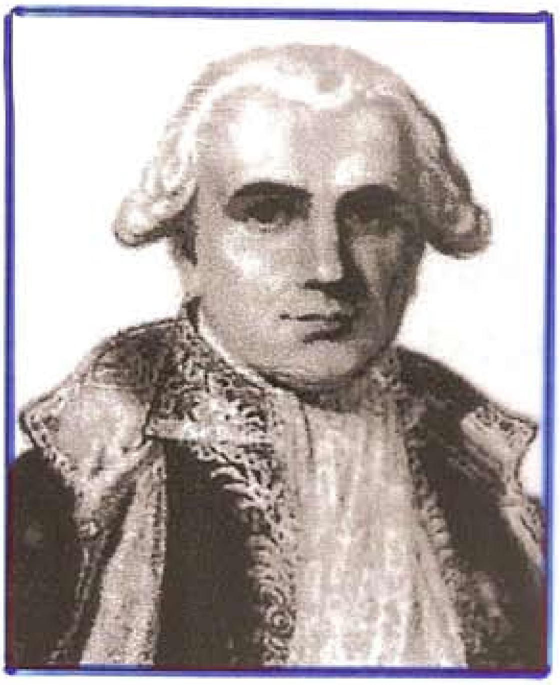
The 19th Century: Revolution or Evolution?

Robin Wilson

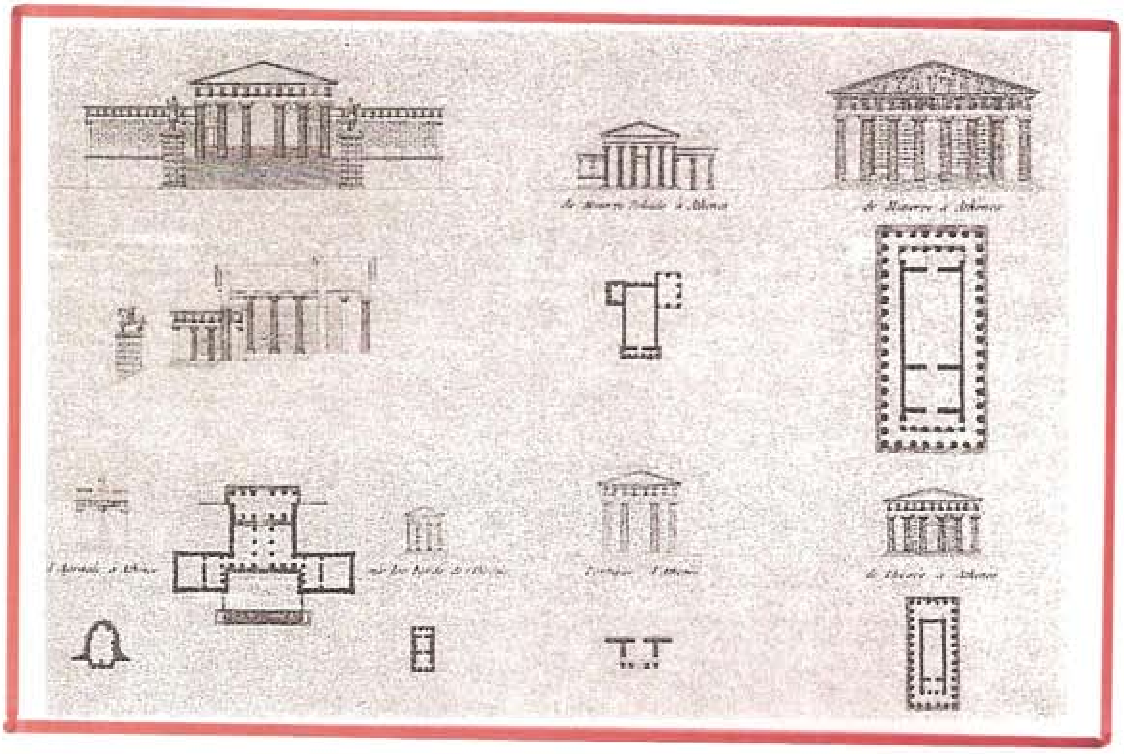
- Algebra
- Geometry
- Calculus / Analysis
- Foundations



