SQUARING THE CIRCLE AND OTHER IMPOSSIBILITIES

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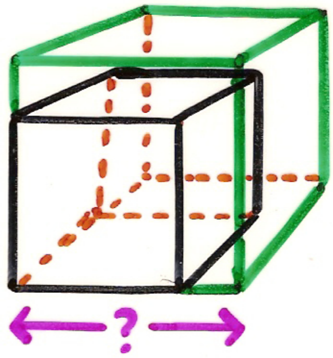
Circles to square,
and Cubes to double,
Would give a Man
excessive Trouble.

Matthew Prior (1718)

The Three Classical Problems

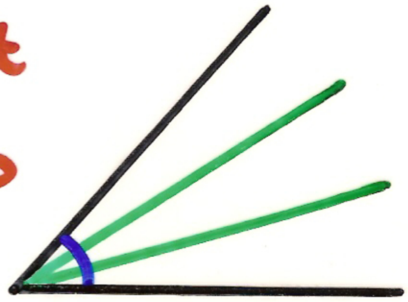
1. Doubling the Cube

Given a cube, construct another with twice the volume.



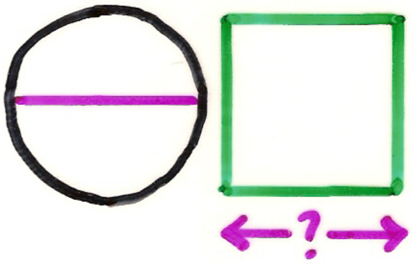
2. Trisecting the Angle

Given any angle, construct two lines that divide it into three equal parts.



3. Squaring the Circle

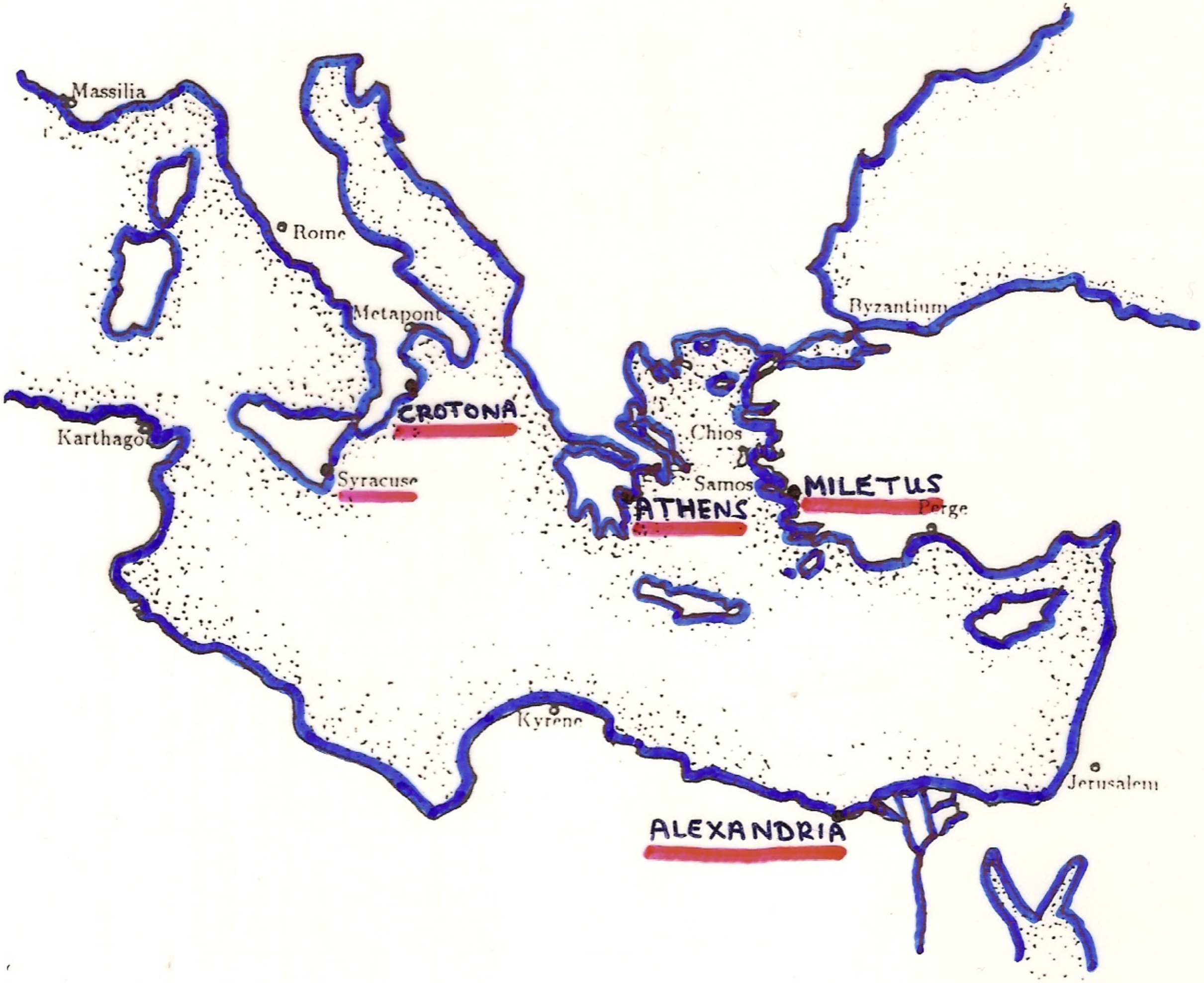
Given a circle, construct a square with the same area.

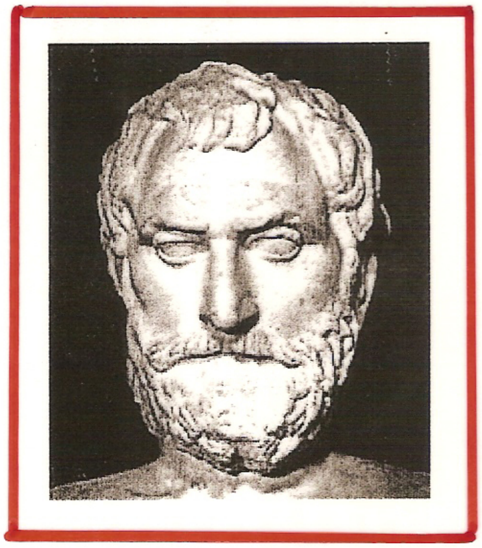


Three Periods

Early:	Thales	600 BC
	Pythagoras	520 BC
Athens:	Plato	387 BC
	Aristotle	350 BC
	Eudoxus	370 BC
Alexandria:	Euclid	300 BC
	(Archimedes)	250 BC
	Apollonius	220 BC
	Ptolemy	150 AD
	Diophantus	250 AD?
	Pappus	320 AD
	Hypatia	400 AD

MAP OF GREECE (c. 300 BC)



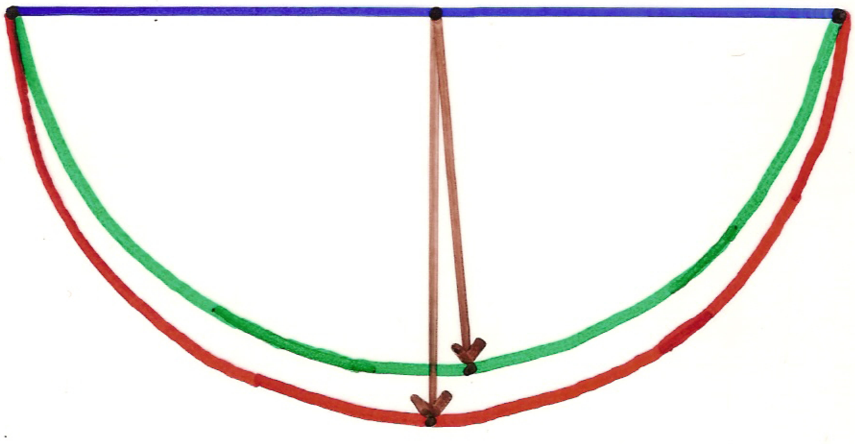


Thales of Miletus

(c. 624 – 547 BC)

The famous Thales is said to have been the first to demonstrate that the circle is bisected by the diameter. If you wish to demonstrate this mathematically, imagine the diameter drawn and one part of the circle fitted upon the other.

