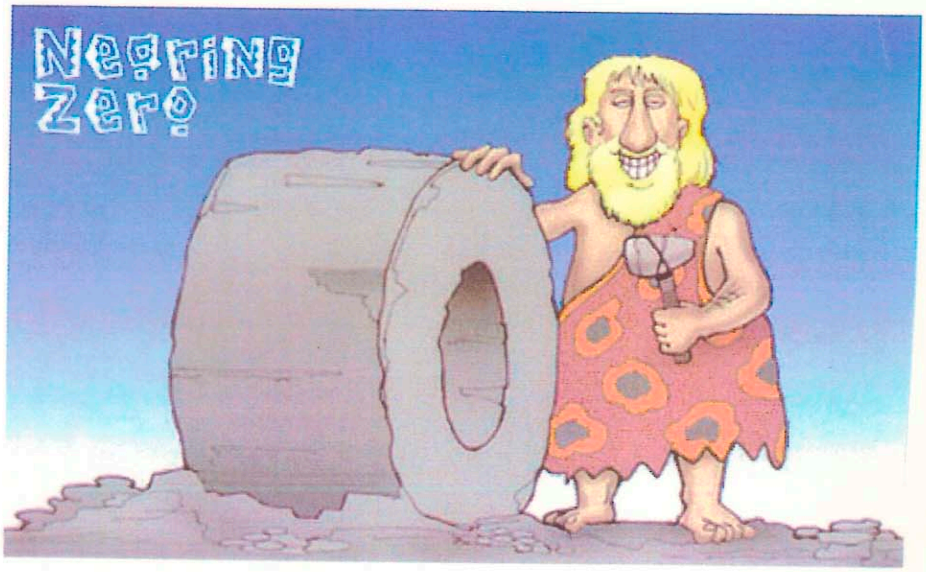
MUCH ADO ABOUT ZERO

Robin Wilson

(Gresham Professor of Geometry)



Early Mathematics Time-line

- 2700-1600 BC : Egypt
- 2000-1600 BC : Mesopotamia ('Babylonian')
- 600 BC - 500 AD : Greece (Three periods)
- 300 BC - 1400 AD : China
- 400 - 1200 AD : India
- 500 - 1000 AD : Mayan
- 750 - 1400 AD : Islamic/Arabic
- 1000 - ... AD : Europe (Middle Ages → Renaissance)

Much ado about zero

We use numbers to count things:

5 cows – but not 0 cows or –20 cows.

Egyptians: grouping system ||| ∩∩∩ ∩∩

Greeks: separate symbols for

1, 2, …, 9, 10, 20, …, 90, 100, 200, …, 900

Mesopotamians: place-value system based on 60

< < < < | 41?   41 × 60?

$(40 \times 60) + 1 = 2401$

Use of context to determine meaning.

(1) 0 as a place-holder:

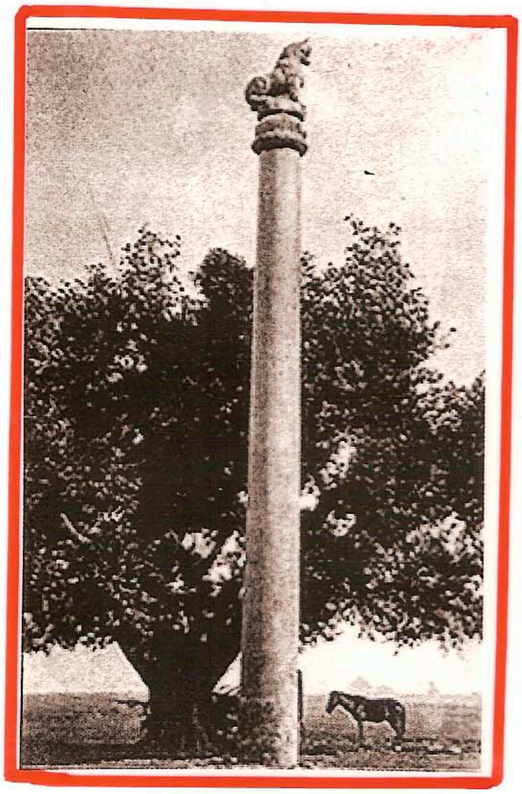
305 different from 35 or 3 5, or 350.

(Later Mesopotamians used ≰ for a zero.)

(2) 0 as a number: $2 - 2 = 0$ (cf $3 - 2 = 1$)

(Much later – India)

Chinese and Indian counting



India : King Ashoka (c. 250 BC)

Numbers inscribed on pillars around the kingdom

Place-value system based on 10.

Chinese counting board:

1	2	3	4	5	6	7	8	9
𝍠	𝍡	𝍢	𝍣	𝍤	𝍥	𝍦	𝍧	𝍨
				or				
𝍩	𝍪	𝍫	𝍬	𝍭	𝍮	𝍯	𝍰	𝍱

⊥	Π	≡	T
6	7	3	6

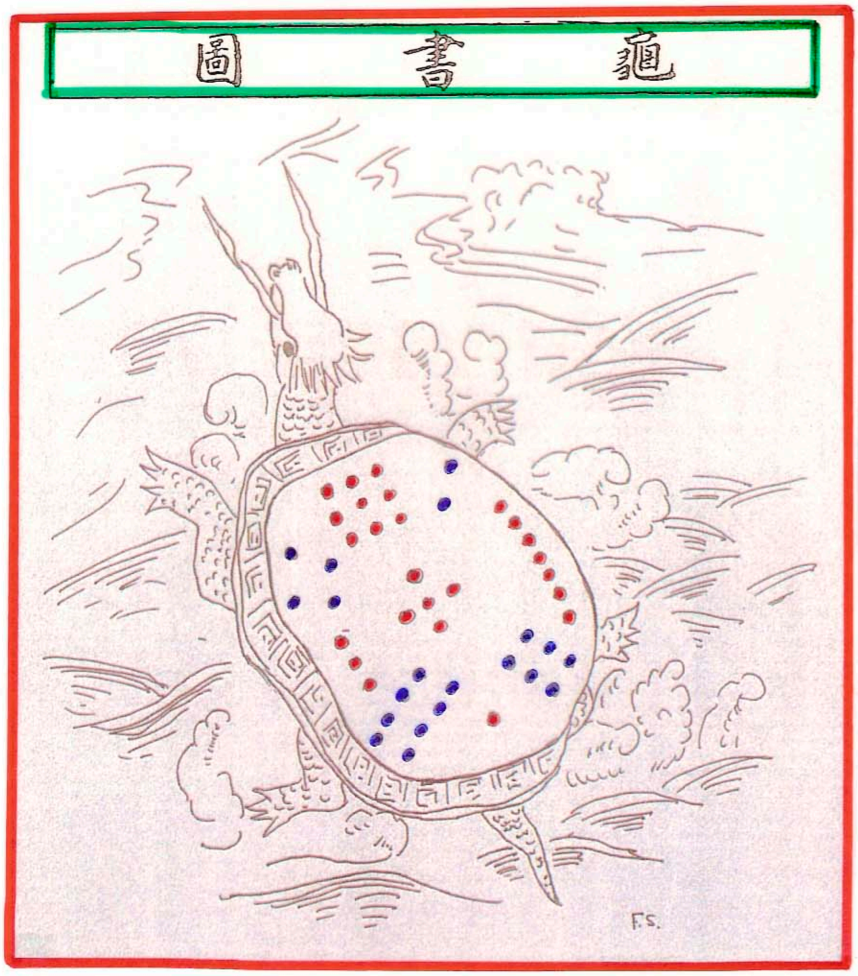
=	|		|
2	1	0	1

Indian number system similar to this :

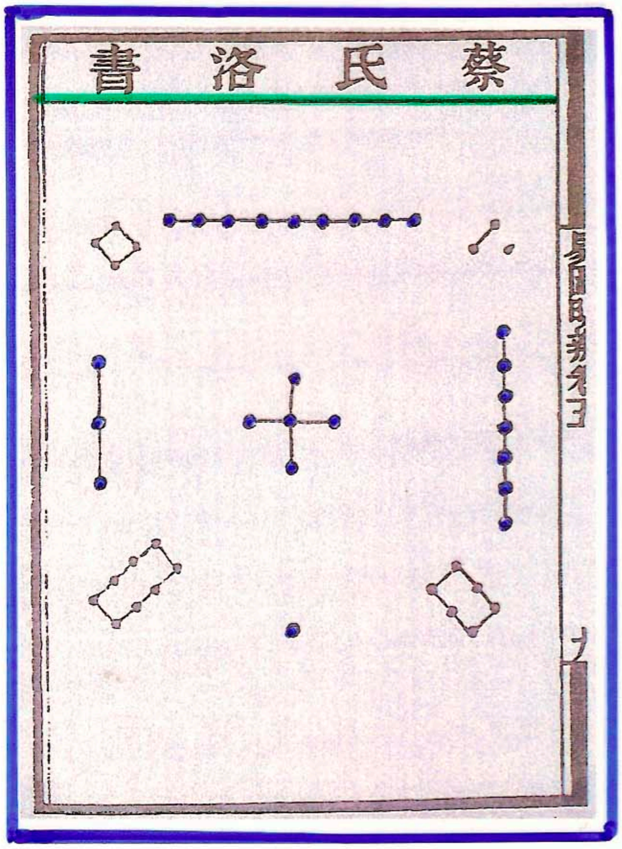
- place-value system based on 10
- used only 1, 2, 3, ···, 9 – and eventually 0.

