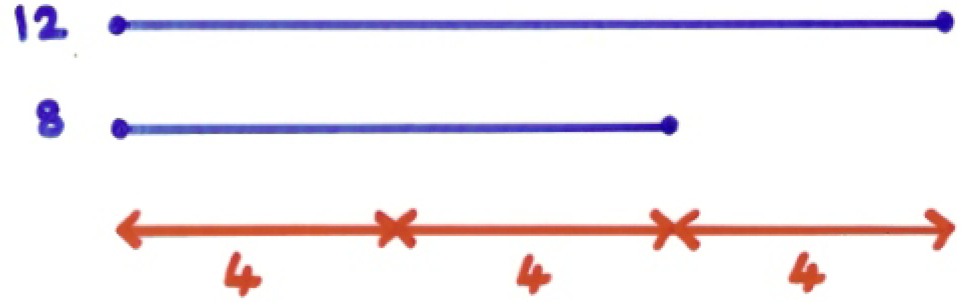


$16 = 1 + 3 + 5 + 7$

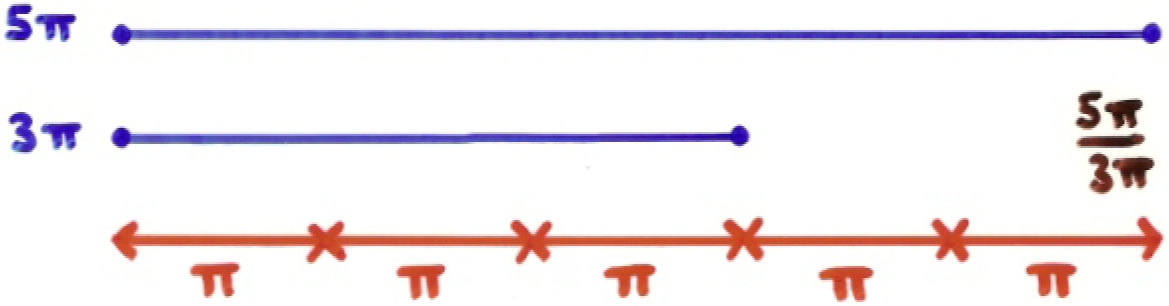


Any square number is the sum of two consecutive triangular numbers.

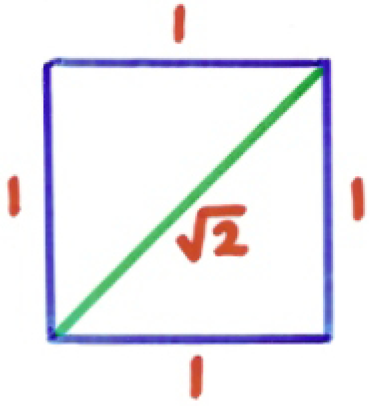
Commensurability



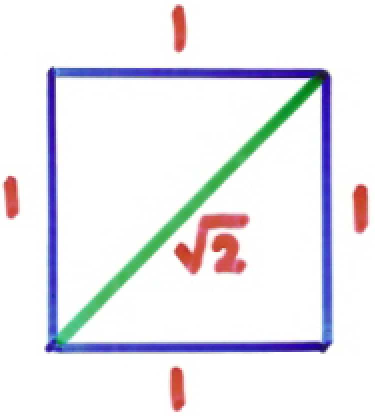
$$\frac{12}{8} = \frac{3}{2}$$



$$\frac{5\pi}{3\pi} = \frac{5}{3}$$



are 1 and $\sqrt{2}$ commensurable?



Let $\sqrt{2} = \frac{a}{b}$

(a/b in lowest terms)

Then: $2 = \frac{a^2}{b^2}$, so $a^2 = 2b^2$.

So a^2 is even and a is even

Let $a = 2k$, so $a^2 = 4k^2 = 2b^2$.

So $b^2 = 2k^2$.

So b^2 is even and b is even.

So a and b are both divisible by 2.

CONTRADICTION

So $\sqrt{2}$ cannot be written as a/b

Greek Counting

1	2	3	4	5	6	7	8	9	10
α	β	γ	δ	ε	ϛ	ζ	Η	θ	ι

10	20	30	40	50	60	70	80	90	100
ι	κ	λ	μ	ν	ξ	ο	π	ϟ	ρ

Multiplication table

μηκος

βάθος

	1	2	3	4	5	6	7	8	9	10
1	α	β	γ	δ	ε	ϛ	ξ	η	ϑ	ι
2	β	δ	ϛ	η	ι	ιβ	ιδ	ιϛ	ιη	κ
3	γ	ϛ	ϑ	ιβ	ιε	ιη	κα	κδ	κξ	λ
4	δ	η	ιβ	ιϛ	κ	κδ	κη	λβ	λϛ	μ
5	ι	ε	ιε	κ	κε	λ	λε	μ	με	ν
6	ϛ	ιβ	ιη	κδ	λ	λϛ	μβ	μη	νδ	ξ
7	ζ	ιδ	κα	κη	λε	μβ	μθ	νϛ	ξγ	ο
8	η	ιϛ	κδ	λβ	μ	μη	νϛ	ξδ	οβ	π
9	ϑ	ιη	κζ	λϛ	με	νδ	ξγ	οβ	πα	ϟ
10	ι	κ	λ	μ	ν	ξ	ο	π	ϟ	ρ

βάθος

μῆκος

Archimedes' 'Sand-reckoner'

Count as far as you can:

a myriad = 10,000

Use this to go to the next stage:

a myriad myriads = 100,000,000

Next go to

$(100{,}000{,}000)^2$, $(100{,}000{,}000)^3$ ...

up to $P = (100{,}000{,}000)^{100{,}000{,}000}$

Form powers of P:

P^2, P^3, ..., $P^{100{,}000{,}000}$, and further

Archimedes reached the number

1 followed by 80,000,000,000,000,000 zeros.

Much ado about zero

We use numbers to count things:

5 cows – but not 0 cows or –20 cows.

Egyptians: grouping system ||| ∩∩∩∩∩

Greeks: separate symbols for

1, 2, ..., 9, 10, 20, ..., 90, 100, 200, ..., 900

Mesopotamians: place-value system based on 60

< < < < | 41?   41 × 60?

(40 × 60) + 1 = 2401

Use of context to determine meaning.

(1) 0 as a place-holder:

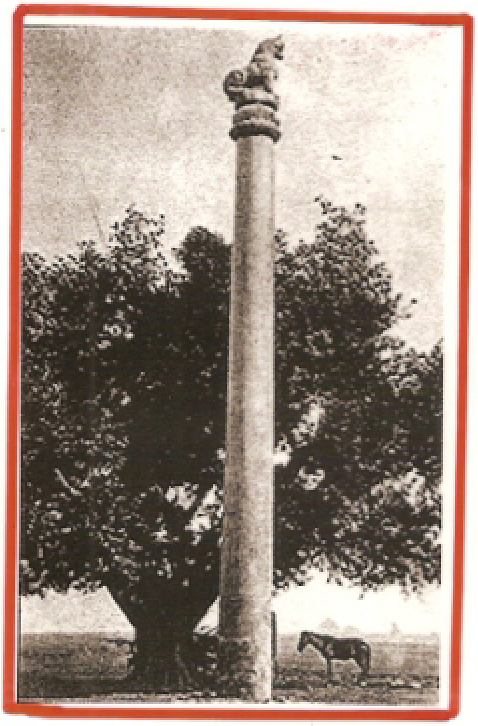
305 different from 35 or 3 5, or 350.

(Later Mesopotamians used ⋞ for a zero.)

(2) 0 as a number: 2 – 2 = 0   (cf 3 – 2 = 1)

(Much later – India)

Chinese and Indian counting

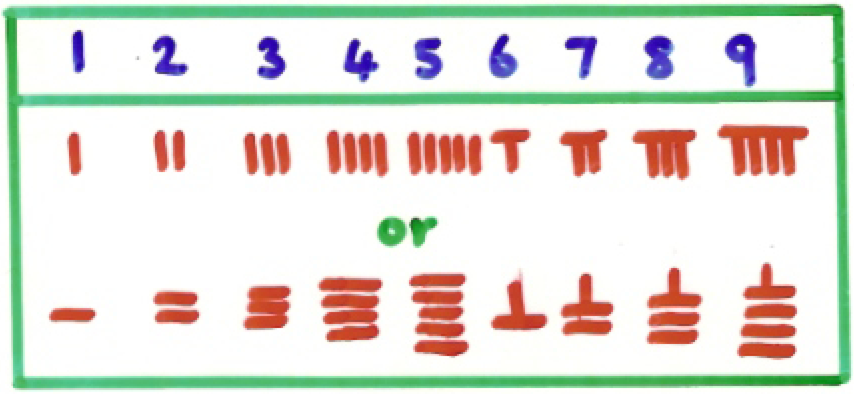


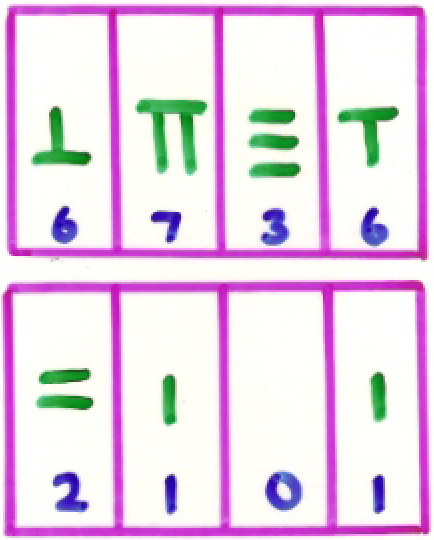
India : King Ashoka (c. 250 BC)

Numbers inscribed on pillars around the kingdom

Place-value system based on 10.

Chinese counting board:





Indian number system similar to this:

- place-value system based on 10
- used only 1, 2, 3, ···, 9 – and eventually 0.

Brahmagupta: zero and negative numbers

The sum of cipher and negative is negative;
of positive and nought, positive;
of two ciphers, cipher.

Negative taken from cipher becomes positive,
and positive from cipher is negative;
cipher taken from cipher is nought.