If it is not equal to the other, it will fall either inside or outside it, and in either case it will follow that a shorter line is equal to a longer. For all the lines from the centre to the circumference are equal, and hence the line that extends beyond will be equal to the line that falls short, which is impossible.



Aristophanes' *The Birds* (414 BC)

METON

These are my special rods
for measuring the air.
You see, the air is shaped
— how shall I put it? —
like a sort of extinguisher;
so all I have to do is to attach this flexible rod
at the upper extremity, take the compasses,
insert the point here, and —
you see what I mean?

PEISTHETAERUS

No.

METON

Well, I now apply the straight rod — so
— thus squaring the circle;
and there you are . . .

PEISTHETAERUS

Brilliant — the man's a Thales.

Lines and Circles

Euclid's postulates 1-3 :

1. Draw a straight-line segment between any two points.
2. Extend a straight-line segment.
3. Draw a circle with given centre and radius.

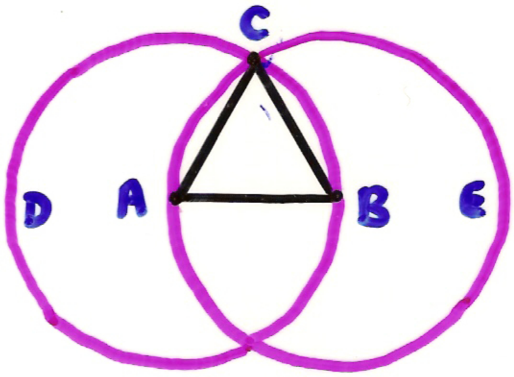
Straight-edge (ruler)

and compasses

(no measuring allowed)

Euclid Book I, Prop. I

On a given straight line to construct an equilateral triangle



Let AB be the line.

With centre A and distance AB, draw the circle BCD. [Post. 3]

With centre B and distance BA, draw the circle ACE. [Post. 3]

Join AC and BC. [Post. 1]

Then the triangle ABC is equilateral.

Proof . . .

Bisecting a line segment

Let AB be the line segment.

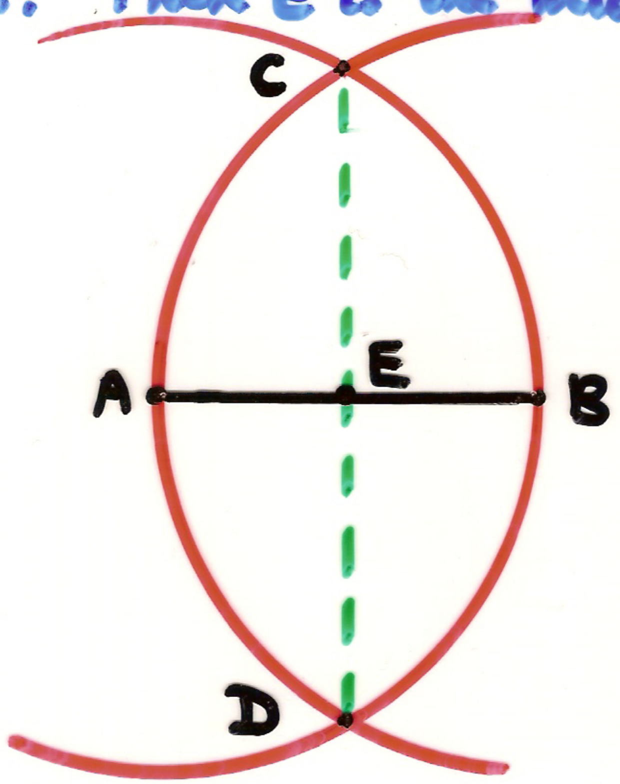
Draw:

the circle centre A, radius AB,

the circle centre B, radius BA.

These circles intersect at C and D.

Draw the line CD, and let it meet AB at E. Then E is the midpoint of AB.



Trisecting a line segment

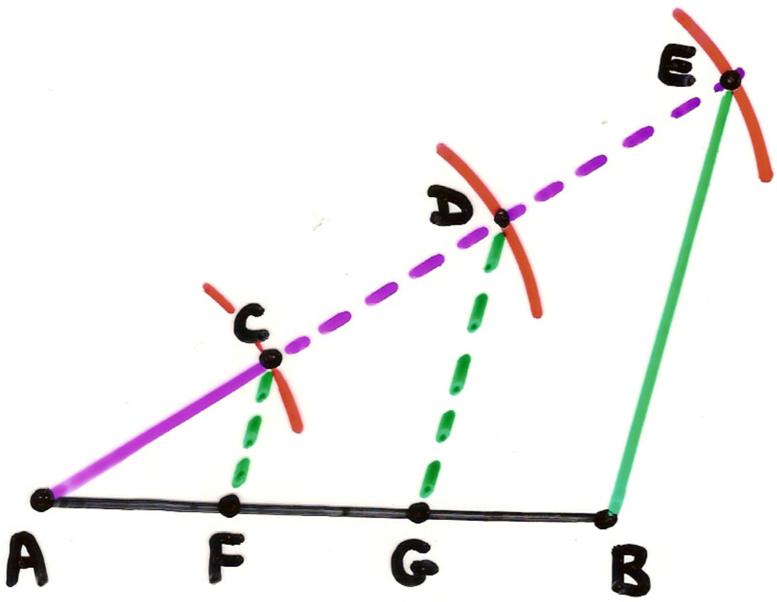
Let AB be the line segment.

Draw any line AC (C not on AB)

Extend it, and mark off AC = CD = DE (using the compasses)

Draw EB, and then parallel to it, CF, DG, where F, G lie on AB.

Then AF = FG = GB.

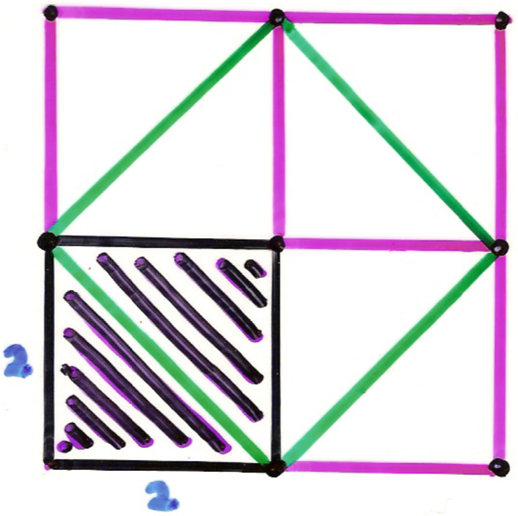


Can be extended to a division of AB into any number of parts...