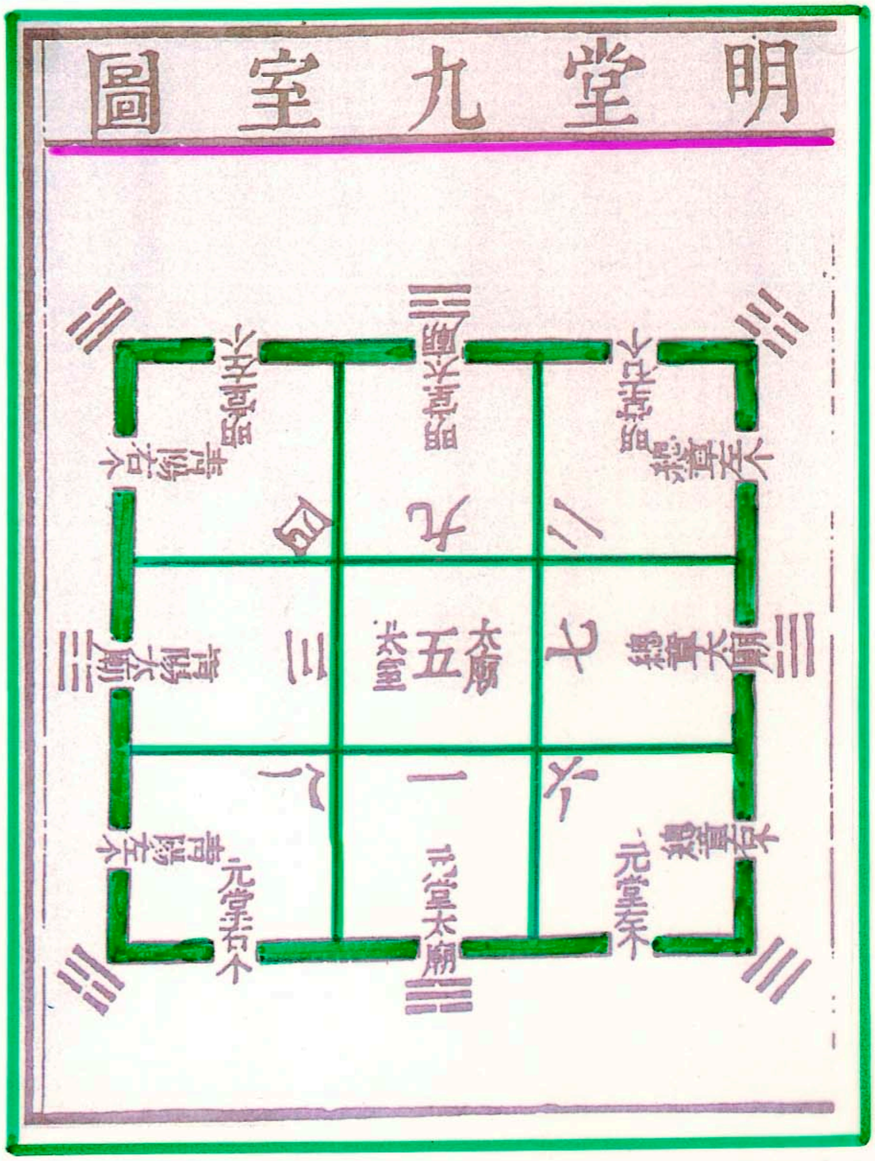
Lo-Shu

Magic Square



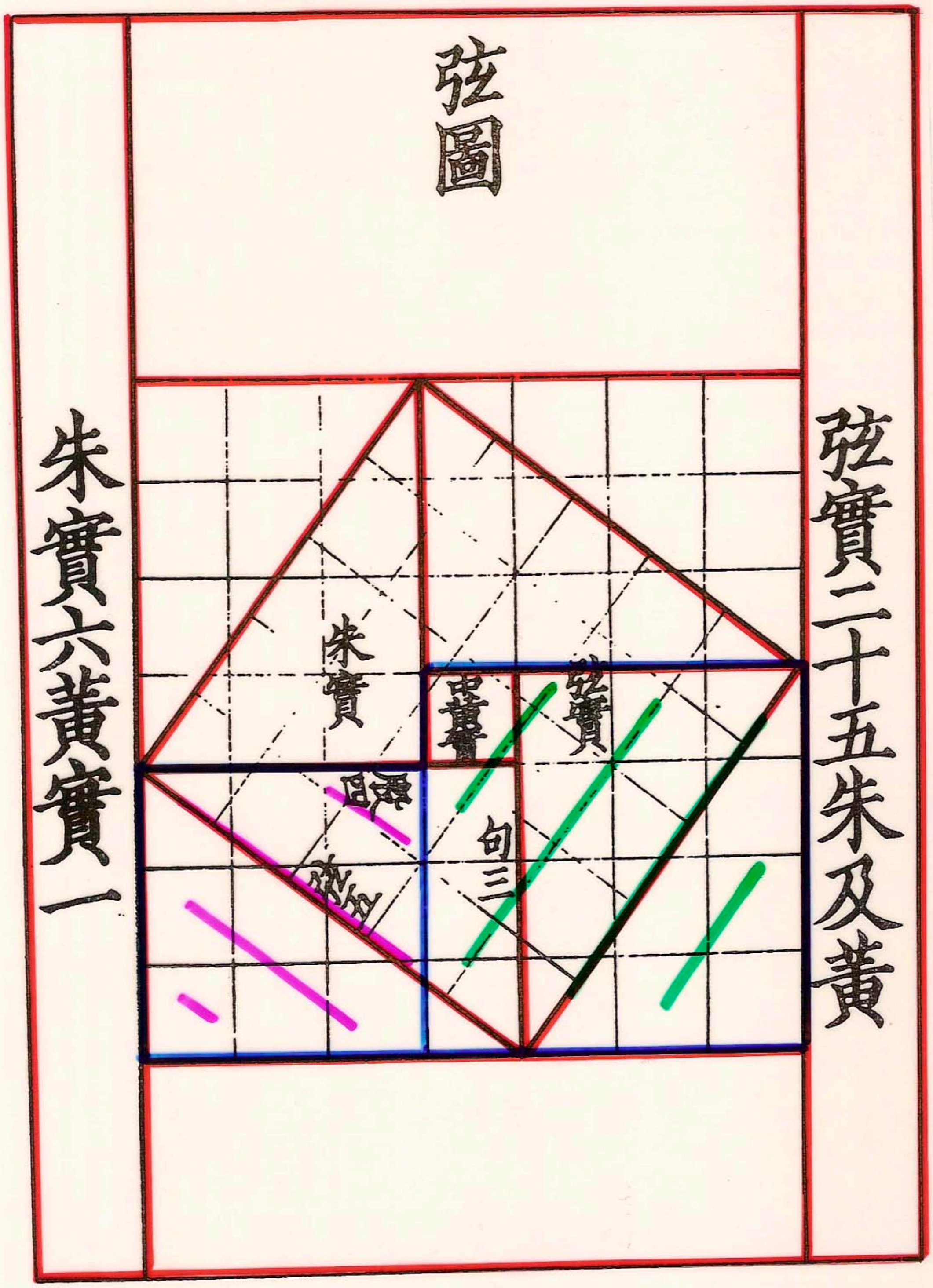
4	9	2
3	5	7
8	1	6





Zhou-bei suanjing

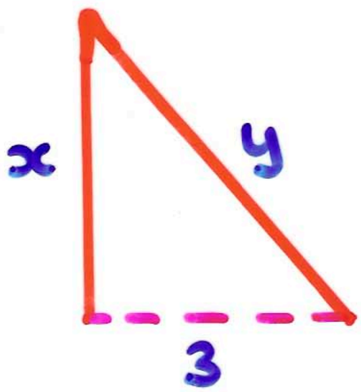
(The arithmetical classic of the gnomon ...)