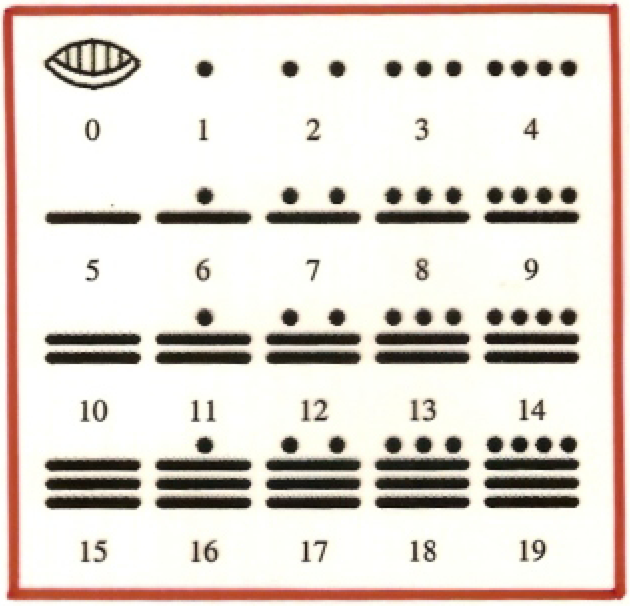
The product of cipher and positive,
or of cipher and negative, is nought;
of two ciphers, it is cipher.

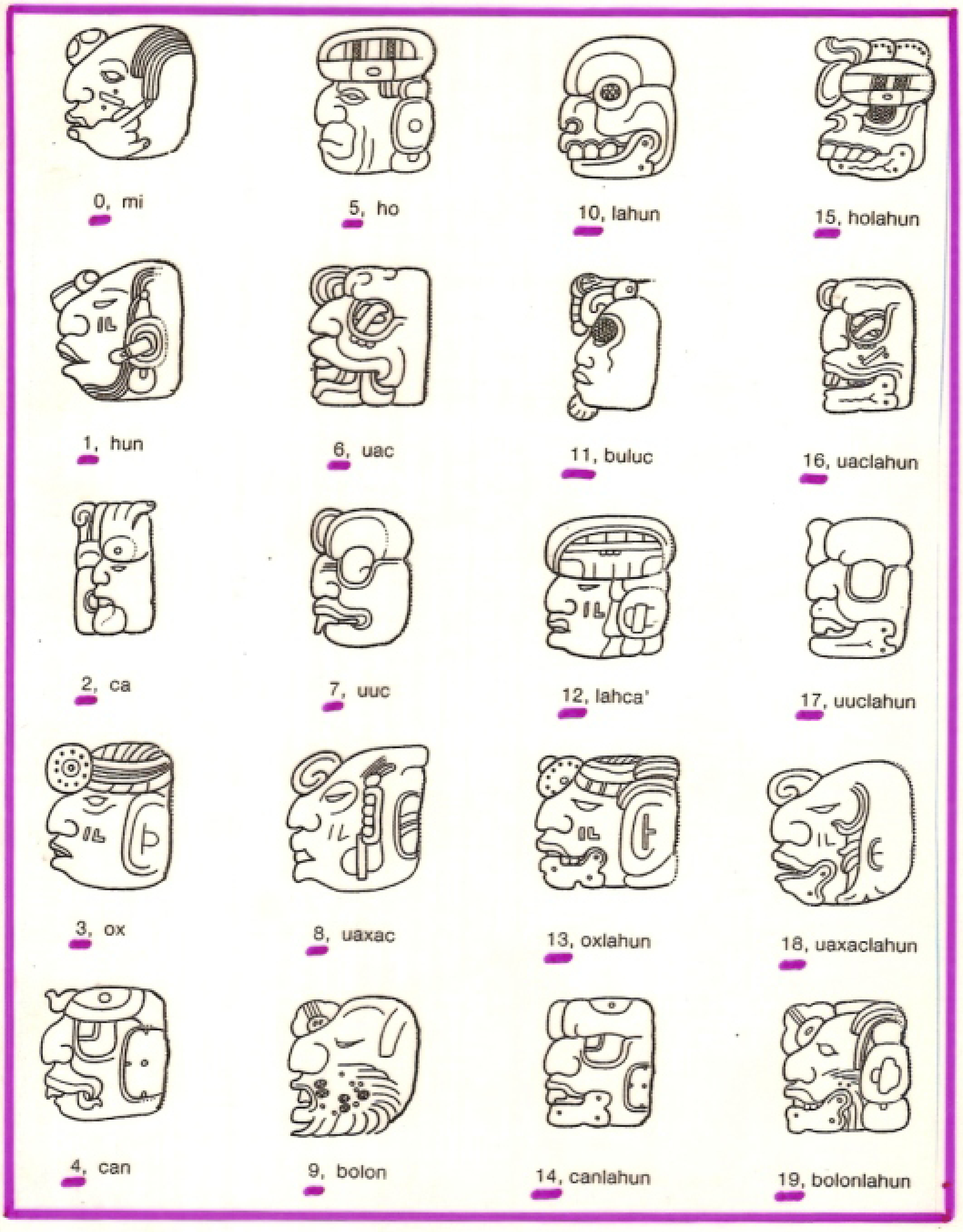
Cipher divided by cipher is nought.

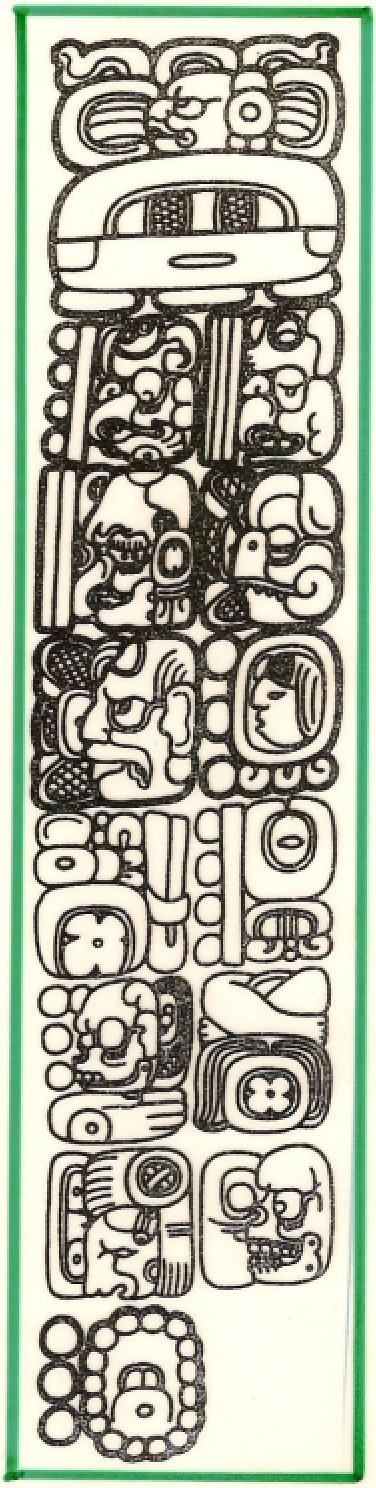
Positive or negative divided by cipher is
a fraction with that as denominator... $\left(\frac{\pm n}{0}\right.$

Cipher divided by positive or negative is... $\left(\frac{0}{\pm n}\right.$

Mayan counting







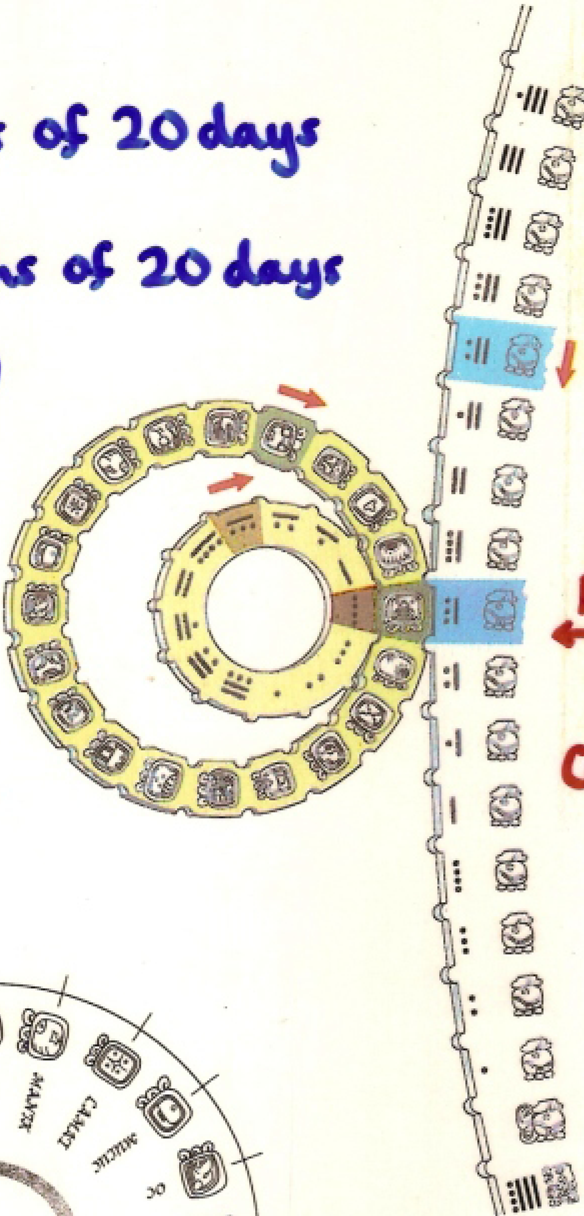
The Mayan calendar

Two forms:

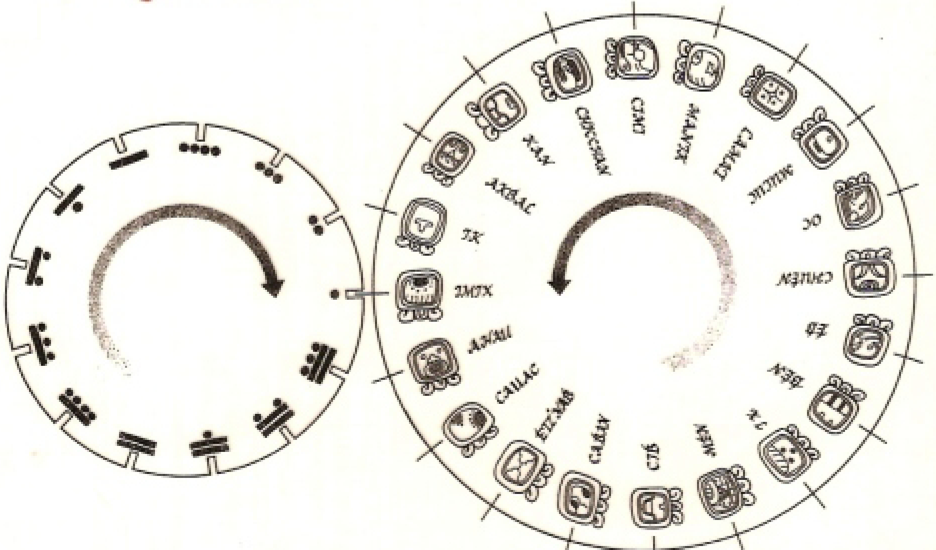
260 days : 13 months of 20 days

365 days : 18 months of 20 days
(+ 5 days)

These combine :
calendar-round of
18980 days
(= 52 years).