Doubling a square

Plato's 'Meno':
Socrates and the slave boy

double the size of the square



$2 \times 2 = 4$

$4 \times 4 = 16$ ✗

$3 \times 3 = 9$ ✗

Take the square on the diagonal ...

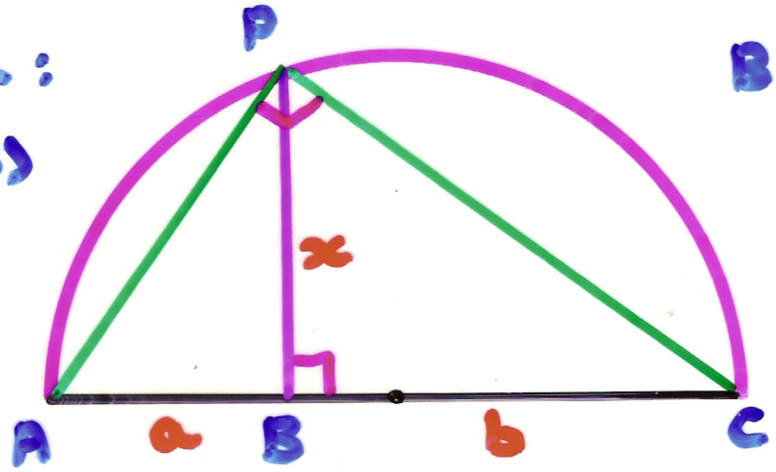
$$(2\sqrt{2}) \times (2\sqrt{2}) = 8 \quad ✓$$

Mean proportionals

A mean proportional of a and b is a number x such that

$$a : x = x : b, \text{ so } x = \sqrt{ab}$$

Euclid:
(IV, Prop13)

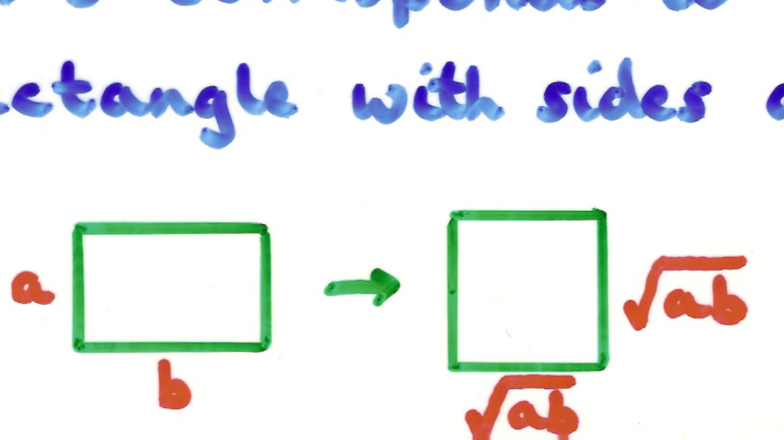


$BP = \sqrt{(AB)(BC)}$

$$\frac{x}{a} = \frac{b}{x}$$

So: a mean proportional of 1 and 2 is $\sqrt{2}$.

Finding the mean proportional of a and b corresponds to 'squaring' the rectangle with sides a and b.



Theon of Smyrna (2nd century AD)

In his work entitled *Platonicus*,
Eratosthenes says that, when the god
announced to the Delians by oracle that
to get rid of a plague they must construct
an altar double of the existing one,
their craftsmen fell into great perplexity
in trying to find how a solid
could be made double of another solid,
and they went to ask Plato about it.

He told them that the god had given
this oracle, not because he wanted an altar
of double the size, but because he wished,
in setting this task before them,
to reproach the Greeks for their neglect
of mathematics and their contempt
for geometry.

Eutocius (6th century AD)

The story goes that one of the ancient
tragic poets represented Minos
having a tomb built for Glaucus,
and that when Minos found that the tomb
measured a hundred feet on every side,
he said:
'Too small is the tomb you have marked out
as the royal resting place.
Let it be twice as large.
Without spoiling the form quickly double
each side of the tomb.'
This was clearly a mistake.
For if the sides are doubled, the surface is
multiplied fourfold and the volume eightfold.
Now geometers, too, sought a way to double
the given solid without altering its form.
This problem came to be known as
the duplication of the cube,
for, given a cube, they sought to double it.

Two mean proportionals

Eutocius: Hippocrates of Chios found that to double a cube, we must find two mean proportionals between given straight lines — one double the other:

So: given a and b, find x and y so that

$$a : x = x : y = y : b, \quad \text{with } b = 2a$$

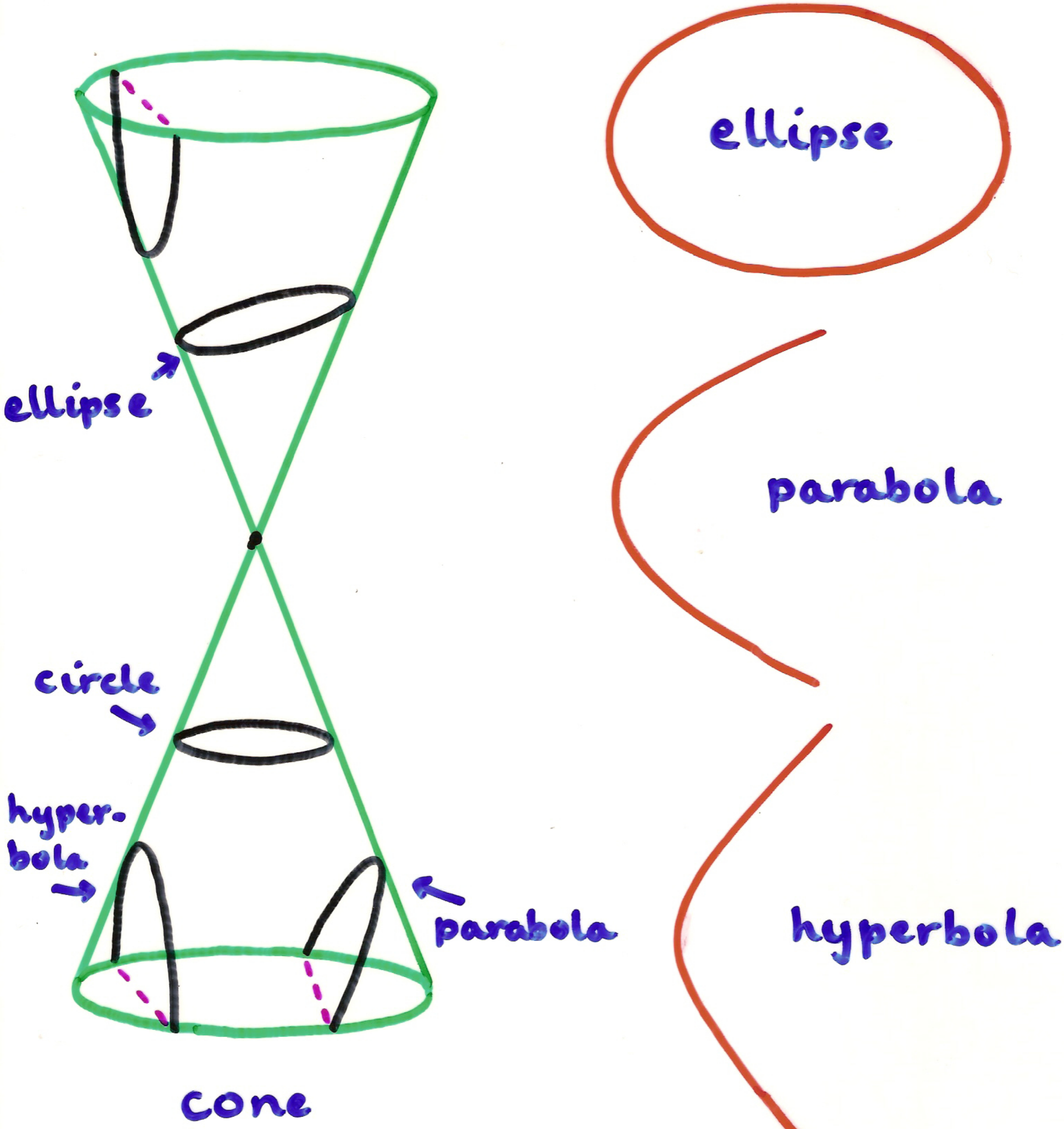
Now $$\frac{x}{a} = \frac{y}{x} = \frac{b}{y}$$