Dissection proof of 'Pythagoras's Theorem'

Broken bamboo problem

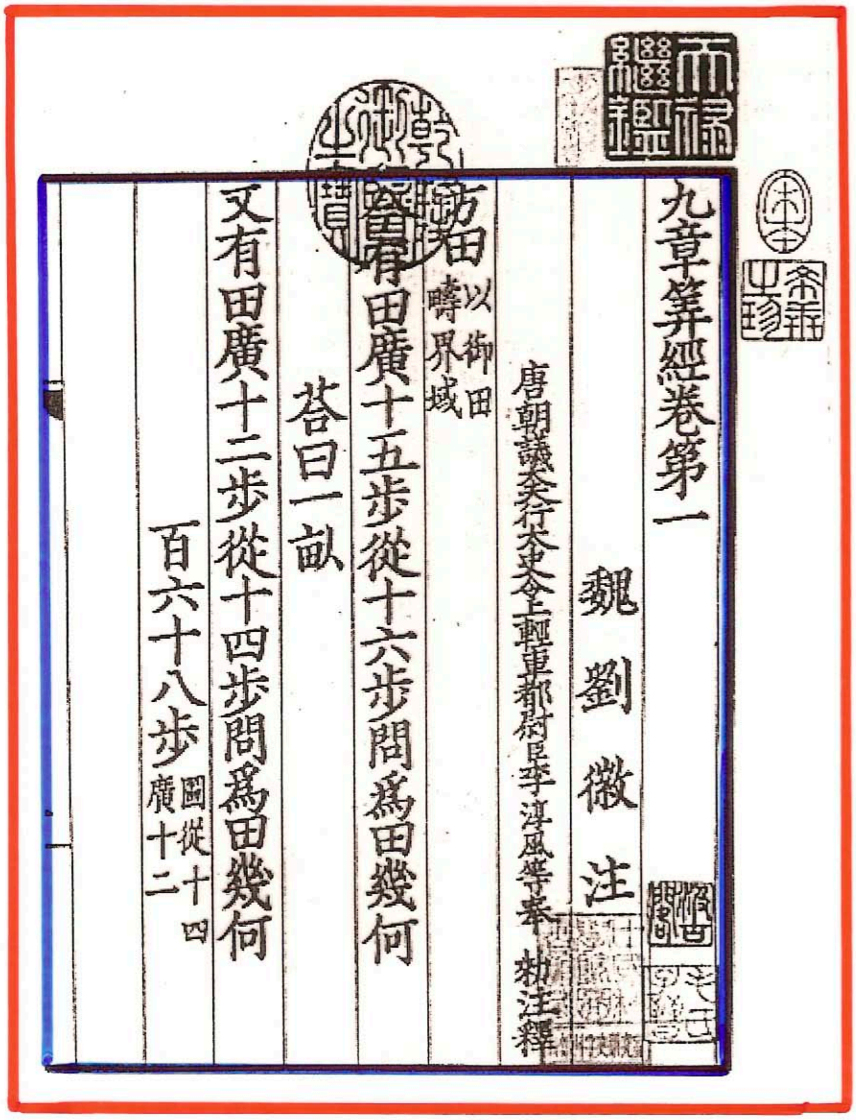
There is a bamboo 10 feet high, the upper end of which being broken reaches the ground 3 feet from the stem. Find the height of the break.



$$x + y = 10$$

$$x^2 + 3^2 = y^2$$

Jiuzhang suanshu


九章筭經卷第一

魏劉徽注

唐朝議大夫行太史令上輕車都尉臣李淳風等奉勑注釋

方田 以御田疇界域

今有田廣十五步從十六步問爲田幾何

荅曰一畝

又有田廣十二步從十四步問爲田幾何

百六十八步 圖從十四廣十二

Opening of Chapter 1

Nine Chapters on the Mathematical Art

(200 BC?)

246 questions with answers (but no 'working')

agriculture, business, surveying, etc

- calculation of areas and volumes
- calculation of square/cube roots
- study of right-angled triangles
- simultaneous equations

A problem involving grain

Three types of grain: A, B and C.

3 bundles of A, 2 of B, 1 of C = 39 measures;

2 bundles of A, 3 of B, 1 of C = 34 measures;

1 bundle of A, 2 of B, 3 of C = 26 measures.

How many measures in one bundle of each type?

(1) $3A + 2B + 1C = 39$

(2) $2A + 3B + 1C = 34$

(3) $1A + 2B + 3C = 26$

I	II	III	(A)
II	III	II	(B)
III	I	I	(C)
=T	≡IIII	≡TIIII	
(3)	(2)	(1)	

1	2	3
2	3	2
3	1	1
26	34	39

↓

0	0	3
0	5	2
36	1	1
99	24	39

→

$36C = 99$

$5B + C = 24$

$3A + 2B + C = 39$

→

$$\begin{cases} C = 2\,^3/_4 \\ B = 4\,^1/_4 \\ A = 9\,^1/_4 \end{cases}$$

The Chinese Remainder Theorem

Sun Zi (250 AD) in Sunzi suanjing

(Master Sun's mathematical manual).

We have things of which we do not know the number.

If we count them by 3s, the remainder is 2.

If we count them by 5s, the remainder is 3.

If we count them by 7s, the remainder is 2.

How many things are there?

$$N \equiv 2 \pmod 3 \equiv 3 \pmod 5 \equiv 2 \pmod 7$$

The answer is 23.

Early values of π

c. 1800 BC: Mesopotamia $\pi = 3\frac{1}{8}$

c. 1800 BC: Egypt $\pi = \frac{256}{81} \sim 3.16$

c. 550 BC: Bible (I Kings) $\Rightarrow \pi = 3$

c. 250 BC: Archimedes:

$$3\frac{10}{71} < \pi < 3\frac{1}{7}$$

→ 96 sides

c. 263 AD: Liu Hui: $\pi = 3.14159$ 3072

c. 500: Zu Changzhi: $\pi \sim 355/113$

$3.1415926 < \pi < 3.1415927$ 24576

c. 1400: al-Kashi (Samarkand) 14 dp

1596...: Ludolph van Ceulen 35 dp 2^{62} sides

1579: François Viète:

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{(2+\sqrt{2})}}{2} \cdot \frac{\sqrt{\{2+\sqrt{(2+\sqrt{2})}\}}}{2} \cdot \ldots$$

1650: John Wallis:

$$\frac{\pi}{2} = \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \ldots}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \ldots}$$

Magic squares

Islamic scholars:

990: Ikhwan-al-Safa (Brethren of Purity)

(elementary constructions for squares of sides 3-6 (and 7-9?)

1200: al-Buni (bordering technique)

1275: Yang Hui: 9 × 9 = nine 3×3 magic squares

31	76	13	36	81	18	29	74	11
22	40	58	27	45	63	20	38	56
67	4	49	72	9	54	65	2	47
30	75	12	32	77	14	34	79	16
21	39	57	23	41	59	25	43	61
66	3	48	68	5	50	70	7	52
35	80	17	28	73	10	33	78	15
26	44	62	19	37	55	24	42	60
71	8	53	64	1	46	69	6	51

1315: Moschopoulos (general construction rules when n (side) is odd or divisible by 4)

Magic square of al-Antaakii (d.987)

62	2	222	220	8	10	214	213	212	16	18	206	204	24	64
126	78	26	198	196	32	11	189	207	34	190	188	40	80	100
128	122	94	42	182	7	35	173	183	203	180	48	96	104	98
50	124	118	110	3	31	51	165	167	179	199	112	108	102	176
52	70	120	201	75	159	155	153	83	87	79	25	106	156	174
54	72	205	181	141	95	135	133	103	99	85	45	21	154	172
170	209	185	169	145	125	111	121	107	101	81	57	41	17	56
211	187	171	163	149	129	109	113	117	97	77	63	55	39	15
168	9	33	49	69	89	119	105	115	137	157	177	193	217	58
60	82	5	29	65	127	91	93	123	131	161	197	221	144	166
66	142	90	1	147	67	71	73	143	139	151	225	136	84	160
158	140	92	114	223	195	175	61	59	47	27	116	134	86	68
152	88	130	184	44	219	191	53	43	23	46	178	132	138	74
76	146	200	28	30	194	215	37	19	192	36	38	186	148	154
162	224	4	6	218	216	12	13	14	210	208	20	22	202	164