4
Ahau
8
Cumku



13 × 20 = 260 days
day names

Mayan timekeeping

1 kin = 1 day

20 kins = 1 uinal = 20 days

18 uinals = 1 tun = 360 days

20 tuns = 1 katun = 7200 days

20 katuns = 1 baktun = 144,000 days ...

= 4 × 2,880,000 = 11,520,000

= 6 × 144,000 = 864,000

= 14 × 7200 = 100,800

= 13 × 360 = 4680

= 15 × 20 = 300

= 1 × 1 = 1

Total: 12,489,781 days

Muhammad ibn Musa al-Khwarizmi

- Arithmetic text

كتاب الخوارزمي

- Algebra text

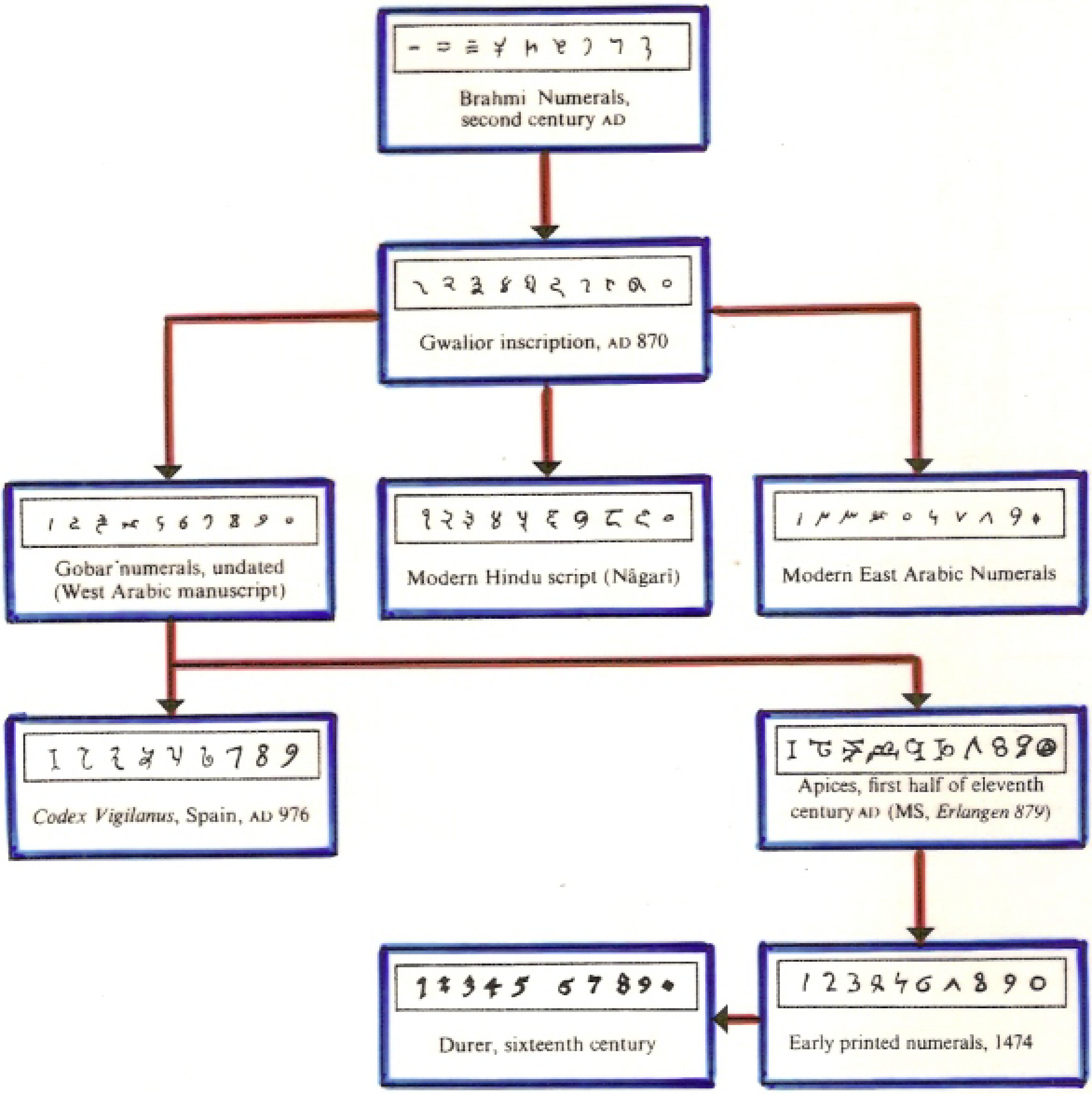
Kitab al-jabr w'al-muqabalah

= Ludus algebrae et almucgrabalaeque

Algorithmi de numero Indorum

'Dixit Algorismi'

The Hindu-Arabic Numerals



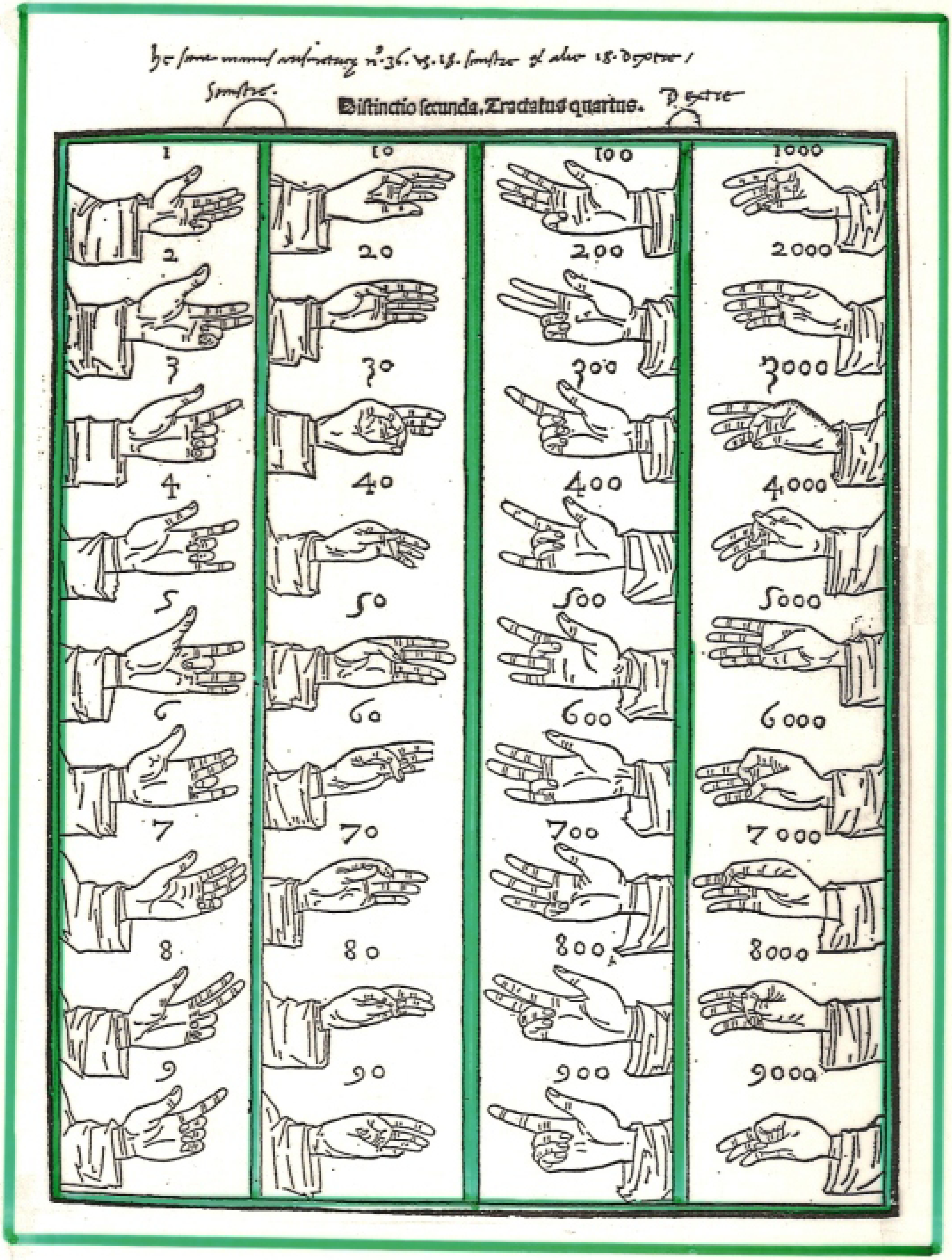
Gerbert of Aurillac (938–1003)

Pope Sylvester II
(999)