so $$\left(\frac{x}{a}\right)^3 = \frac{\cancel{x}}{a} \times \frac{\cancel{y}}{\cancel{x}} \times \frac{b}{\cancel{y}} = \frac{b}{a} = 2,$$

giving $x^3 = 2a^3$.

This corresponds to taking a cube with side a and doubling its volume.

Conic Sections



Menaechmus (4th century BC)

Apollonius ('Conics': 250 BC)

Mean proportionals and conics

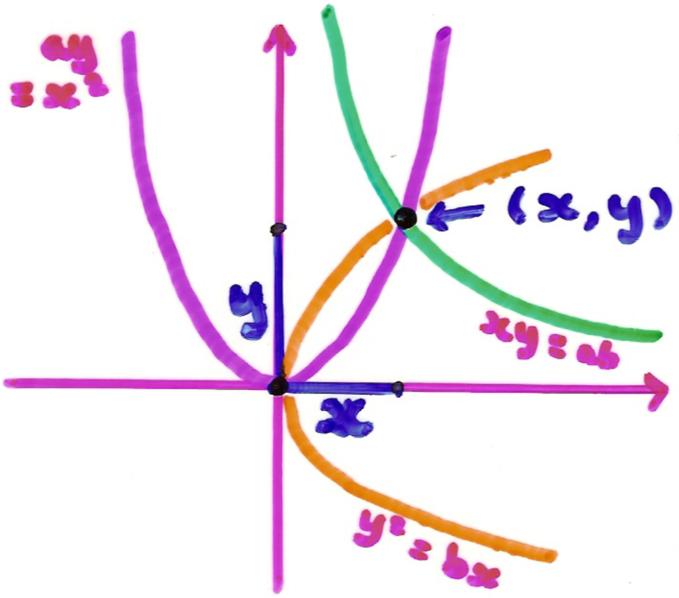
Menaechmus:

if $a : x = x : y = y : b$, then:

$\frac{x}{a} = \frac{y}{x}$, so $ay = x^2$ (parabola)

$\frac{y}{x} = \frac{b}{y}$, so $y^2 = bx$ (parabola)

$\frac{x}{a} = \frac{b}{y}$, so $xy = ab$ (hyperbola)



Bisecting an angle

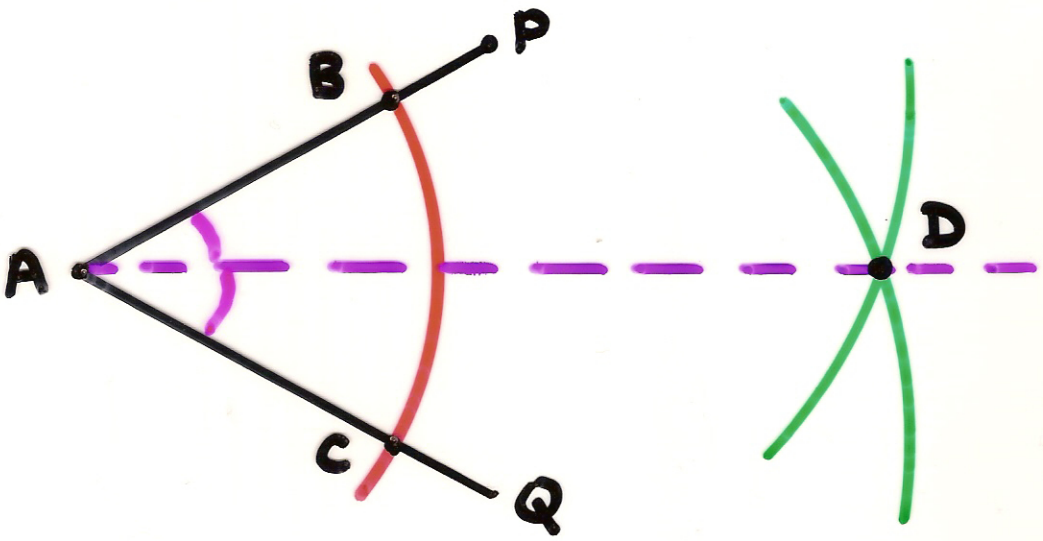
Let the angle be PAQ.

Draw a circle with centre A, any radius.

This crosses AP and AQ at B and C.

Draw circles, centres B and C, with the same radius. These meet at D.

The line segment AD bisects PAQ.

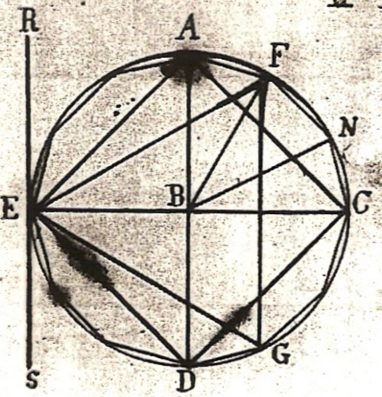


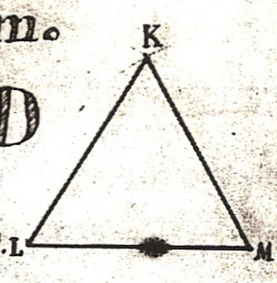
Euclid : after finding B and C, join them, and draw the perpendicular bisector of BC.

Problem.

C.L.D

Dec: 1844.





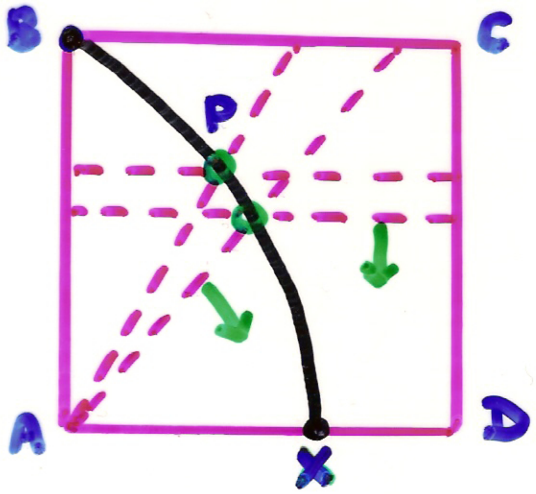
To trisect a right angle, that is, to divide it into three equal parts.