Benjamin Franklin's magic square

200	217	232	249	8	25	40	57	72	89	104	121	136	153	168	185
58	39	26	7	250	231	218	199	186	167	154	135	122	103	90	71
198	219	230	251	6	27	38	59	70	91	102	123	134	155	166	187
60	37	28	5	252	229	220	197	188	165	156	133	124	101	92	69
201	216	233	248	9	24	41	56	73	88	105	120	137	152	169	184
55	42	23	10	247	234	215	202	183	170	151	138	119	106	87	74
203	214	235	246	11	22	43	54	75	86	107	118	139	150	171	182
53	44	21	12	245	236	213	204	181	172	149	140	117	108	85	76
205	212	237	244	13	20	45	52	77	84	109	116	141	148	173	180
51	46	19	14	243	238	211	206	179	174	147	142	115	110	83	78
207	210	239	242	15	18	47	50	79	82	111	114	143	146	175	178
49	48	17	16	241	240	209	208	177	176	145	144	113	112	81	80
196	221	228	253	4	29	36	61	68	93	100	125	132	157	164	189
62	35	30	3	254	227	222	195	190	163	158	131	126	99	94	67
194	223	226	255	2	31	34	63	66	95	98	127	130	159	162	191
64	33	32	1	256	225	224	193	192	161	160	129	128	97	96	65

sum of any row or column = 2056

sum of any half-row or half-column = 1028

sum of four corners

+ sum of four central squares = 1028

Three Indian Mathematicians

500 AD: Aryabhata the elder:

- first systematic treatment of Diophantine equations (continued fractions)
- trigonometry (sine tables)

628 AD: Brahmagupta:

- used 0 and negative numbers
- 'completing the square' for quadratic equations
- Diophantine equations – 'Pell's equation'

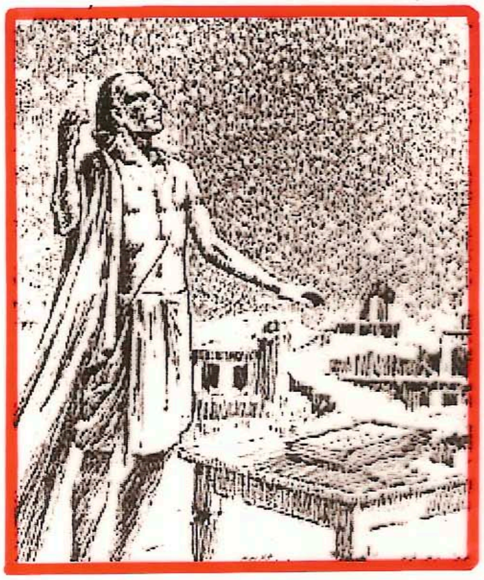
1150 AD: Bhaskara:

- wrote 'Lilavati' (arithmetic)
- Diophantine equations
- rules for dealing with irrationals:

$$\sqrt{a \pm \sqrt{b}} = \sqrt{\{a + \sqrt{a^2 - b}\}/2} \pm \sqrt{\{a - \sqrt{a^2 - b}\}/2} :$$

for example, $\sqrt{17 + \sqrt{240}} = \sqrt{12} + \sqrt{5}$.

Aryabhata : Summing series



Arithmetic progressions :

$5 + 7 + 9 + \cdots + 31$

$10 + 13 + 16 + \cdots + 100$

$a + (a+d) + (a+2d) + \cdots + a + (n-1)d$

The desired number of terms, minus one, halved, multiplied by the common difference between the terms, plus the first term, is the middle term. This multiplied by the number of terms desired is the sum of the desired number of terms. Or the sum of the first and last terms is multiplied by half the number of terms.

$$\text{sum} = n\left\{\left(\frac{n-1}{2}\right)d + a\right\}$$

$$= \frac{n}{2}\left\{a + (a + (n-1)d)\right\}.$$

Brahmagupta: zero and negative numbers

The sum of cipher and negative is negative;
of positive and nought, positive;
of two ciphers, cipher.

Negative taken from cipher becomes positive,
and positive from cipher is negative;
cipher taken from cipher is nought.

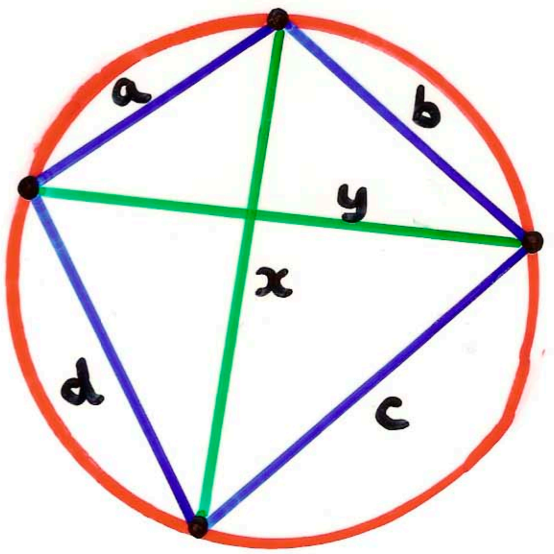
The product of cipher and positive,
or of cipher and negative, is nought;
of two ciphers, it is cipher.

Cipher divided by cipher is nought.

Positive or negative divided by cipher is
a fraction with that as denominator... $\left(\frac{\pm n}{0}\right.$

Cipher divided by positive or negative is... $\left(\frac{0}{\pm n}\right.$

Brahmagupta : Cyclic Quadrilaterals



Area A :

if $s = (a+b+c+d)/2$, then

$$A = \sqrt{(s-a)(s-b)(s-c)(s-d)} .$$

Diagonals :

$$x^2 = \frac{(ad+bc)(ac+bd)}{ab+cd}, \quad y^2 = \frac{(ab+cd)(ac+bd)}{ab+cd}$$

Pythagorean triples :

if $a^2+b^2=c^2$ and $A^2+B^2=C^2$,

then aC, cB, bC and cA are the lengths of the sides of a cyclic quadrilateral ;

e.g. $3^2+4^2=5^2$ and $7^2+24^2=25^2$

give 75, 120, 100, 35, or 15, 24, 20, 7.

Also, the diagonals are perpendicular.

Brahmagupta / Bhaskara: 'Pell's equation'

For a given C, solve $Cx^2 + 1 = y^2$.

Example (C = 3): $3x^2 + 1 = y^2$

Solutions: $x = 1, y = 2$ or $x = 4, y = 7$.

	x		y		x		
Write	1	×	2	→	2	add to give	$x = 4$,
	1		2	→	2		$y = 7$.
	1	×	2	→	8	add to give	$x = 15$,
	4		7	→	7		$y = 26$.
	1	×	2	→	30	add to give	$x = 56$
	15		26	→	26		$y = 97$

etc.

Tell me, O mathematician, what is that square which multiplied by 8 becomes – together with unity – a square? $8x^2 + 1 = y^2$

$67x^2 + 1 = y^2$: $x = 5967$, $y = 48,842$.

Indian Combinatorics

6th century BC : Susruta -
combinations of tastes, taken 1, 2, 3, ...
at a time : $\binom{6}{2} = 15$, $\binom{6}{3} = 20$, ... Total : 63

c. 300 BC : Jainas (Bhagabati Sutra):
combinations of five senses,
or of men, women and eunuchs ...

c. 200 BC : Pingala (Chandrasutra):
combinations of short/long sounds in
a metrical poem (— ∪ ∪ — ∪ , etc.)

c. 550 AD: Varahamihira (Brhat samhita):
make perfumes from 4 ingredients out of 16:
total number = $\binom{16}{4} = 1820$.

1150 AD: Bhaskara (Lilavati):
gave general rules for $n!$, $\binom{n}{k}$, etc.

An example of Bhaskara

How many are the variations of form of the god Sambhu, by the exchange of his ten attributes held reciprocally in his several hands — namely:

the rope, the elephant's hook, the serpent, the tabor, the skull, the trident, the bedstead, the dagger, the arrow and the bow?

Statement: Number of places: 10

The variations of form are found to be 3 628 800.

[Possibly known earlier by Brahmagupta.]

Pascal's triangle

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 2 & & 1 & & & & & \\
 & & & & 1 & & 3 & & 3 & & 1 & & & & \\
 & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \\
 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1
\end{array}
$$

. . . .