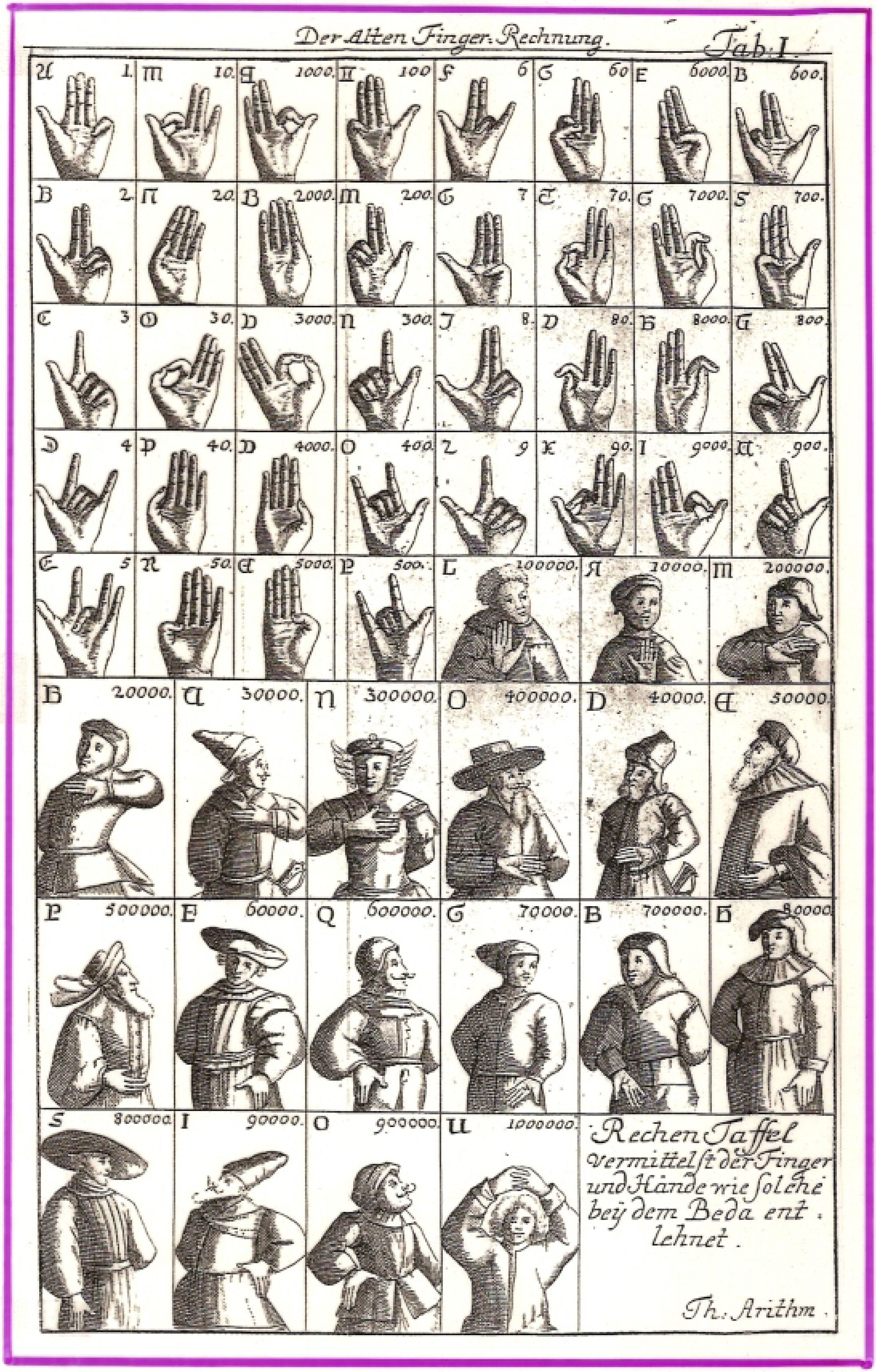
Trained in Catalonia

Introduced Hindu-Arabic numerals
to Christian Europe

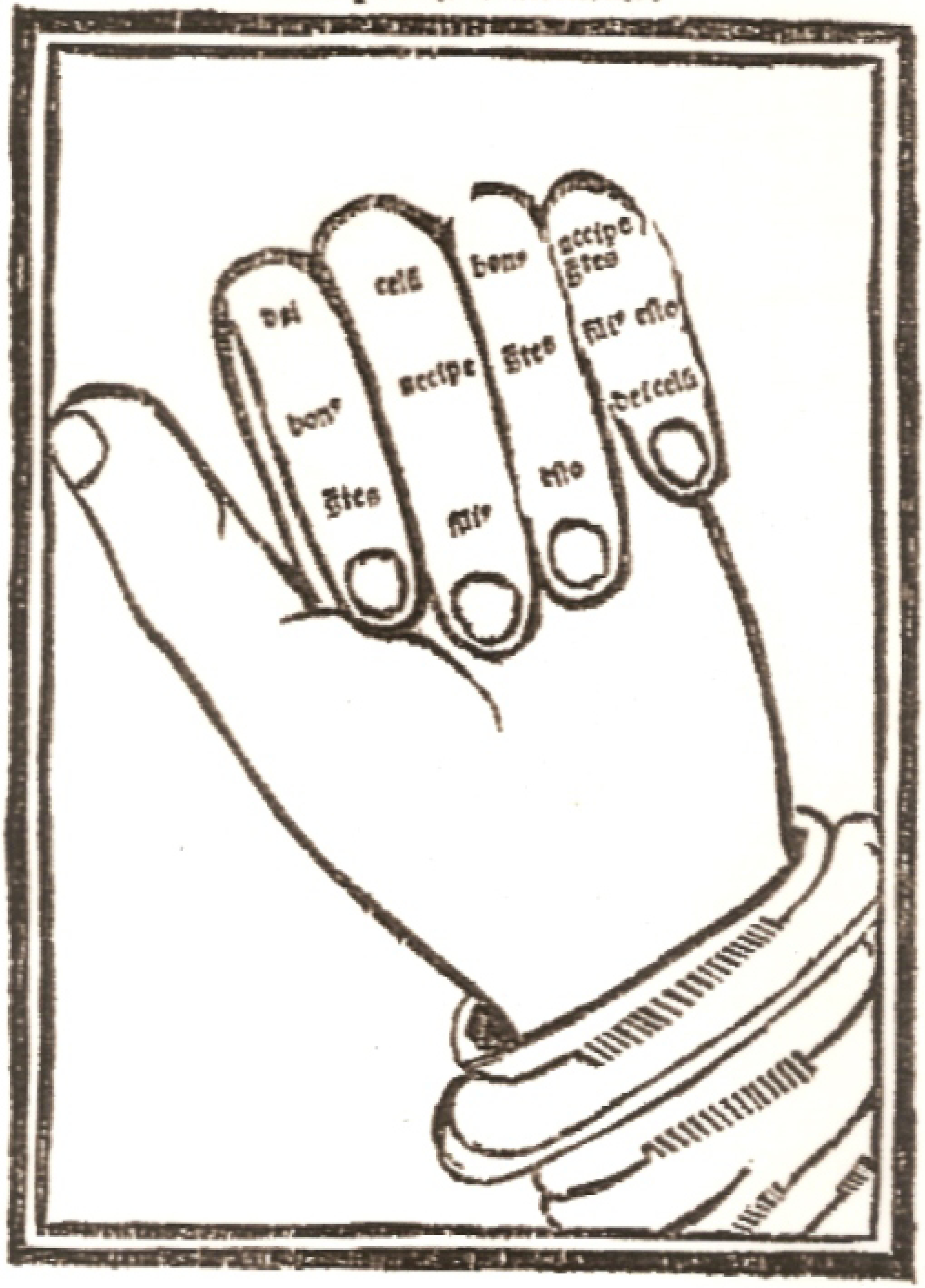
Finger counting



Finger counting



Compotus manualis.



Si forsan nescis que littera tu dñil sit
Et quotus est solis annus cognoscere queris
Annos a dñi: demas ãnis prius octo
Perq; quater septem dñi diuiseris annos
Dat quotus est solis ãnus quicquid remanebit

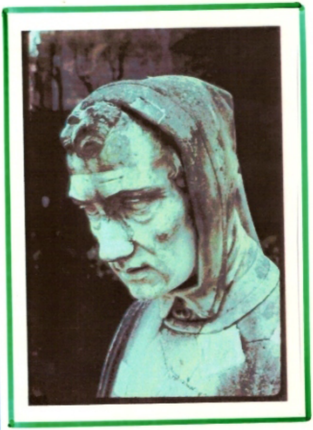
Compotus manualis ad usum Oxoniensium, 1520.

Fibonacci (Leonardo of Pisa)

Liber abaci (1202)

Book of squares

Hindu-Arabic numerals



The first arithmetic book in English (1537)

An introduction

for to lerne to recken with the pen, or with the counters accordyng to the trewe cast of Algorisme, in hole numbers or in broken, newly corrected. And certayne notable and goodly rules of false positions therevnto added, not before sene in oure Englyshe tonge, by the which all maner of difficile questions may easely be dissolued and assoyled. Anno. 1546.

The groũd of artes

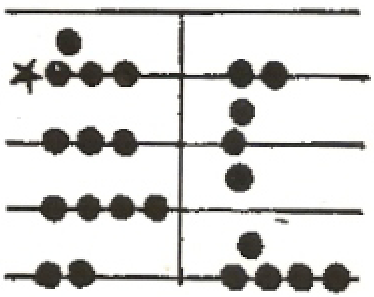
teachyng the worke and pra-
ctise of Arithmetike, moch necessary
for all states of men. After a more
easyer & exacter sorte, then any
lyke hath hytherto ben set
forth: with dyuers newe
additions, as by the
table doth partly
appeare.

ROBERT RECORDE.

ADDITION.

Maſter.

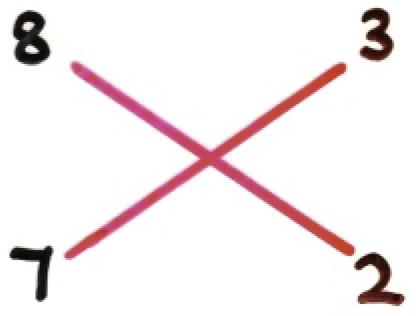
The eaſieſt way in this arte, is to adde but two ſummes at ones togyther: how be it, you maye adde more, as I wil tel you anone. therefore whenne you wylle adde two ſummes, you ſhall fyrſte ſet downe one of them, it forceth not whiche, and then by it draw a lyne croſſe the other lynes. And afterwarde ſette doune the other ſumme, ſo that that lyne maye be betwene them: as if you woulde adde 2659 to 8342, you muſt ſet your ſūmes as you ſee here.