Gaspard Monge

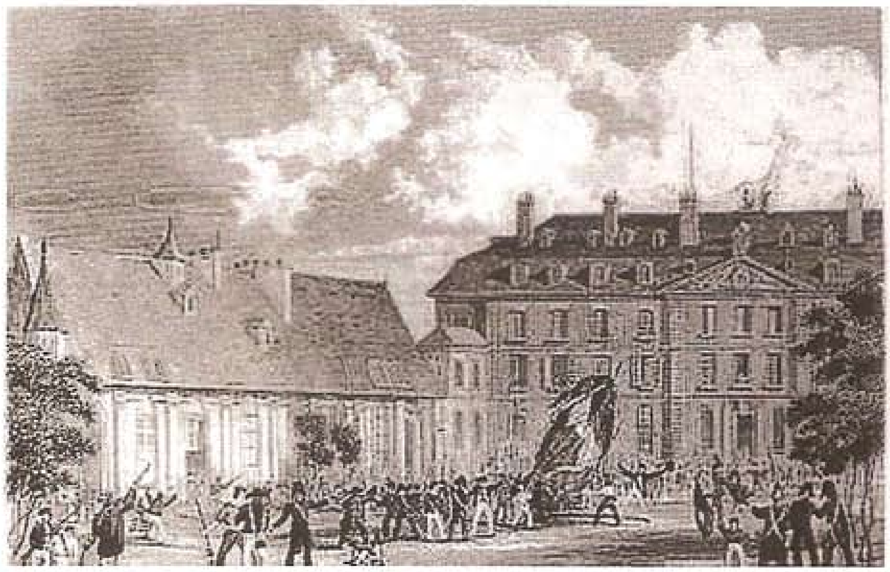
(1746 - 1818)

'descriptive geometry'