Combinations : $\binom{n}{r} = \frac{n!}{r!\,(n-r)!}$

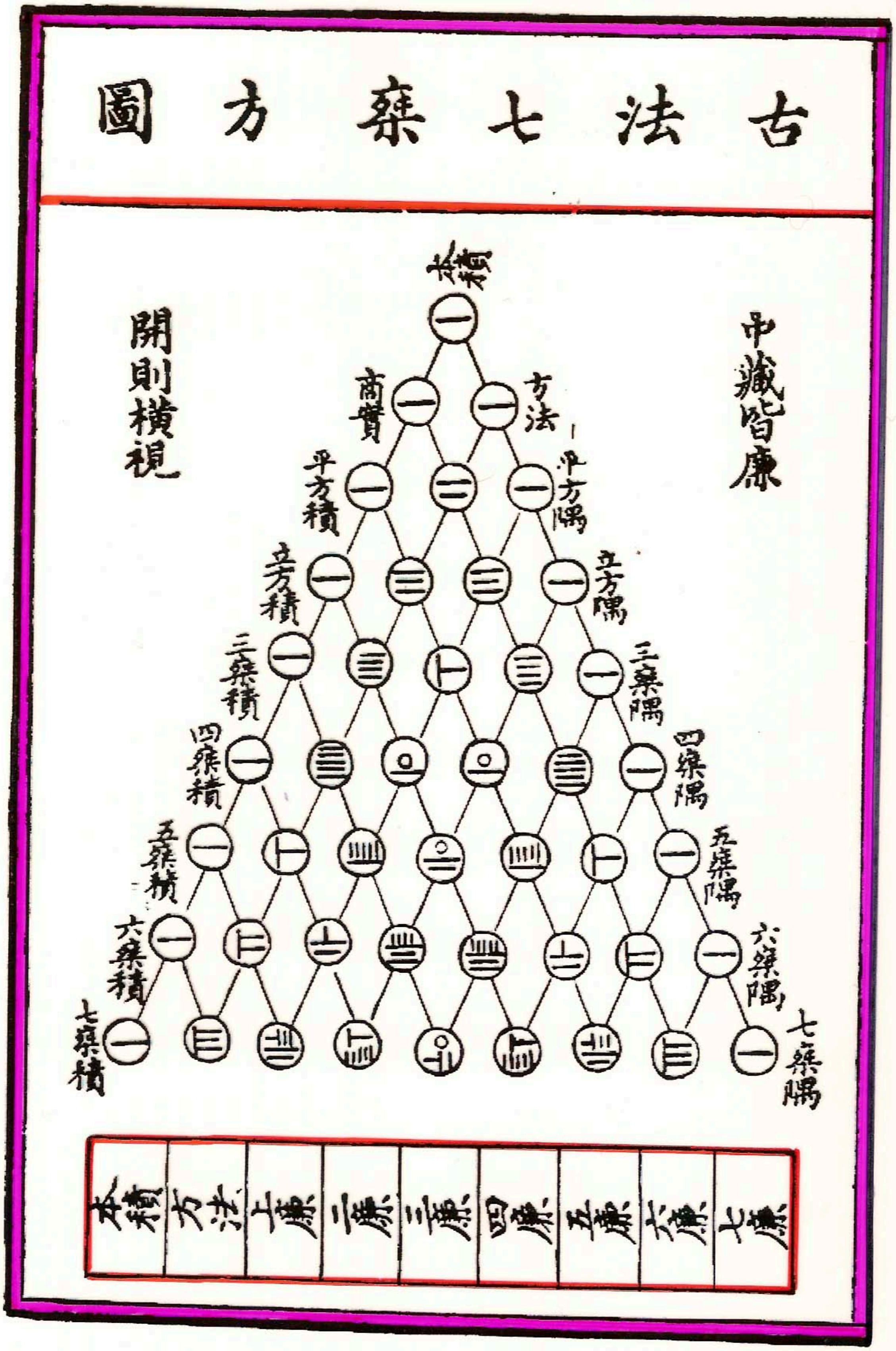
binomial coefficients of $(a+b)^n$

figurate numbers (eg triangular)

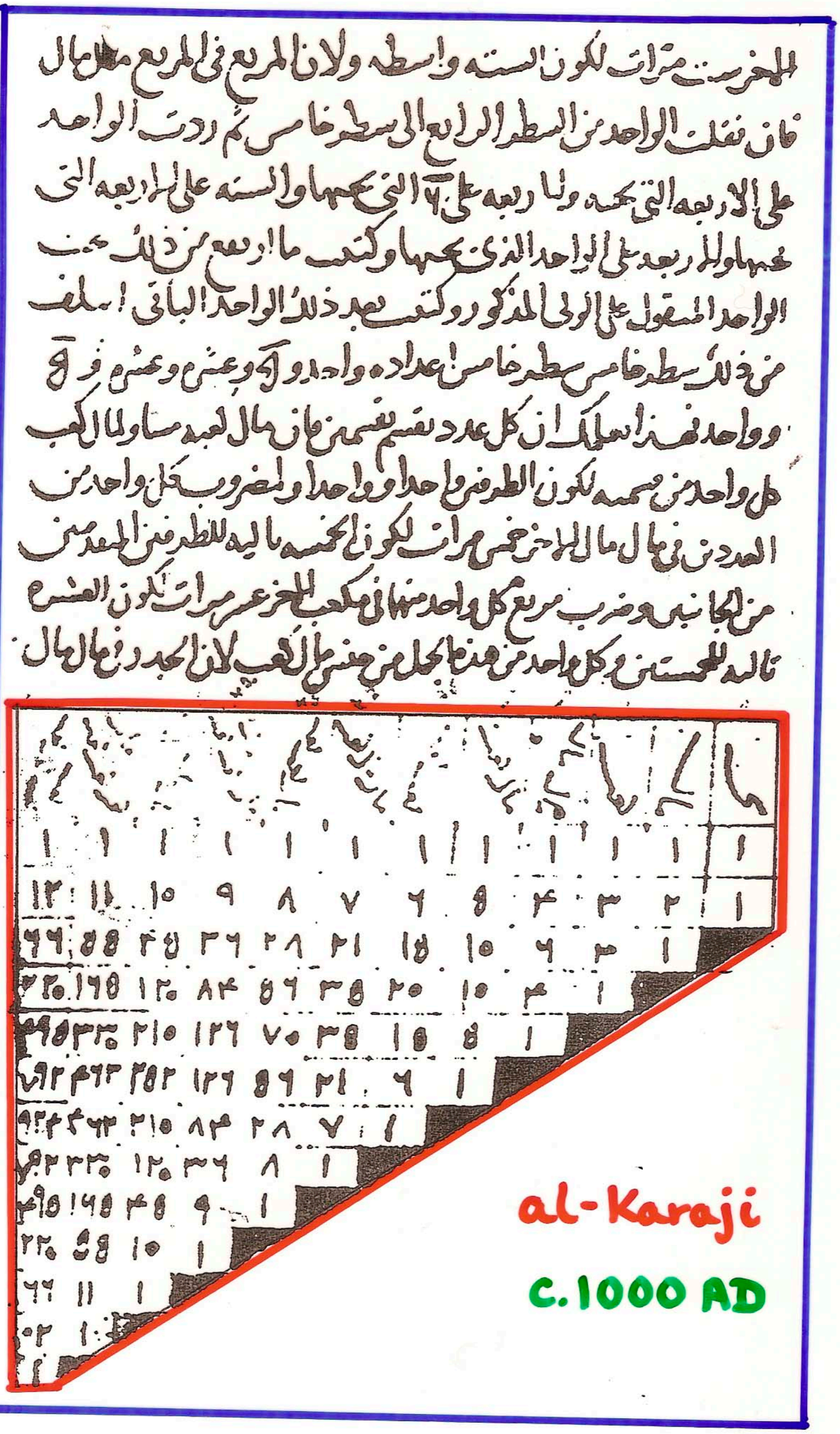
$$(a+b)^n = (a+b)(a+b)\cdots(a+b)$$

$$= \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n}b^n$$

1303: Zhu Shijie: 'Sijuan yujian'
(Precious mirror of the four elements)



The earliest 'Pascal's triangle' ?