And then if you lyſt, you maye adde the one to the other in the ſame place, or els you may adde them bothe tohither in a new place: which way, bycauſe it is moſt plyueſt

Multiplication

$8 \times 7 = 56$



(10-7)

(10-8)

$(8-3)$ or $(7-2) = 5$

$3 \times 2 = 6$

} 56

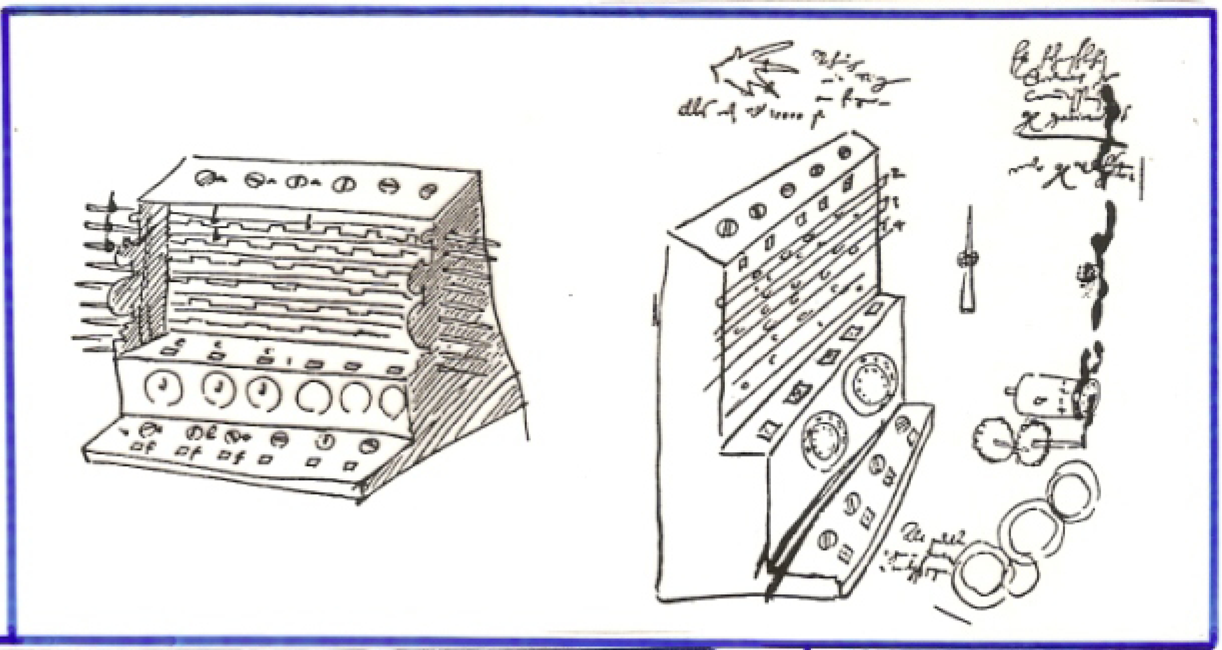
DE
THIENDE

Leerende door onghehoorde lichticheyt allen rekeningen onder den Menschen noodich vallende, afveerdighen door heele ghetalen sonder ghebrokenen.

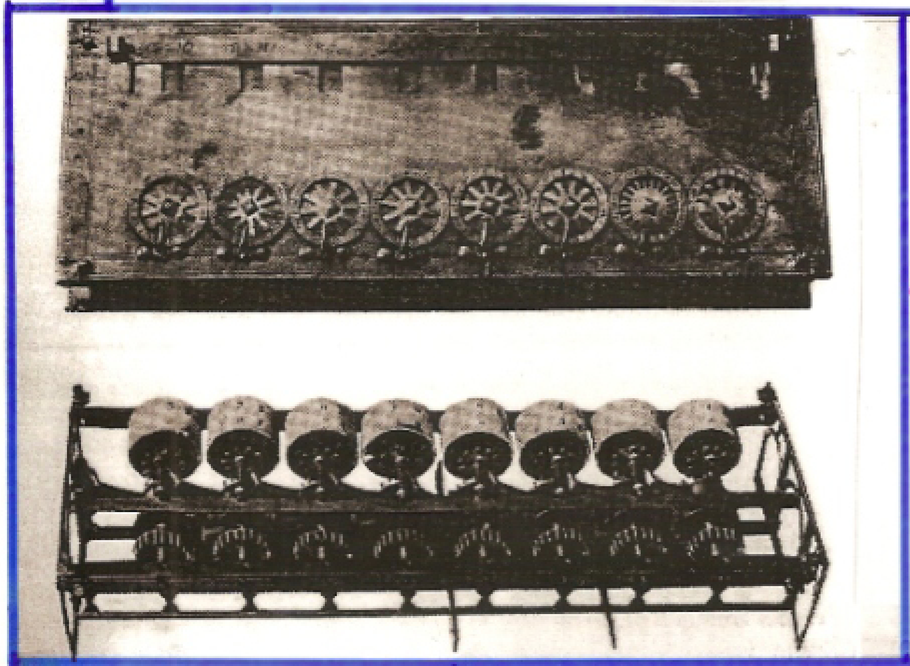
Beschreven door SIMON STEVIN· *van Brugghe*.

TOT LEYDEN,
By Christoffel Plantijn.
M. D. LXXXV.

Calculating Machines

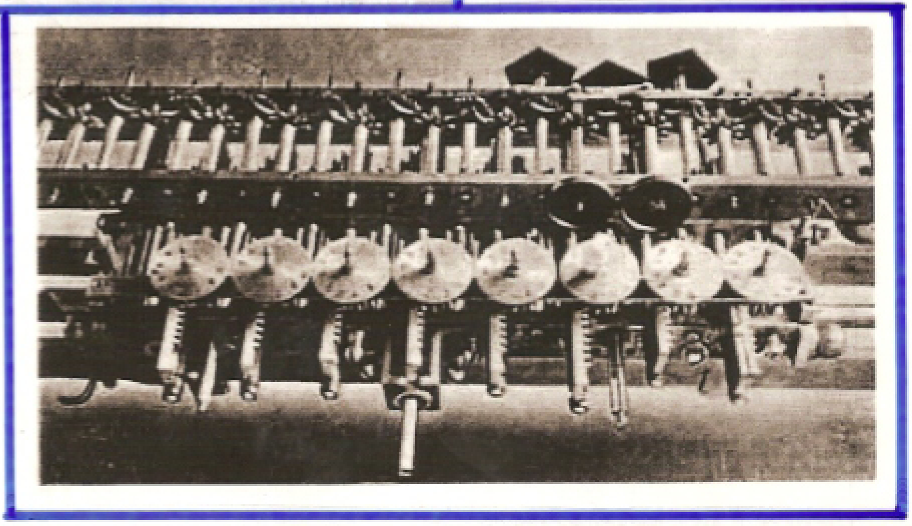


↑Schickhard



←Pascal

↓Leibniz

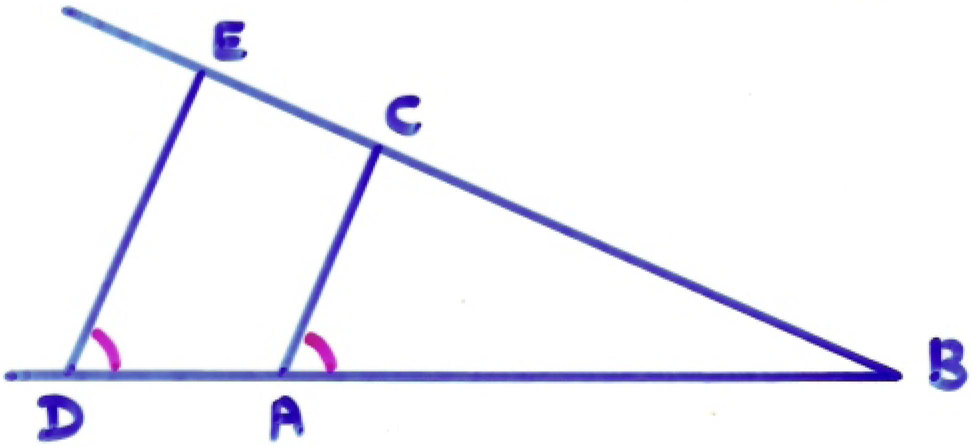


9. Pascal's Calculating Machine, 1642. Engraving from *Machines approuvées par l'Académie Royale des Sciences*, Vol. 4. Paris 1735

Descartes and Dimension

What is meant by $x^2 + x$?

How can we multiply lengths?



Take a unit line AB:

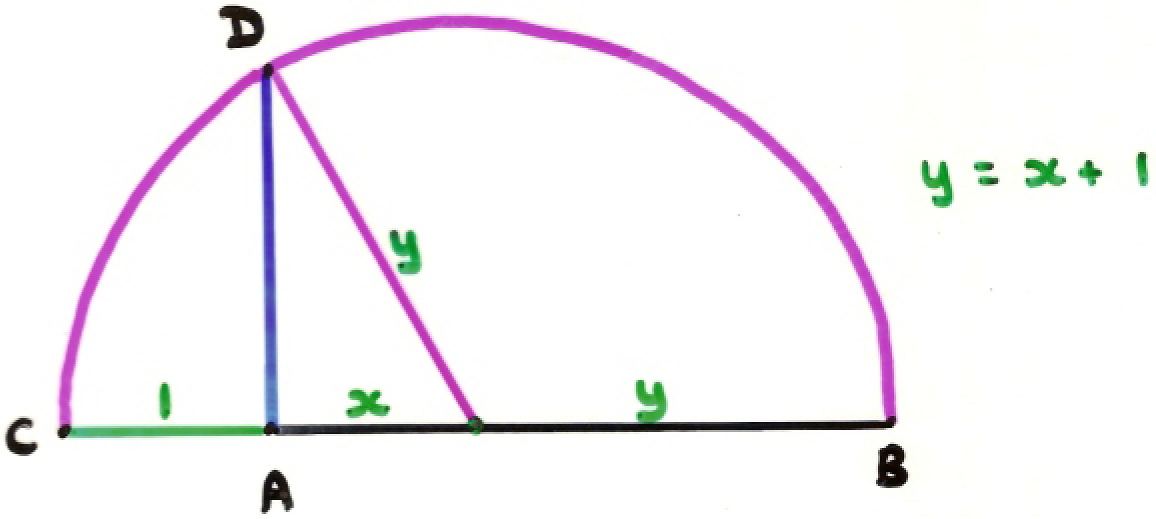
what is $BD \times BC$?

Draw ED parallel to AC:

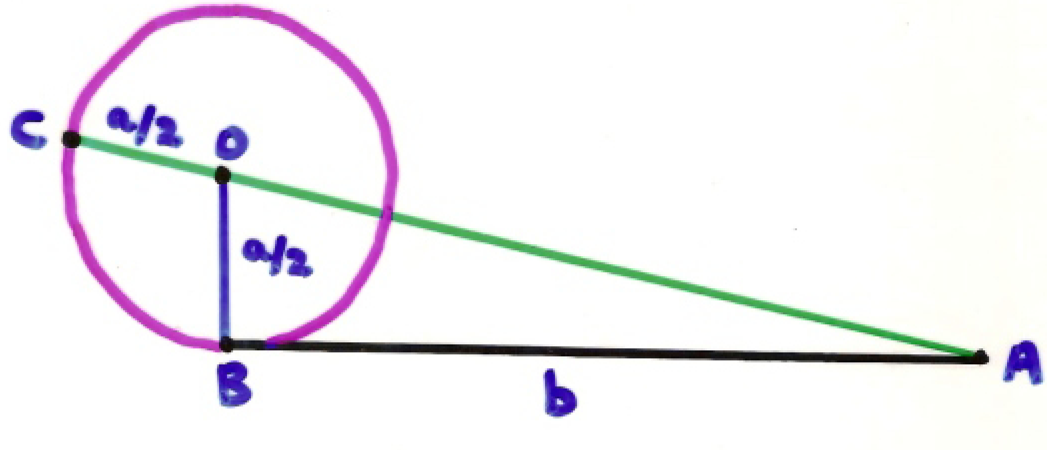
$BE / BD = BC / AB = BC$,

so $BD \times BC = BE$.