Let there be a right angle ABC, it is required to trisect it.

Produce AB ~~to~~ D and make BD equal to AB, and make BC equal to AB and produce CB to E and make EB equal to BC, and join AE, ED, DC, CA. Because AB is equal to BD, and BE is common to the two triangles ABE, DBE, and the angle ABE is equal to the angle DBE, therefore the base AE is equal to the base ED; and in like manner it may be proved that all the four AE, ED, DC, CA are equal, therefore AEDC is equilateral, and because the ~~two~~ three angles of a triangle are equal to two right angles, and that the angle ABE is a right angle, (for ABC is a right angle, and EC is a straight line) therefore the angles BAE, BEA are equal to one right angle and because BA is equal to BE, therefore the angle BAE is ½ a right angle, and in like manner it may be proved that the angle BAC is ½ a right angle, therefore the angle EAC is a right angle, and in like manner it may be proved that the angles AED, EDC, DCA are also right angles, therefore AEDC ~~a quadrangular that is~~ has all its angles right angles, and it was proved

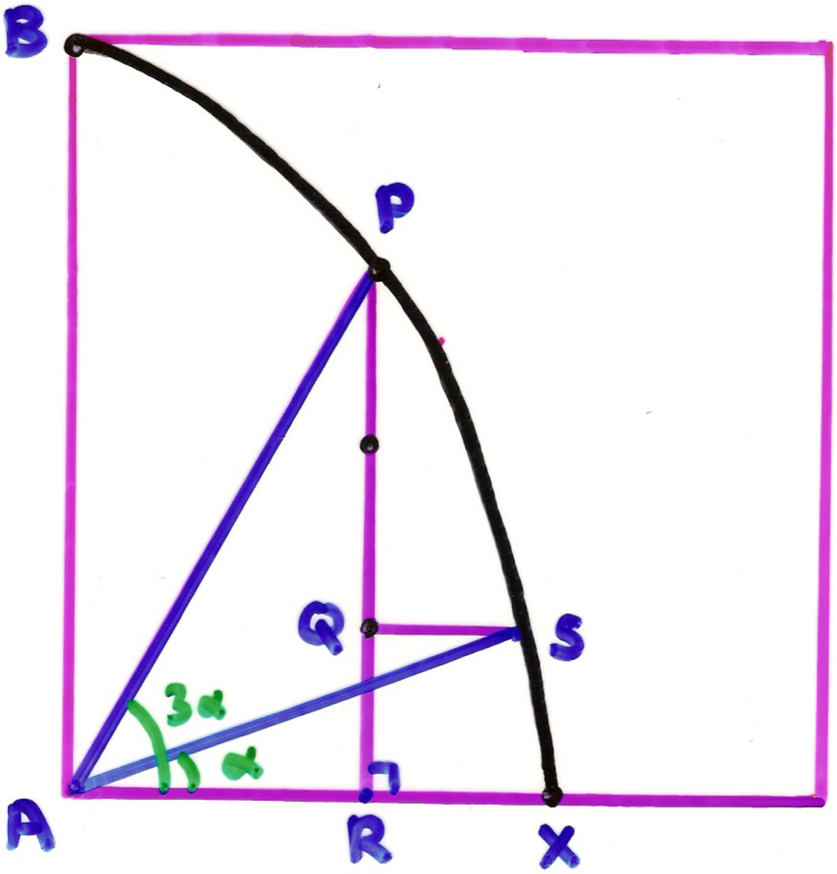
Trisecting an angle

Use a quadratrix (= trisectrix): two moving lines –



AB sweeps round a quarter-circle from B

BC moves downwards, ending along AD.



$QR = \frac{1}{3} PR$

$\angle S\overset{\alpha}{A}X = \frac{1}{3} \times \angle P\overset{3\alpha}{A}X.$

Similarly, by dividing PR into 4, 5, … parts, we can divide the angle into any number of parts.

To construct a square

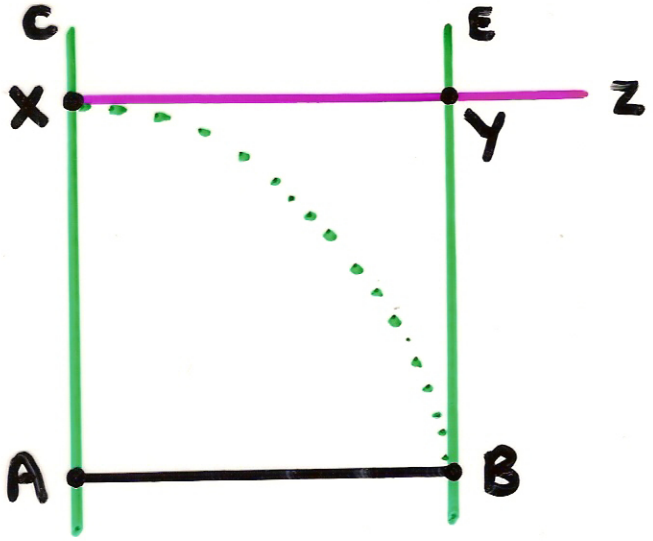
Start with a line segment AB.

Construct AC and BE perpendicular to AB.

With compasses at A, mark X on AC equal to AB.

Construct XZ perpendicular to AC.

Let XZ meet BE at Y: ABYX is a square.