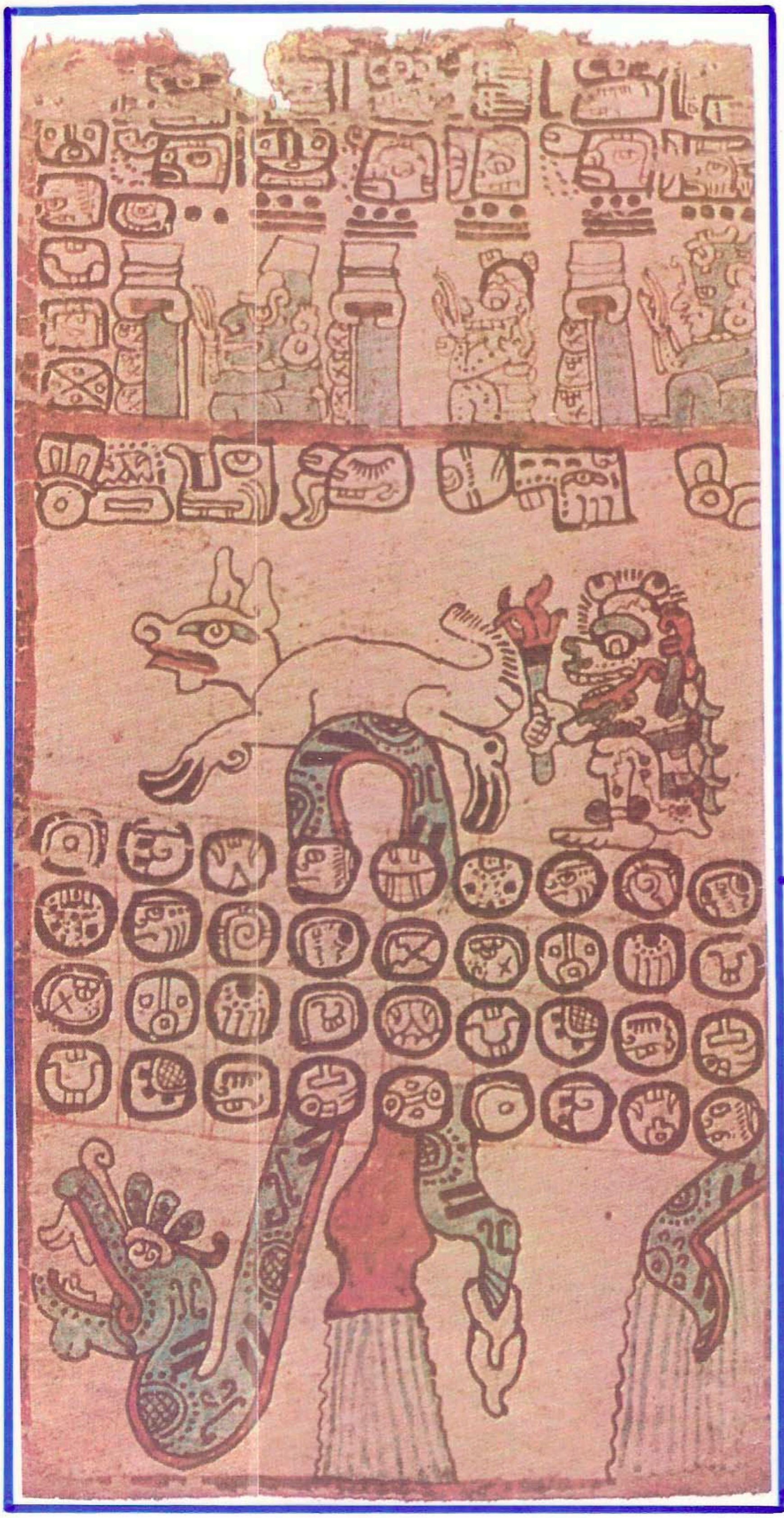


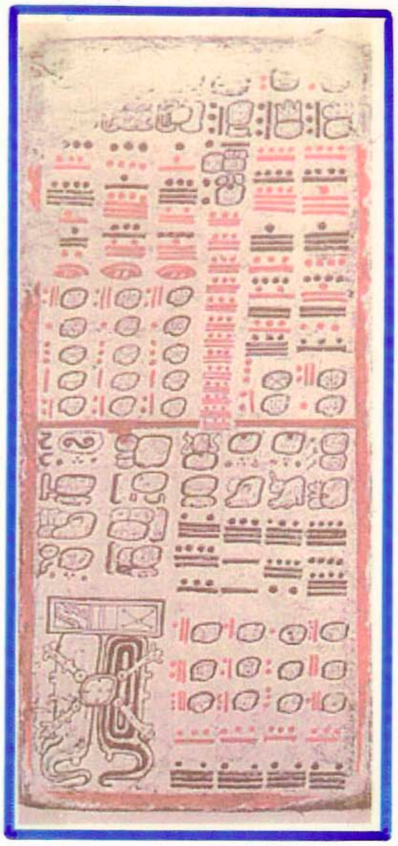
١	١	١	١	١	١	١	١	١	١	١	١
١٢	١١	١٠	٩	٨	٧	٦	٥	٤	٣	٢	١
٦٦	٥٥	٤٥	٣٦	٢٨	٢١	١٥	١٠	٦	٣	١	
٢٢٠	١٦٥	١٢٠	٨٤	٥٦	٣٥	٢٠	١٠	٤	١		
٤٩٥	٣٣٠	٢١٠	١٢٦	٧٠	٣٥	١٥	٥	١			
٧٩٢	٤٦٢	٢٥٢	١٢٦	٥٦	٢١	٦	١				
٩٢٤	٤٦٢	٢١٠	٨٤	٢٨	٧	١					
٧٩٢	٣٣٠	١٢٠	٣٦	٨	١						
٤٩٥	١٦٥	٤٥	٩	١							
٢٢٠	٥٥	١٠	١								
٦٦	١١	١									
١٢	١										
١											