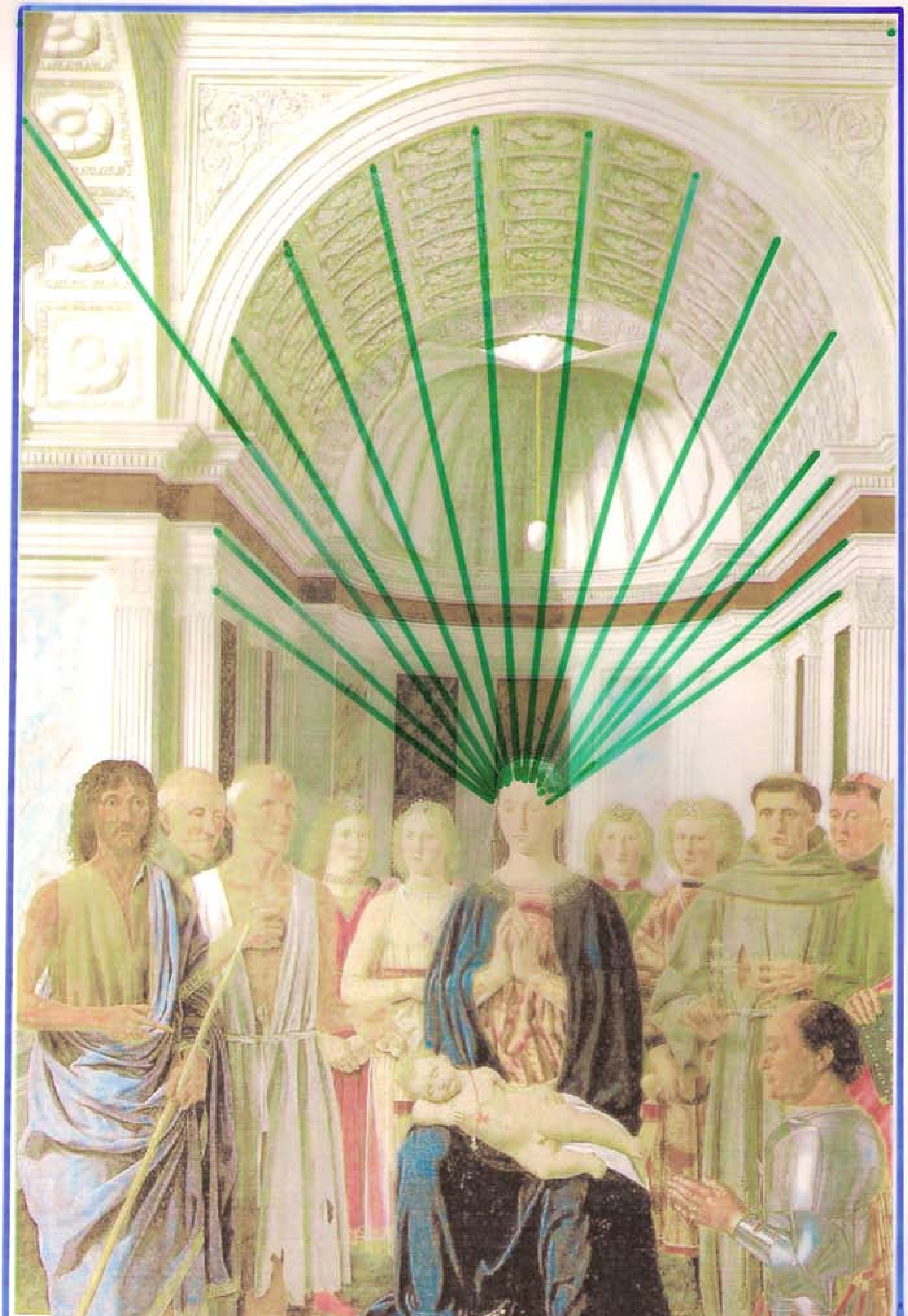
representing buildings on a plane

The École Polytechnique, Paris

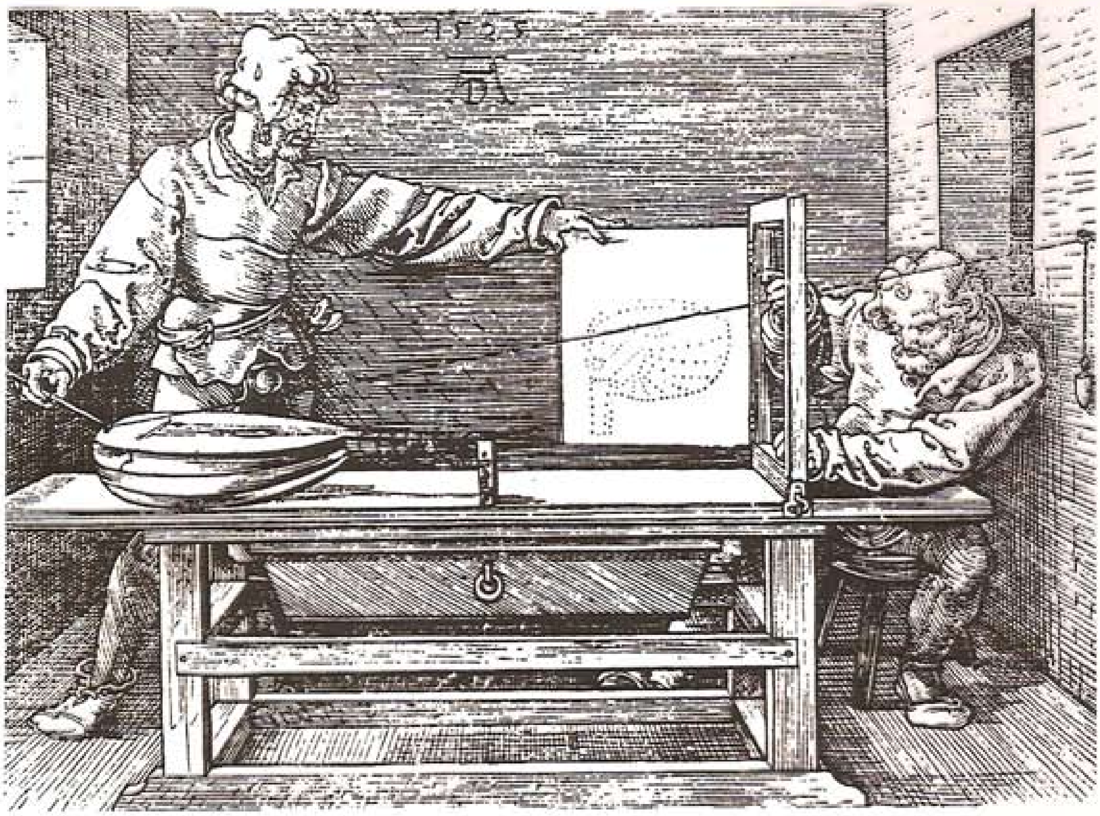


Monge, Laplace, Lagrange,

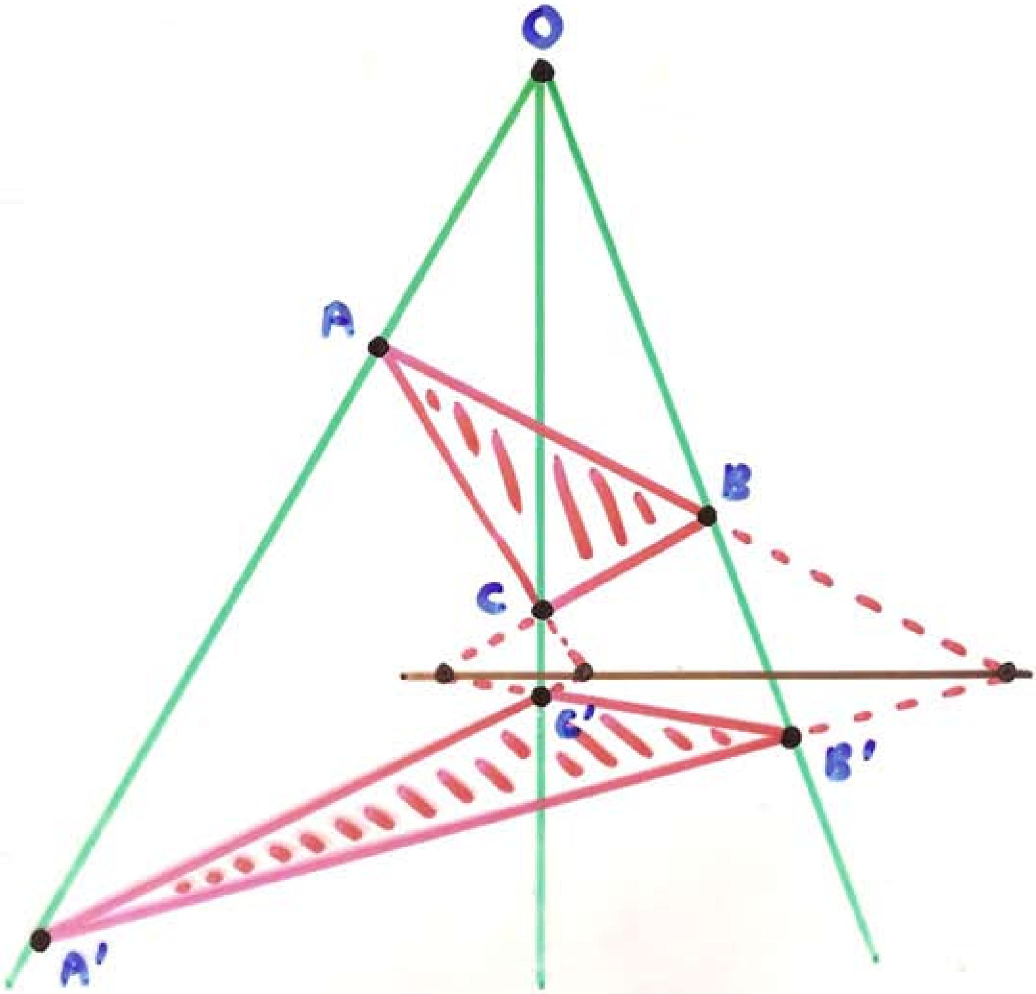
Cauchy



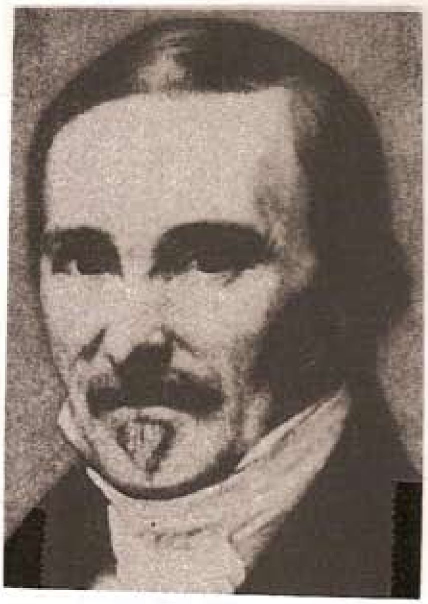
Dürer on perspective drawing