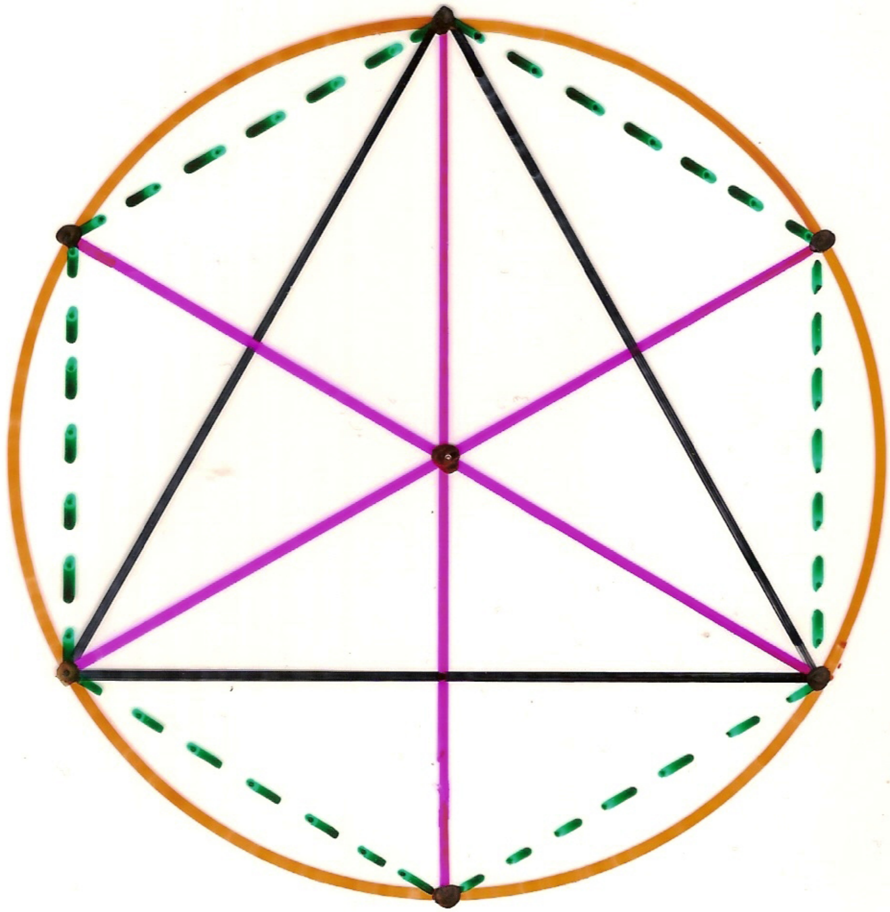


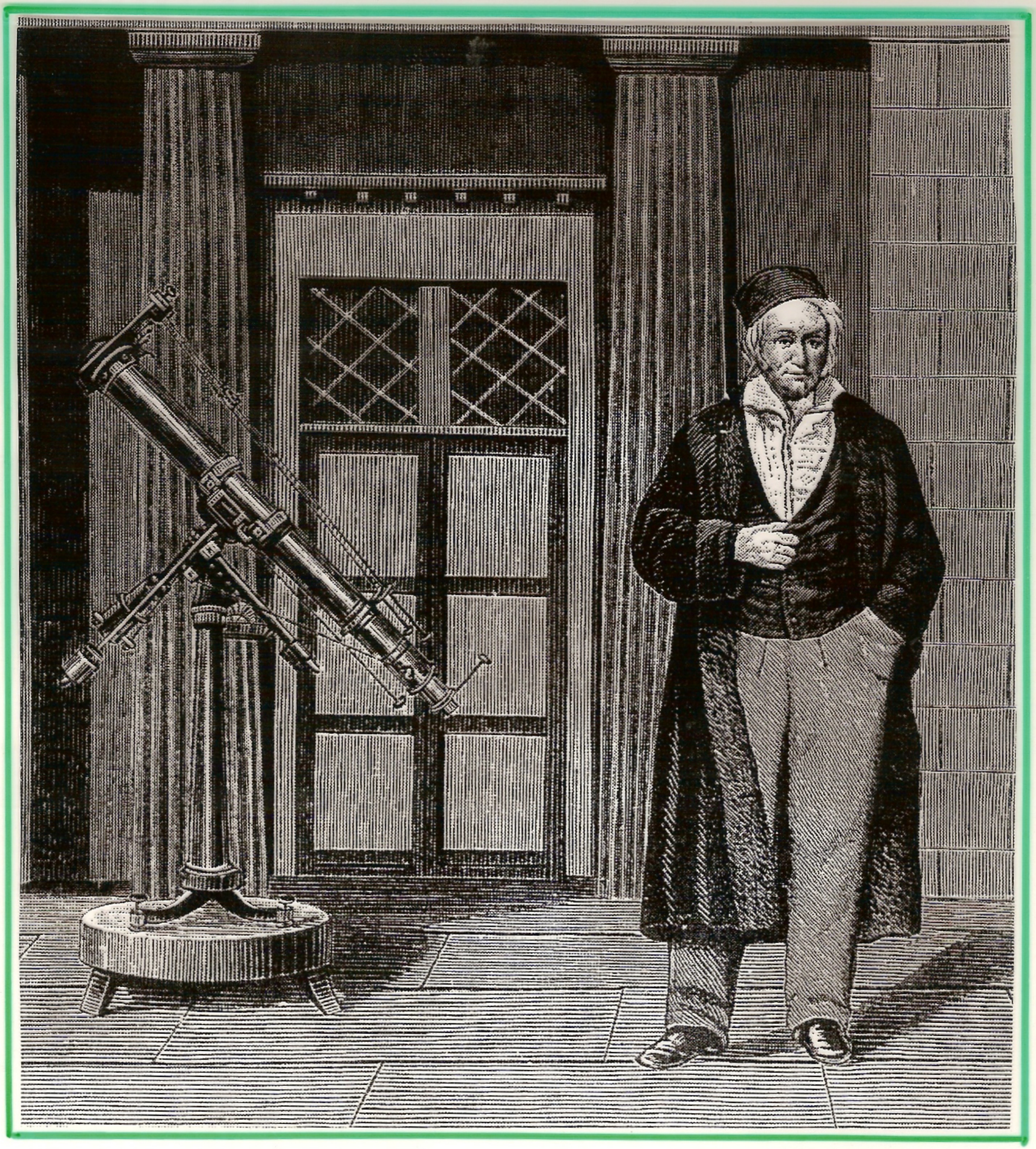
Euclid I, 46

Euclid also shows how to construct a regular pentagon, hexagon and 15-gon.

(Book IV)

Constructing a hexagon





Fermat Prime Numbers

Fermat: $F_n = 2^{2^n} + 1$ is prime (?)

$F_0 = 2^1 + 1 = 3$, $F_1 = 2^2 + 1 = 5$,

$F_2 = 2^4 + 1 = 17$, $F_3 = 2^8 + 1 = 257$,

$F_4 = 2^{16} + 1 = 65{,}537$

$F_5 = 4{,}294{,}967{,}297$? $\leftarrow \boxed{2^{32} + 1}$

Euler: F_5 is divisible by 641

Proof: $641 = 5^4 + 2^4 = (5 \times 2^7) + 1$.

So $2^{32} + 1 = \underline{2}^{28}(5^4 + \underline{2}^4) - (5.2^7)^4 + \underline{1}$

$= 2^{28}.641 - (641 - 1)^4 + 1$

$= 641 \times K$, so $641 \mid 2^{32} + 1$.

Constructing Polygons

Gauss: A regular polygon with n sides can be constructed with straight-edge and compasses if and only if n has the form

$$n = 2^k \times p_1 \times p_2 \times \cdots .$$

distinct 'Fermat primes' $2^{2^k}+1$

3, 4, 5, 6, 8, 10, 12, 15, 16,

17, 20, 24, 30, 32, 34, 40, 48,

51, 60, 64, 68, 80, 85, 96,

102, ..., 257, ..., 65537, ...

Squaring things

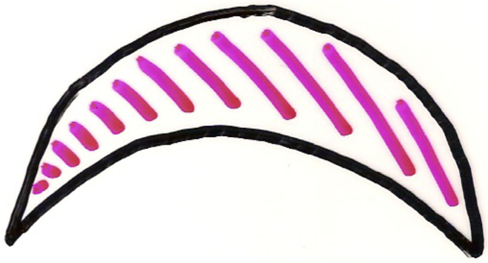
Euclid:

I, 44 Given a triangle, we can construct a parallelogram of equal area.

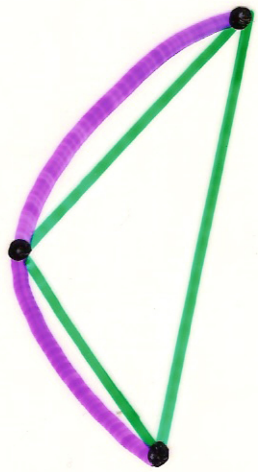
I, 45 Given any polygon, we can construct a parallelogram of equal area.

II, 14 Given any polygon, we can construct a square of equal area.

We can square any polygon!