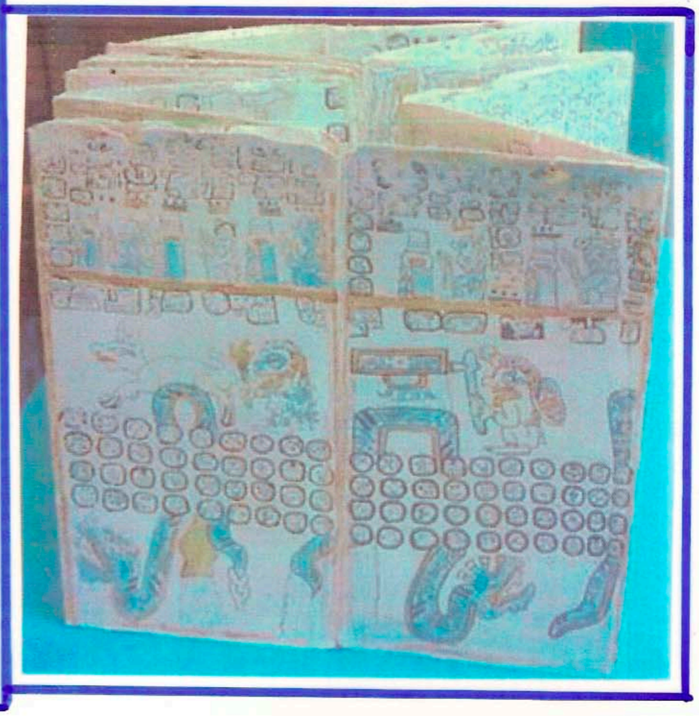
al-Karaji

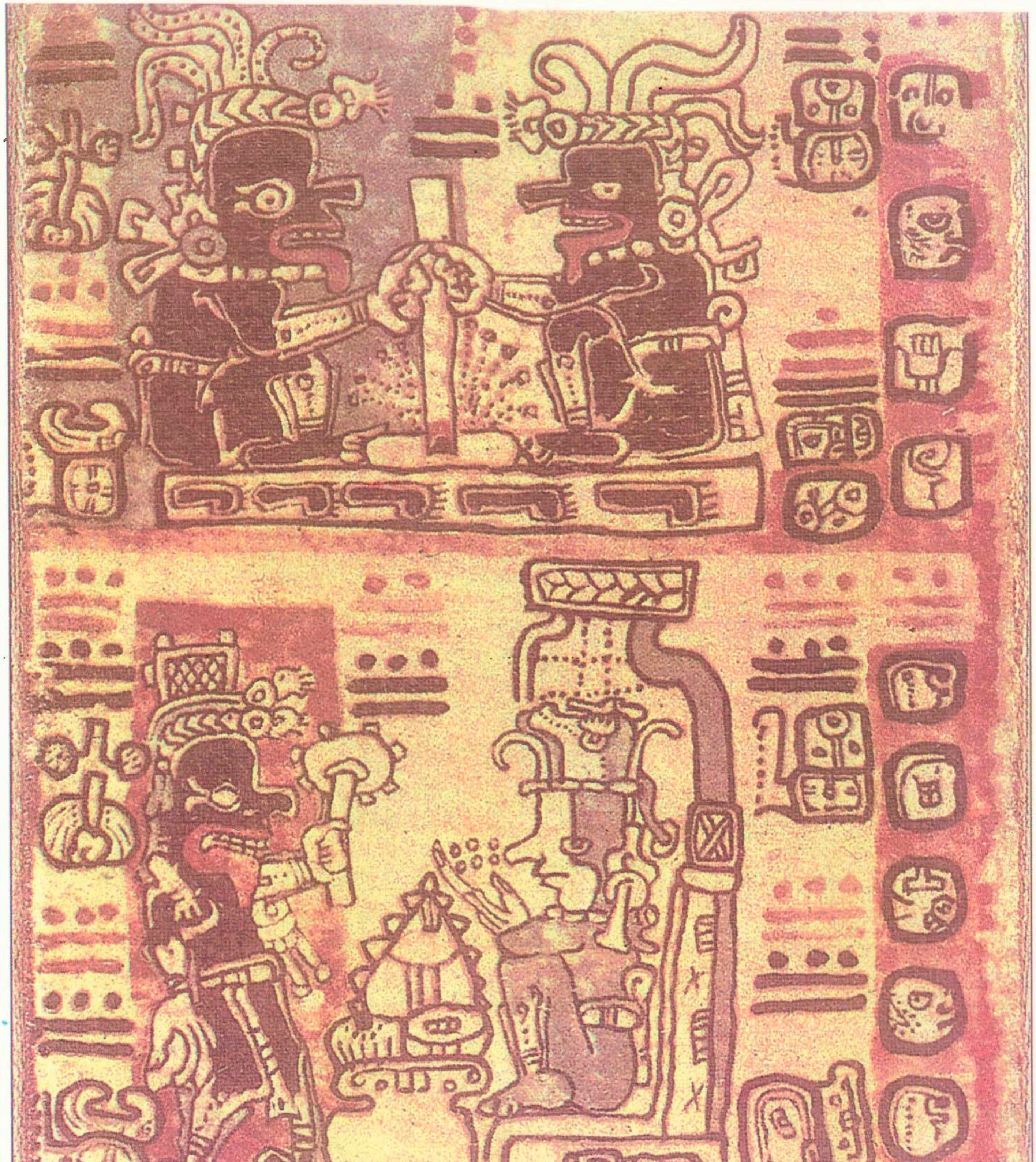
c. 1000 AD

The Mayans of Central America (c.300-900AD)