Descartes' constructions



The square root of AB is AD.



AC is the positive root of

$$x^2 = ax + b^2.$$

Some Numbers

Natural numbers:

$1, 2, 3, 4, 5, \cdots$

Integers:

$\cdots, -2, -1, 0, 1, 2, 3, \cdots$

Rational numbers:

$5/7\ ,\ 11/3\ ,\ -1/7\ , \cdots$

Real numbers:

$\sqrt{2}\ ,\ \sqrt[3]{7}\ ,\ \sqrt{2} + \sqrt{3}, \cdots$

$\pi\ ,\ e\ ,\ \cdots$

Complex numbers:

$\sqrt{-1}\ ,\ 3 - 4\sqrt{-1}, \cdots$

Some Numbers

Natural numbers:

$1, 2, 3, 4, 5, \ldots$ $\qquad x + 3 = 7$

Integers:

$\ldots, -2, -1, 0, 1, 2, 3, \ldots$ $\qquad x + 7 = 3$

Rational numbers:

$5/7, 11/3, -1/7, \ldots$ $\qquad 7x = 5$

Real numbers:

$\sqrt{2}, \sqrt[3]{7}, \sqrt{2} + \sqrt{3}, \ldots$ $\qquad x^3 = 7$

$\pi, e, \ldots$

Complex numbers:

$\sqrt{-1}, 3 - 4\sqrt{-1}, \ldots$ $\qquad x^2 = -1$

Cardano's problem

Divide 10 into two parts whose product is 40.

If the parts are x and $10-x$, then $x(10-x) = 40$.

Cardano: $x = 5+\sqrt{-15}$ or $5-\sqrt{-15}$

'Nevertheless we will operate, putting aside the mental tortures involved.'

$$x(10-x) = (5+\sqrt{-15})(5-\sqrt{-15})$$

$$= 25 - (-15) = 40.$$

George Airy: 'I have not the smallest confidence in any result which is essentially obtained by the use of imaginary symbols.'

Augustus De Morgan: 'We have shown the symbol $\sqrt{-1}$ to be void of meaning, or rather self-contradictory and absurd.'

Complex Numbers

$a + b\sqrt{-1}$, or $a + bi$, $i^2 = -1$.

Addition

$(2 + 3\sqrt{-1}) + (4 + 5\sqrt{-1})$

$= (2 + 4) + (3 + 5)\sqrt{-1} = 6 + 8\sqrt{-1}$;

or: $(2 + 3i) + (4 + 5i) = 6 + 8i$.

Multiplication

$(2 + 3\sqrt{-1}) \times (4 + 5\sqrt{-1})$

$= (2 \times 4) + (3 \times 4)\sqrt{-1} + (2 \times 5)\sqrt{-1} + (3 \times 5)(-1)$

$= (8 - 15) + (12 + 10)\sqrt{-1} = -7 + 22\sqrt{-1}$;

or: $(2 + 3i) \times (4 + 5i) = -7 + 22i$.

What is the square root of i ?

If $x^2 = i$ and $x = a + bi$,

then $(a+bi)^2 = i$,

so $(a^2 - b^2) + 2abi = i$.

So $a^2 - b^2 = 0$ and $2ab = 1$.

This has solutions $a = b = \pm\sqrt{\frac{1}{2}}$

so $x = \pm\frac{1}{\sqrt{2}}(1+i)$.

A Sextic Polynomial

$$x^6 - 12x^5 + 60x^4 - 160x^3 + 239x^2 - 188x + 60 = 0$$

$$= (x^2 - 4x + 3)(x^2 - 4x + 4)(x^2 - 4x + 5)$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$= (x-1)(x-3) \times (x-2)^2 \times (x^2 - 4x + 5)$$

$$\downarrow \qquad \downarrow$$

$$= (x-1)(x-3)(x-2)^2 \times (x-2-i)(x-2+i),$$

so $x = 1, 3, 2$ (twice) and $2 \pm i$.

Hamilton explains imaginaries

Define complex numbers as pairs

$$a + bi \longrightarrow (a, b)$$

which are combined as follows:

Addition:

$$(a, b) + (c, d) = (a+c, b+d)$$

$$[(a+bi) + (c+di) = (a+c) + (b+d)i]$$

Multiplication

$$(a, b) \times (c, d) = (ac-bd, ad+bc)$$

$$[(a+bi) \times (c+di) = (ac-bd) + (ad+bc)i]$$

Note that $(a, 0)$ corresponds to a (real)

and $(0, 1)$ corresponds to i,

and $(0,1) \times (0,1) = (-1, 0)$ $[i^2 = -1]$

Hamilton tries triples

Look at: $a + bi + cj, \; i^2 = j^2 = -1.$

Addition

$$(a + bi + cj) + (d + ei + fj)$$

$$= (a+d) + (b+e)i + (c+f)j.$$

Multiplication

$$(a + bi + cj) \times (d + ei + fj)$$

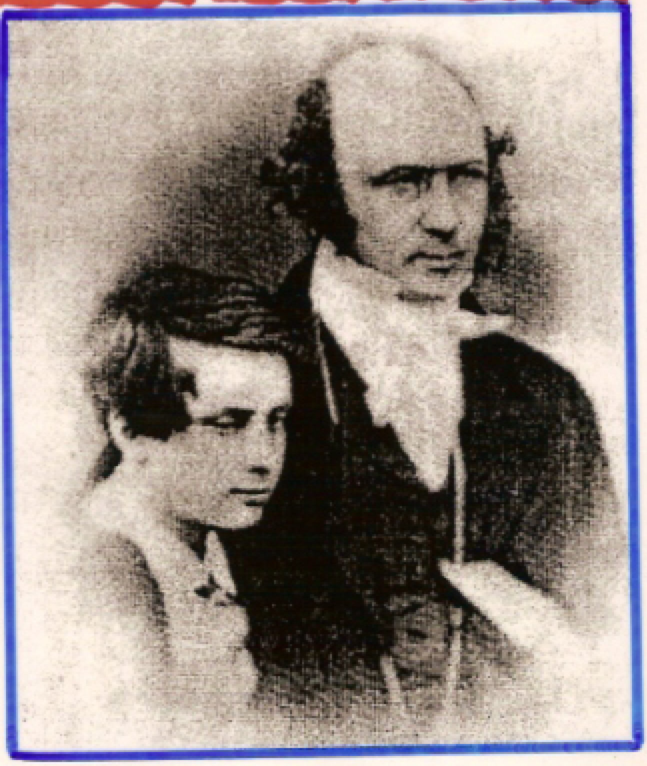
$$= (ad - be - cf) + (ae + bd)i$$

$$+ (af + cd)j + (bf + ce)ij.$$

$ij = 0$: no, since $(ij)^2 = i^2 j^2 = 1$.

$ij = 1$ or -1: these don't work either ...

Hamilton writes to his son



Every morning, on my coming down to breakfast, your little brother William Edwin and yourself used to ask me, 'Well Papa, can you multiply triples? Whereto I was obliged to reply, with a shake of the head: 'No, I can only add and subtract them'.

Hamilton takes a walk

As I was walking with Lady Hamilton to Dublin, and came up to Brougham Bridge, I then and there felt the galvanic circuit of thought close; and the sparks which fell from it were the fundamental equations exactly as I have used them ever since.

I pulled out on the spot a pocket book and made an entry … it is fair to say that this was because I felt a problem to have been at that moment solved — an intellectual want relieved which had haunted me for at least fifteen years since.

Hamilton's Quaternions

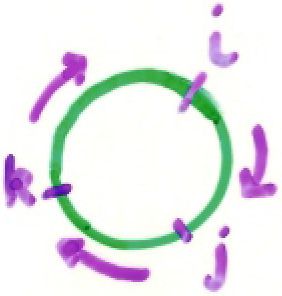
$a + bi + cj + dk$, $\quad i^2 = j^2 = k^2 = -1$:

Addition: ✓

Multiplication:

$ij = k, \; jk = i, \; ki = j$

$ji = -k, \; kj = -i, \; ik = -j$



More concisely:

$$\boxed{i^2 = j^2 = k^2 = ijk = -1.}$$

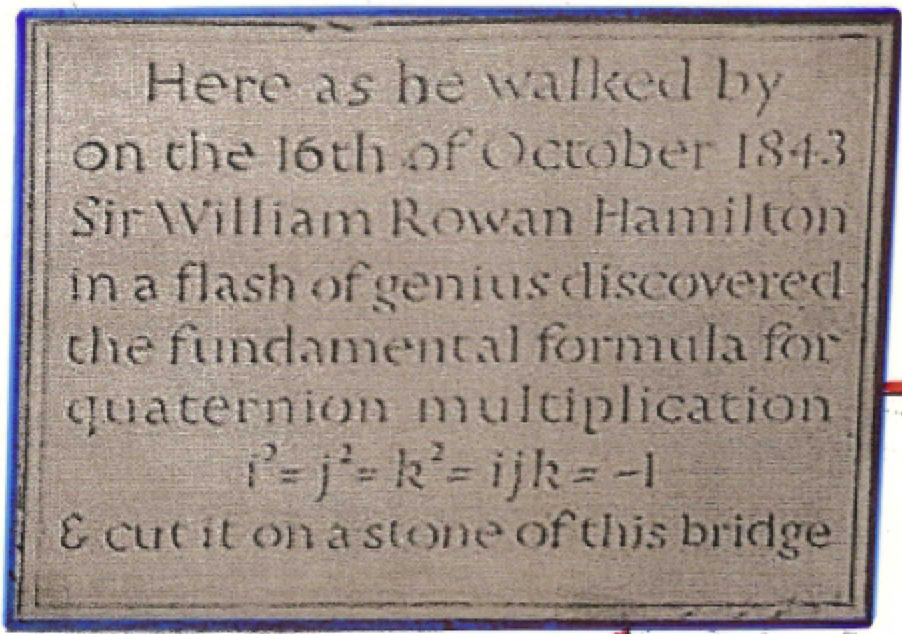
non-commutative system

Pauli matrices:

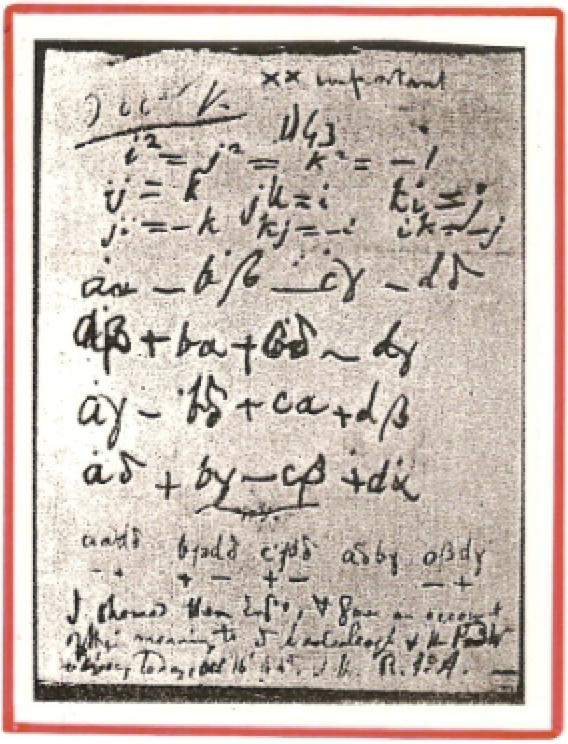
$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

$1 \qquad\qquad i \qquad\qquad j \qquad\qquad k$

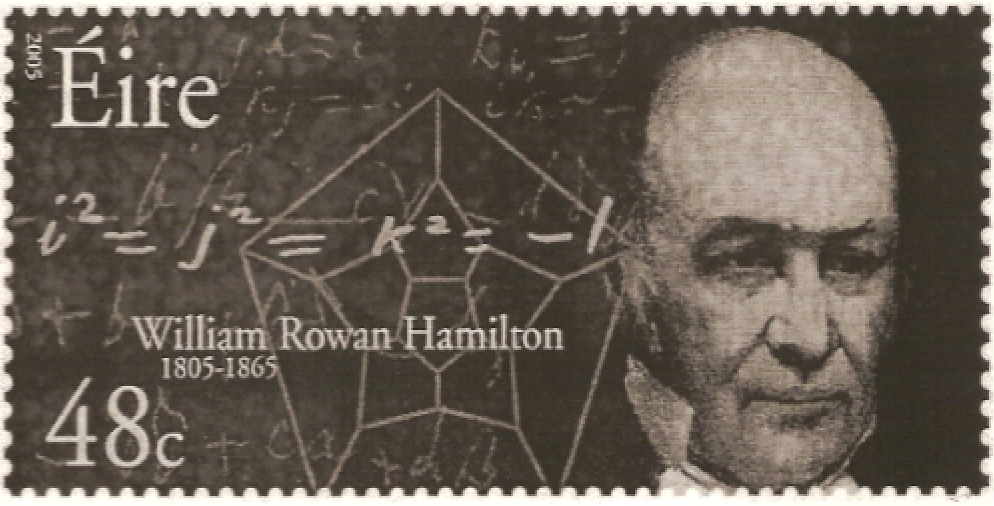
Hamilton at Brougham Bridge

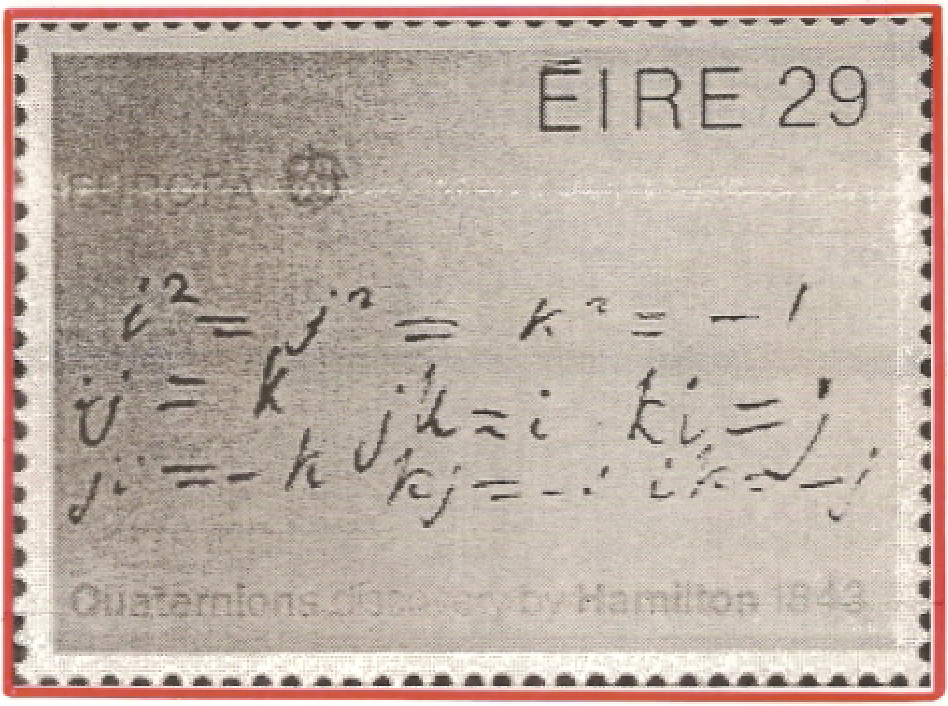


plaque on bridge



pocket book


2005
Éire
$i^2 = j^2 = k^2 = -1$
William Rowan Hamilton
1805-1865
48c


ÉIRE 29
$i^2 = j^2 = k^2 = -1$
$ij = k \quad jk = i \quad ki = j$
$ji = -k \quad kj = -i \quad ik = -j$

Octonions

$$\alpha + \beta \underline{i} + \gamma \underline{j} + \delta \underline{k} + \varepsilon \underline{l} + \zeta \underline{m} + \eta \underline{n} + \theta \underline{o},$$

where $i^2 = j^2 = k^2 = \cdots = o^2 = -1$.

Addition as before

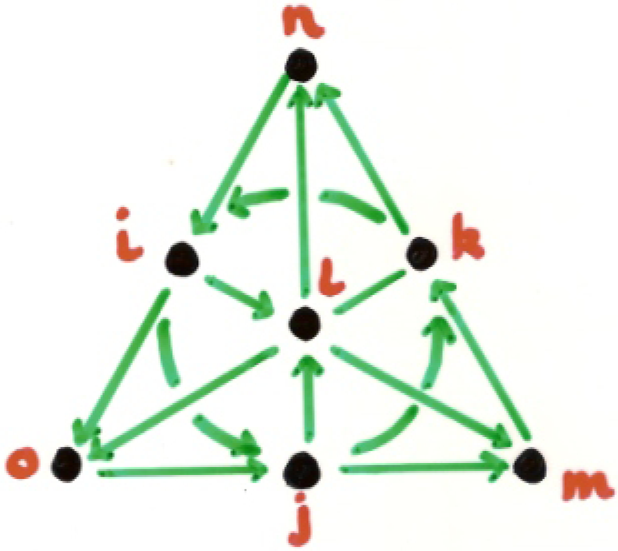
Multiplication:

$ni = o$

$mk = n$

$in = -o$

$mj = -o$, etc



Three systems: $xy = yx$ $(xy)z = x(yz)$

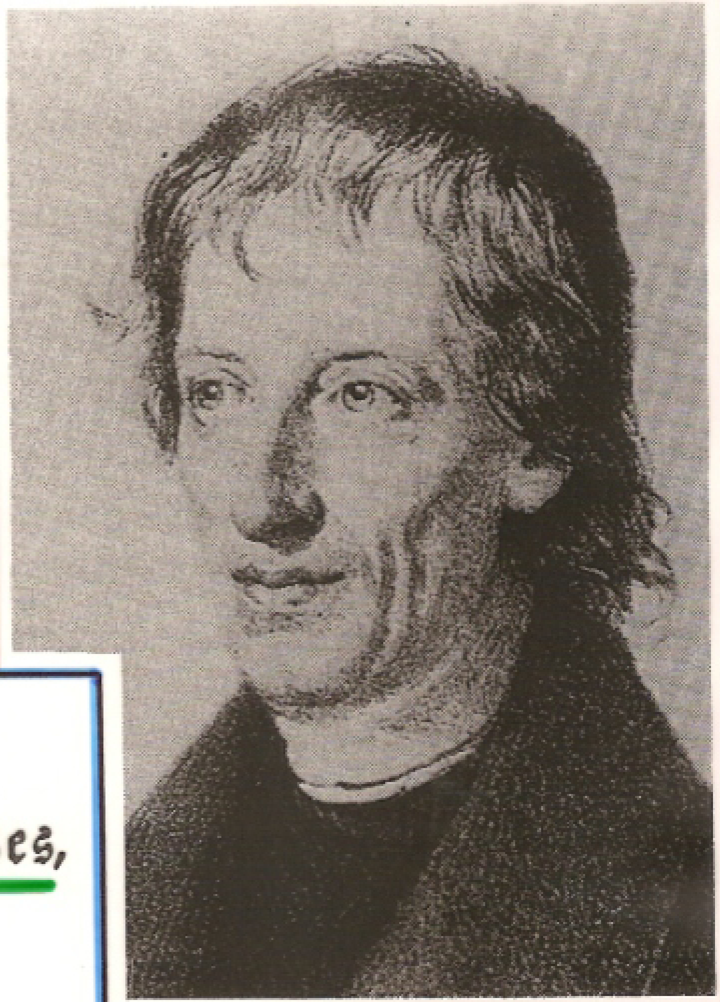
complexes : commutative, associative

quaternions : not commutative, associative

octonions : neither

Bernard Bolzano

(1781-1848)



Rein analytischer

Beweis des Lehrsatzes,

daß

zwischen je zwey Werthen, die ein entgegengesetztes Resultat gewähren, wenigstens eine reelle Wurzel der Gleichung liege;

von

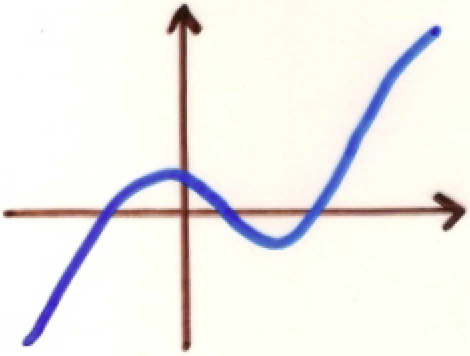
Bernard Bolzano,

Weltpriester, Doctor der Philosophie, k. k. Professor der Religionswissenschaft, und ordentlichem Mitgliede der k. Gesellschaft der Wissenschaften zu Prag.