Hippocrates gave a construction for squaring 'lunes'



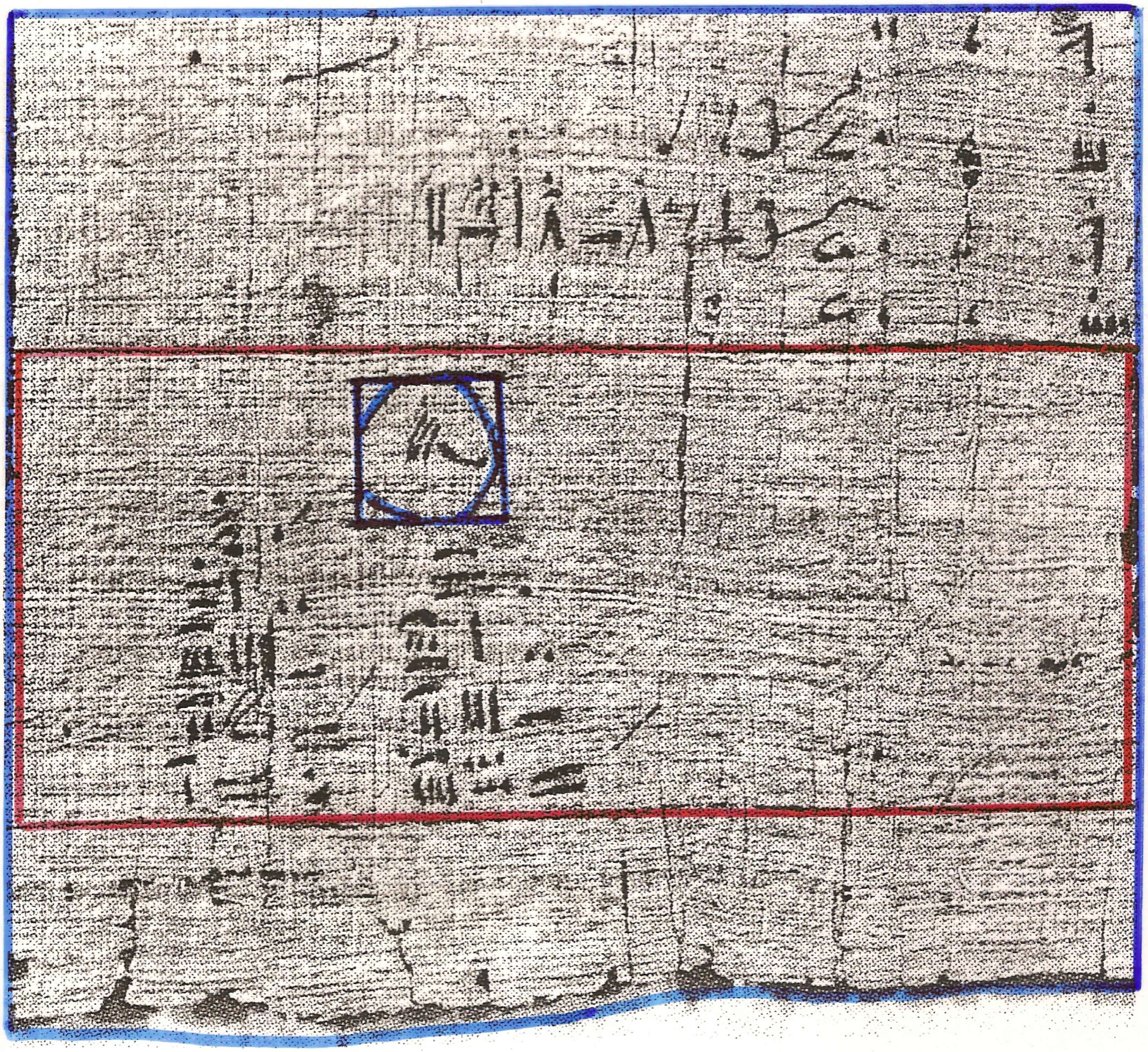
Archimedes: the area of a parabolic segment

$= \frac{4}{3} \times$ (area of enclosed Δ)

$\rightarrow$ quadrature of the parabola

An Egyptian Geometry Problem

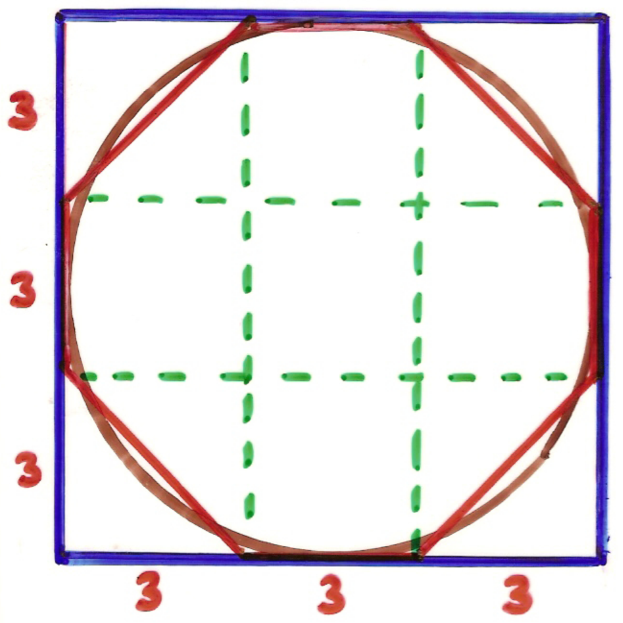
Problem 48. Compare the areas of a circle and its circumscribing square.



A Problem in Geometry (c. 1650 BC)

Problem 48. Compare the areas of a circle and its circumscribing square.

The circle of diameter 9		The square of side 9	
1	8 setat	1	9 setat
2	16 setat	2	18 setat
4	32 setat	4	36 setat
8	64 setat	8	72 setat
		Total	81 setat



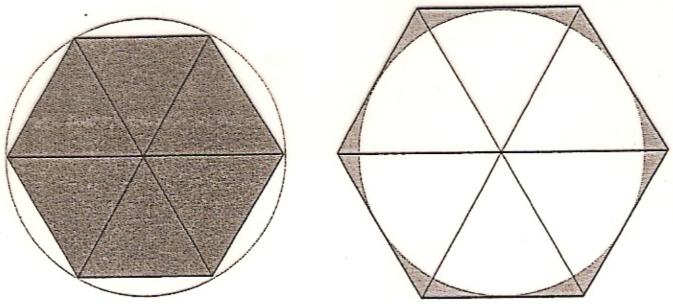
$$\text{Area} = \left(d - \frac{d}{9}\right)^2$$