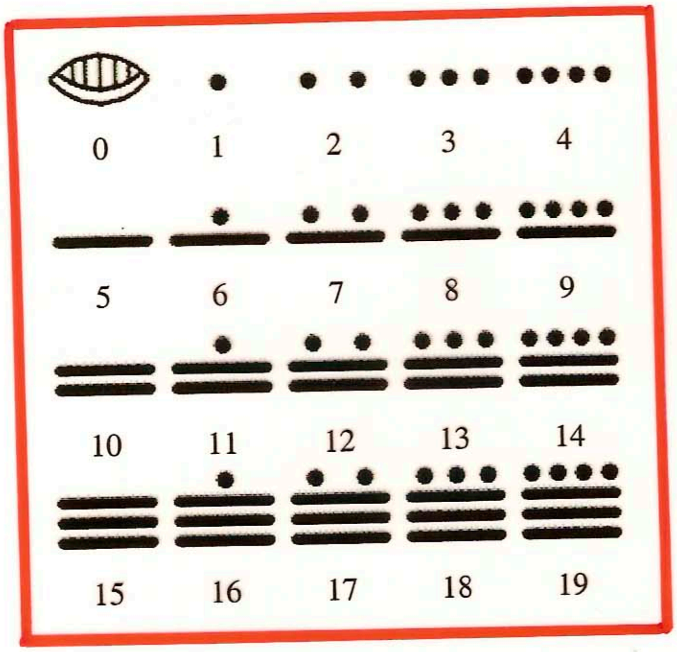


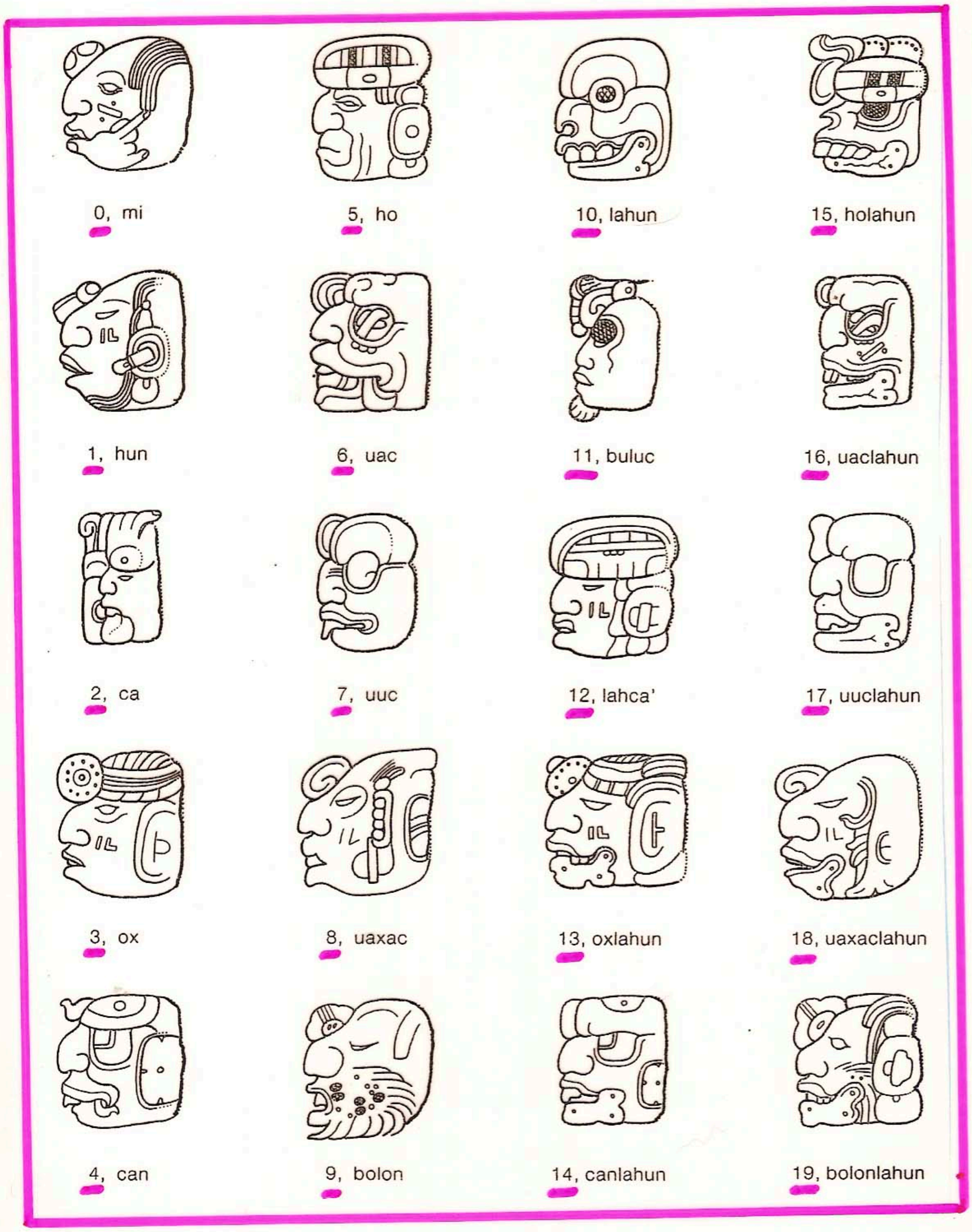


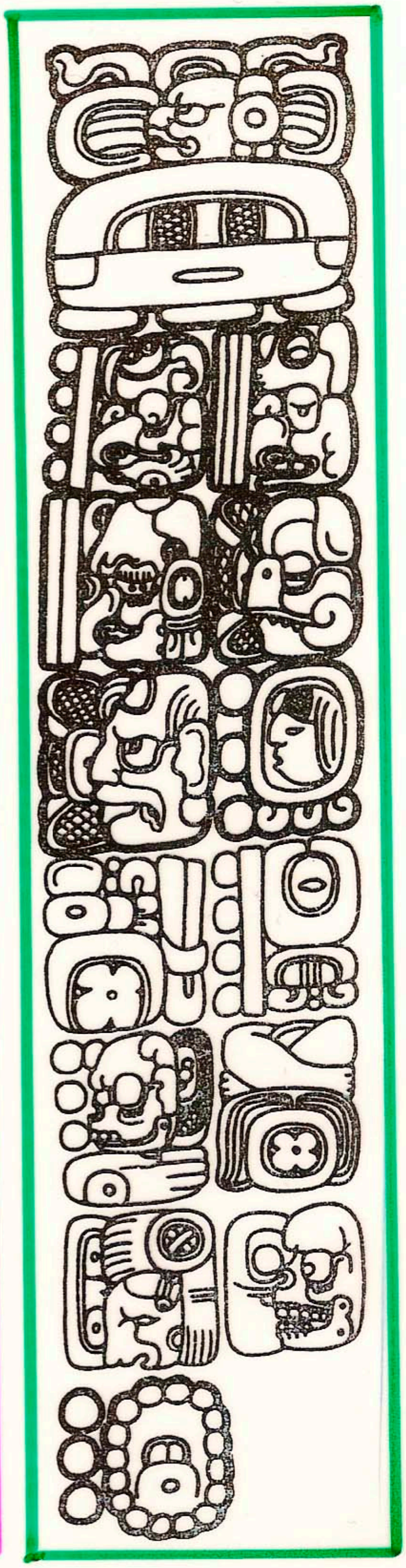




Mayan counting







11 February 526 AD

Mayan timekeeping

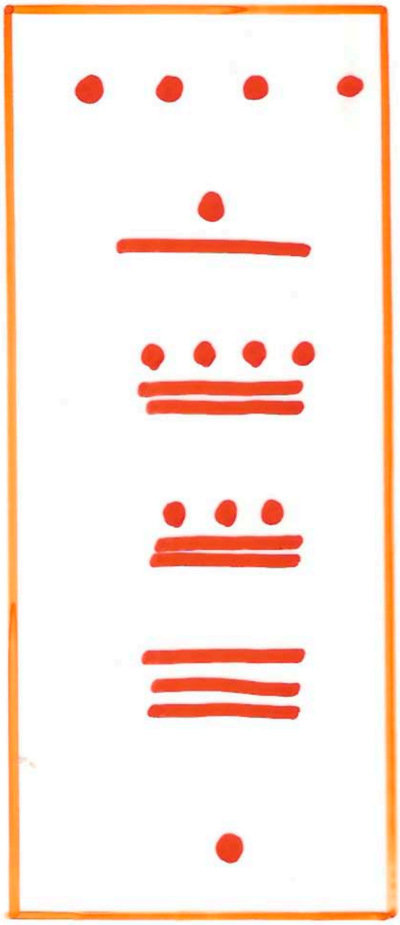
1 kin = 1 day

20 kins = 1 uinal = 20 days

18 uinals = 1 tun = 360 days

20 tuns = 1 katun = 7200 days

20 katuns = 1 baktun = 144,000 days . . .



= 4 × 2,880,000 = 11,520,000

= 6 × 144,000 = 864,000

= 14 × 7200 = 100,800

= 13 × 360 = 4680

= 15 × 20 = 300

= 1 × 1 = 1

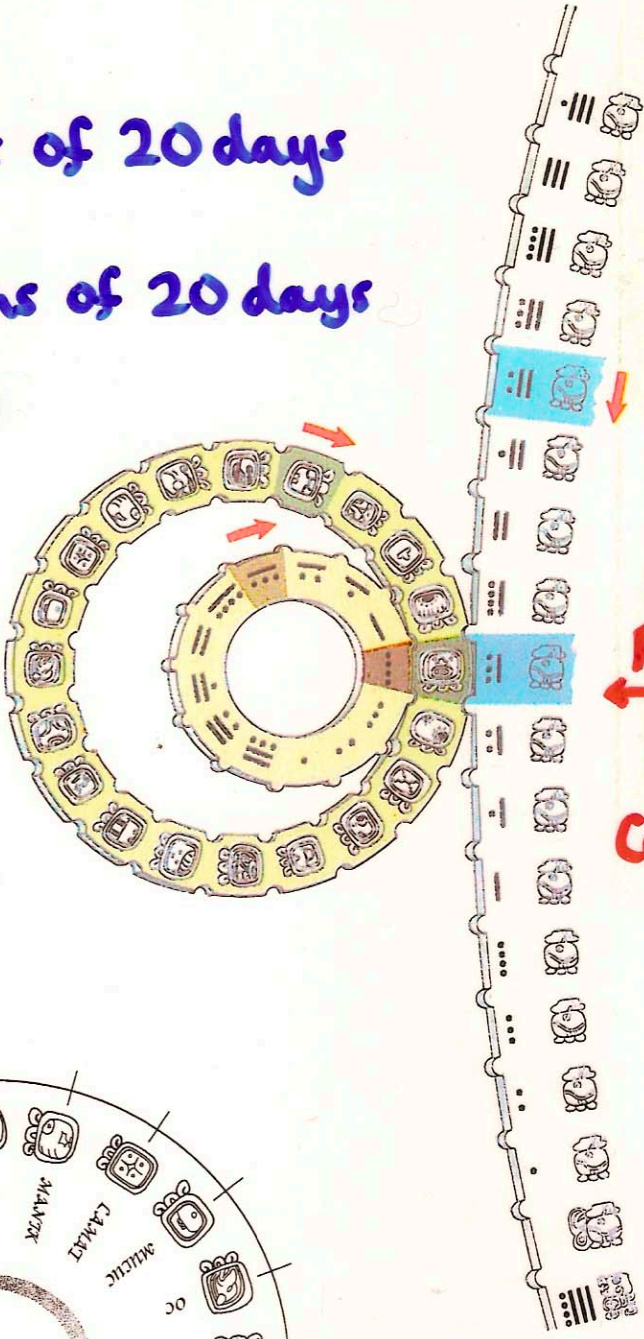
Total: 12,489,781 days

The Mayan calendar

Two forms :

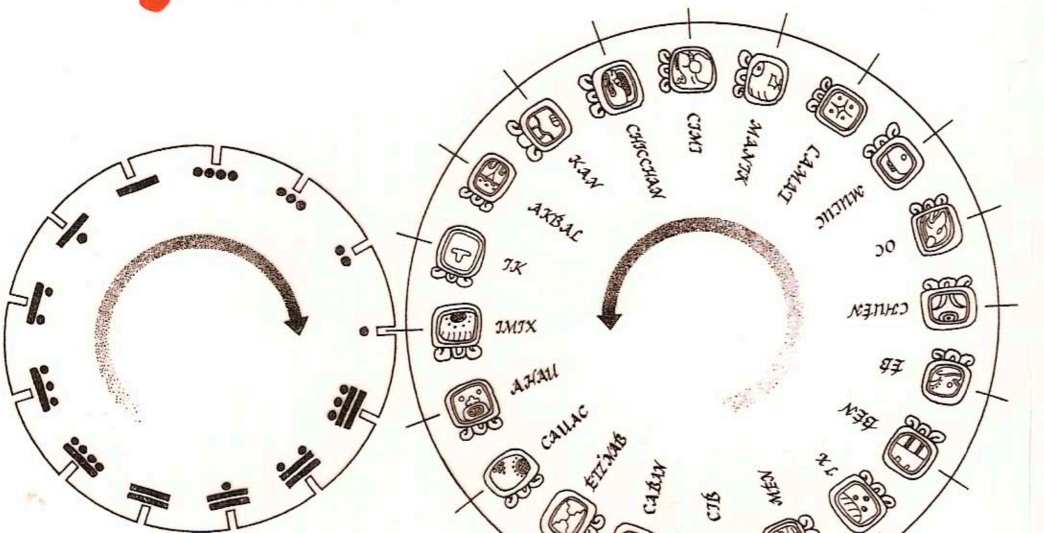
260 days : 13 months of 20 days

365 days : 18 months of 20 days
(+ 5 days)



4
Ahau
←
8
Cumku

These combine :
calendar-round of
18980 days
(= 52 years).



13 × 20 day names = 260 days

Forthcoming Lectures

How to earn a million dollars.

2 February at 6 p.m.

Prime-time mathematics

9 March at 6 p.m.

How hard is a hard problem?

Royal Institution Sixth-Form event

8 February (all day)

Do we need maths now that we have computers? (6 speakers)