Für die Abhandlungen der k. Gesellschaft der Wissenschaften.

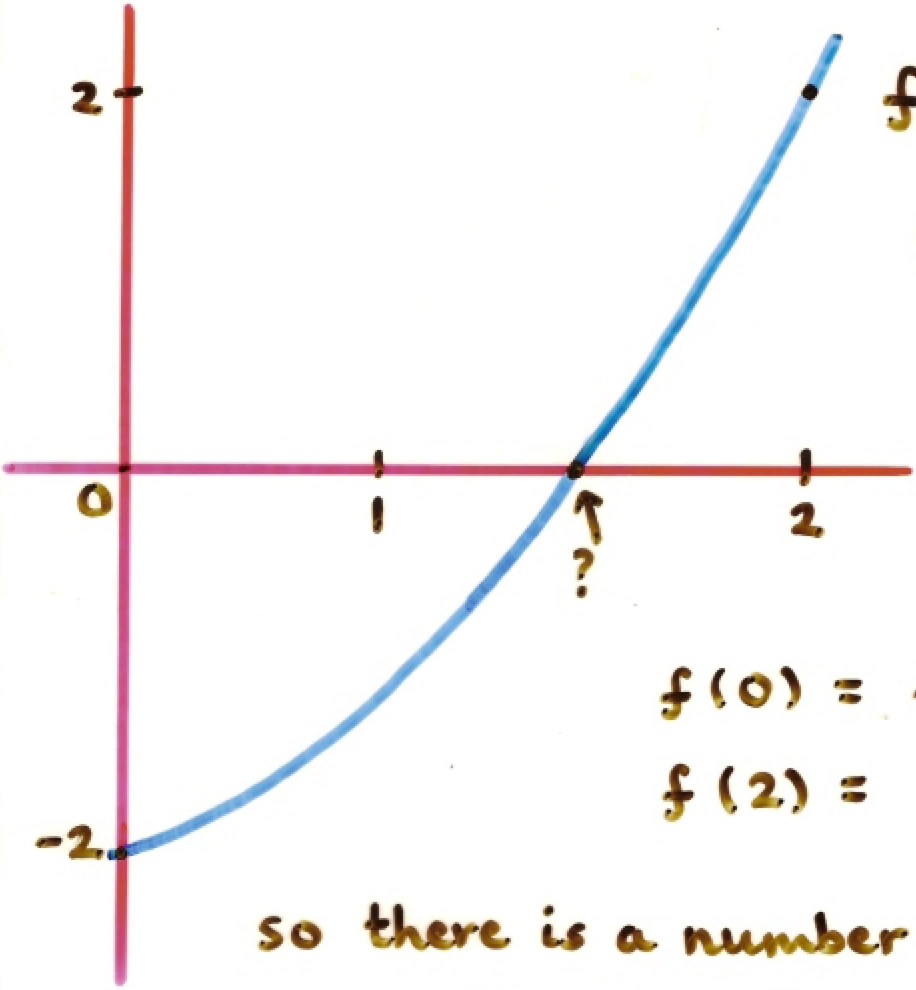
Prag, 1817,
gedruckt bei Gottlieb Haase.

Bolzano's 1817 pamphlet



Intermediate Value Theorem

The 'number' $\sqrt{2}$



$f(x) = x^2 - 2$

on the interval

$[0, 2]$

$f(0) = -2$

$f(2) = 2$

so there is a number x such that

$x^2 - 2 = 0$.

Rationals and Irrationals

$3/4 = 0 \cdot 75$

$1/3 = 0 \cdot 3333\cdots$

$2/7 = 0 \cdot \underline{285714}28571428\cdots$

Add:

$115/84 = 1 \cdot 36\underline{904761}90476190\cdots$

Every rational number has a finite or recurring decimal, and every finite or recurring decimal can be written as a fraction.

$$x = 0 \cdot 242424\cdots$$

$$100x = 24 \cdot 242424\cdots$$

$99x = 24$,

so $x = \frac{24}{99} = \frac{8}{33}$

Defining real numbers

Define a real number as an infinite decimal – for example:

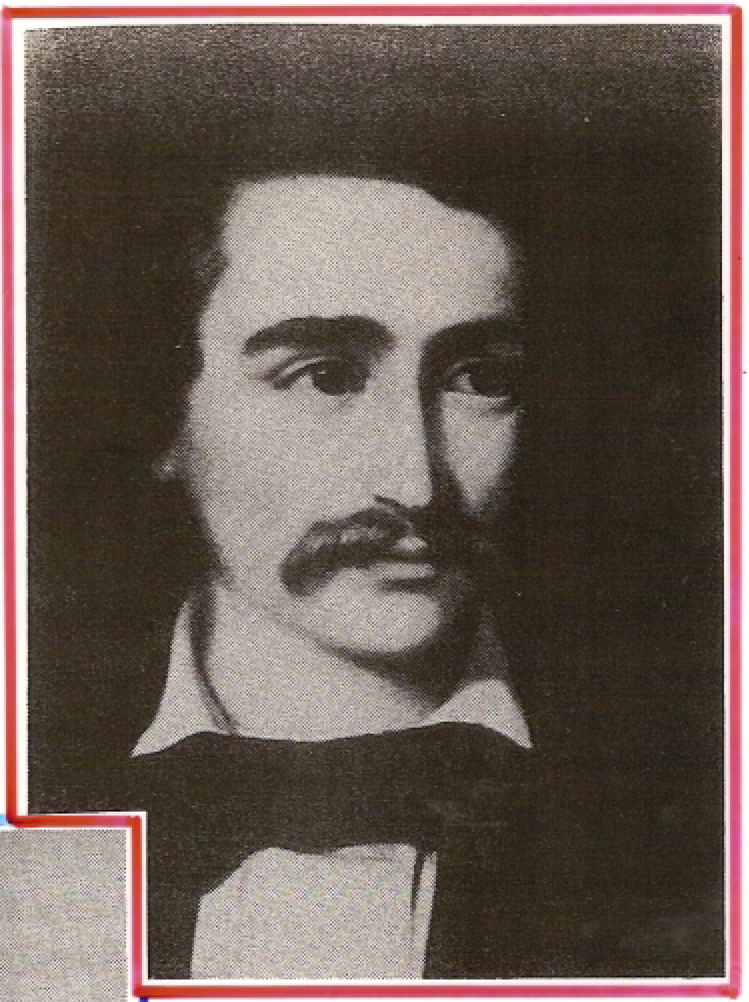
$$\sqrt{2} = 1\cdot4142 13\cdots$$

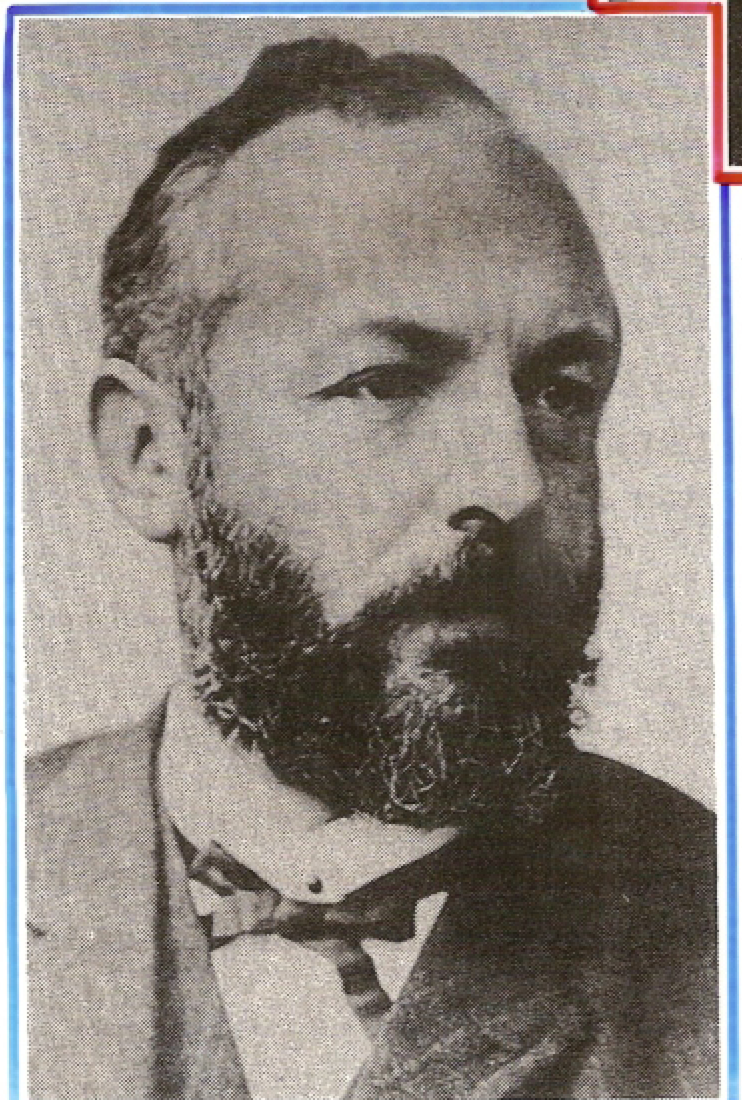
$$\pi = 3\cdot1415926\cdots$$

Problem: how do we prove that

$$\sqrt{2} \times \sqrt{2} = 2 \ ?$$

Richard Dedekind (1831-1916)





Georg Cantor (1845-1918)

Dedekind cuts

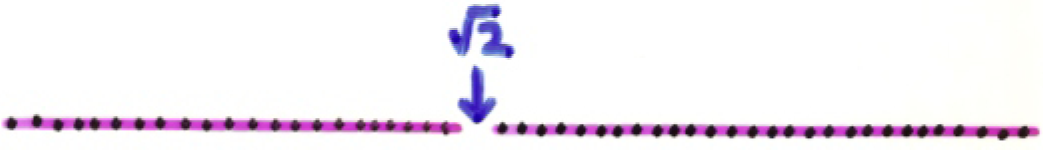
1858, 1872: Dedekind cut: definition of a real number

Basic idea: the real line and the rationals differ – the latter has gaps (e.g. at $\sqrt{2}, \pi, \dots$)

Aim: fill the gaps with numbers

Dedekind: each gap is a number

It is defined by all rationals less than it, and greater than it:



all rationals less than $\sqrt{2}$ | all rationals greater than $\sqrt{2}$

[i.e. all rationals with square < / > 2]