$$= \frac{256}{81} r^2 \simeq 3{\cdot}16 r^2$$

Archimedes' polygons

semi-perimeters

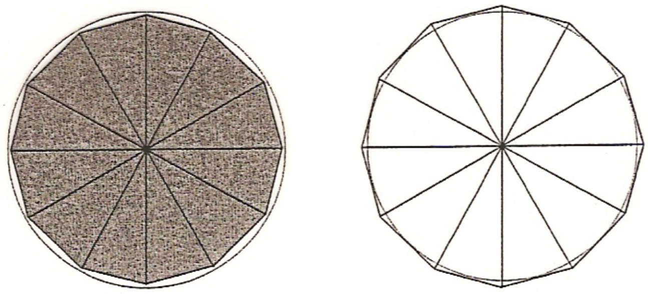
n = 6



l = 3

L = 3·464

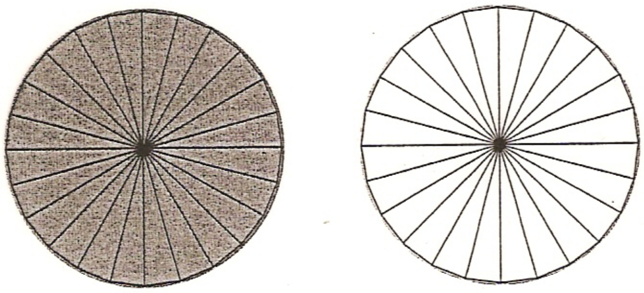
n = 12



l = 3·105

L = 3·215

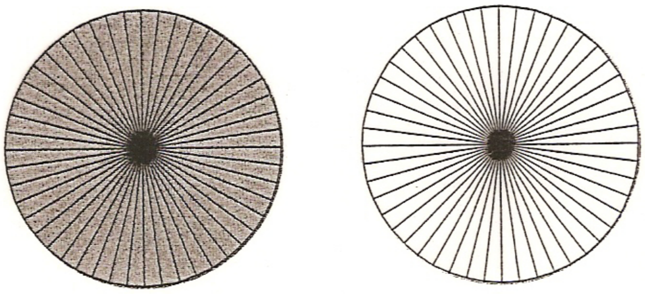
n = 24



l = 3·133

L = 3·160

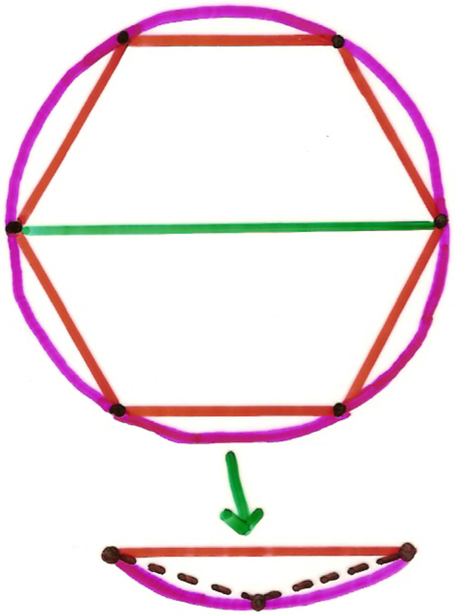
n = 48



l = 3·139

L = 3·146

The Value of π



perimeter of inscribed 6-gon

< circumference of circle

< perimeter of exscribed 6-gon

double the number of sides:

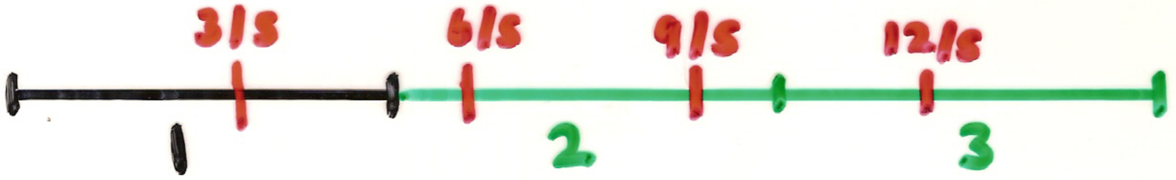
6, 12, 24, 48, 96.

Archimedes obtained the estimates:

$$3\frac{10}{71} < \pi < 3\frac{1}{7}$$

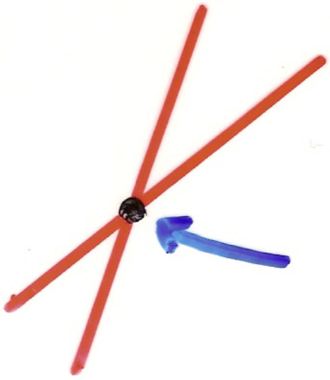
3.14084 3.14286

Using algebra



$\frac{3}{5}$

Two lines:



$$\begin{cases} y = 2x + 3 \\ y = 5x - 1 \end{cases}$$

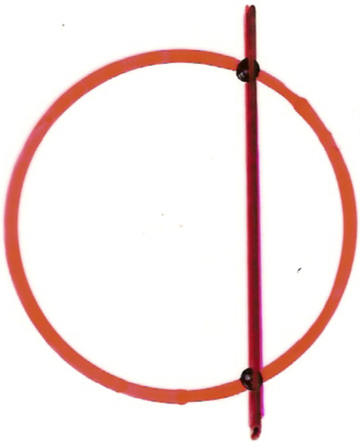
$(4/3, 17/3)$

fractions

Line + circle:

$x = 2$

$(x-1)^2 + y^2 = 4$

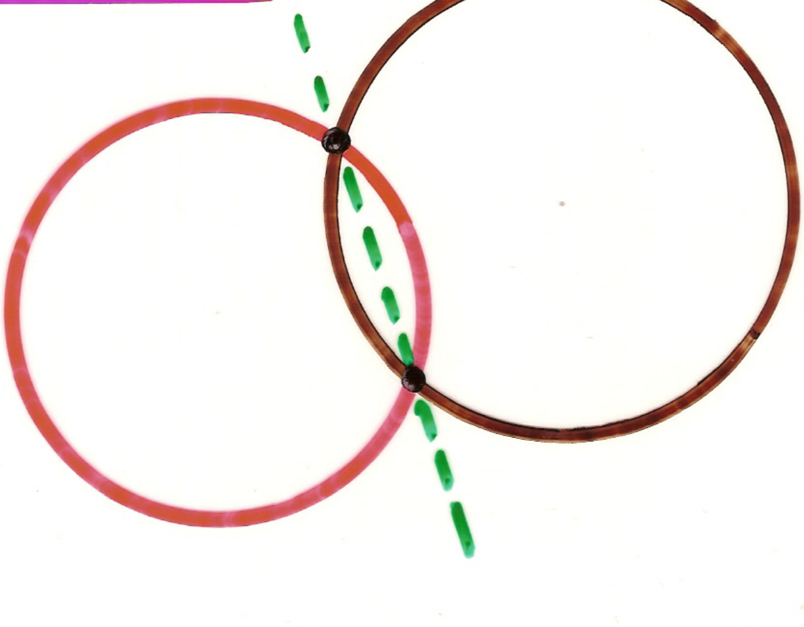


$(2, \sqrt{3})$

$(2, -\sqrt{3})$

square roots

Two circles:



again: fractions and square roots

Constructible numbers

integers, fractions, square roots, and any number arising from adding, subtracting, multiplying, dividing, or taking square roots of these ...

Doubling a square: $x^2 = 8$,
so $x = 2\sqrt{2}$

Constructing a 17-gon:

$$x = \frac{-1 + \sqrt{17} - \sqrt{34 - 2\sqrt{17}}}{4}, \ldots$$

The three Classical Problems

Doubling the cube:

$x^3 = 2$, so $x = \sqrt[3]{2}$ – impossible

Trisecting the angle:

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$$

Take $x = \cos 20°$ $(\cos 60° = 1/2)$

$1/2 = 4x^3 - 3x$, so $8x^3 - 6x - 1 = 0$

If this factorizes into a linear and a quadratic, the linear factor is:

$8x \pm 1,\ 4x \pm 1,\ 2x \pm 1,\ x \pm 1$ $\Big\}$ none works – impossible

Squaring the circle:

Is π constructible?

F. Lindemann (1882): – no,

so squaring the circle is impossible.