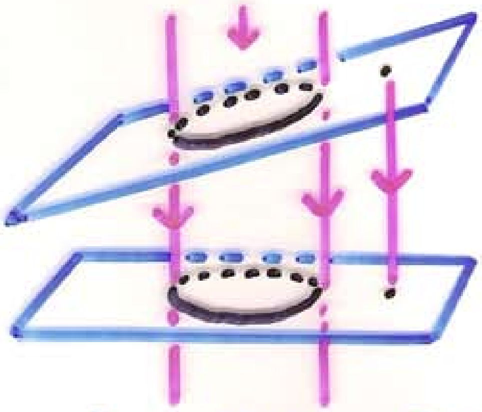


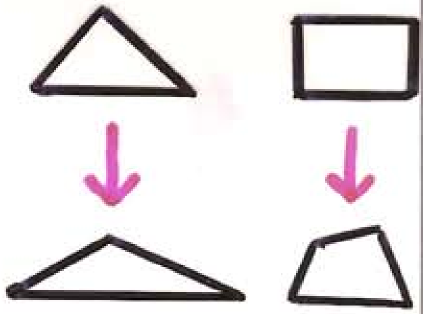
Desargues' theorem



Poncelet: projective transformation

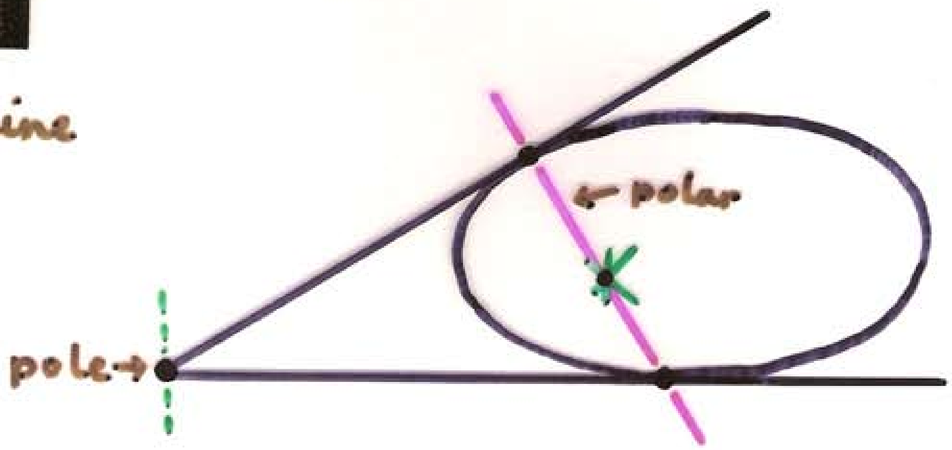






duality (Gergonne) (1827)

point ⟷ line



Gergonne on duality, 1827

Let there be a plane figure composed in any way one wishes of points, lines and curves ...

One then constructs in the same plane another figure ...

1. If there is a system of a certain number of points on a line, then in the other figure there will be a system of exactly as many lines meeting in a point.

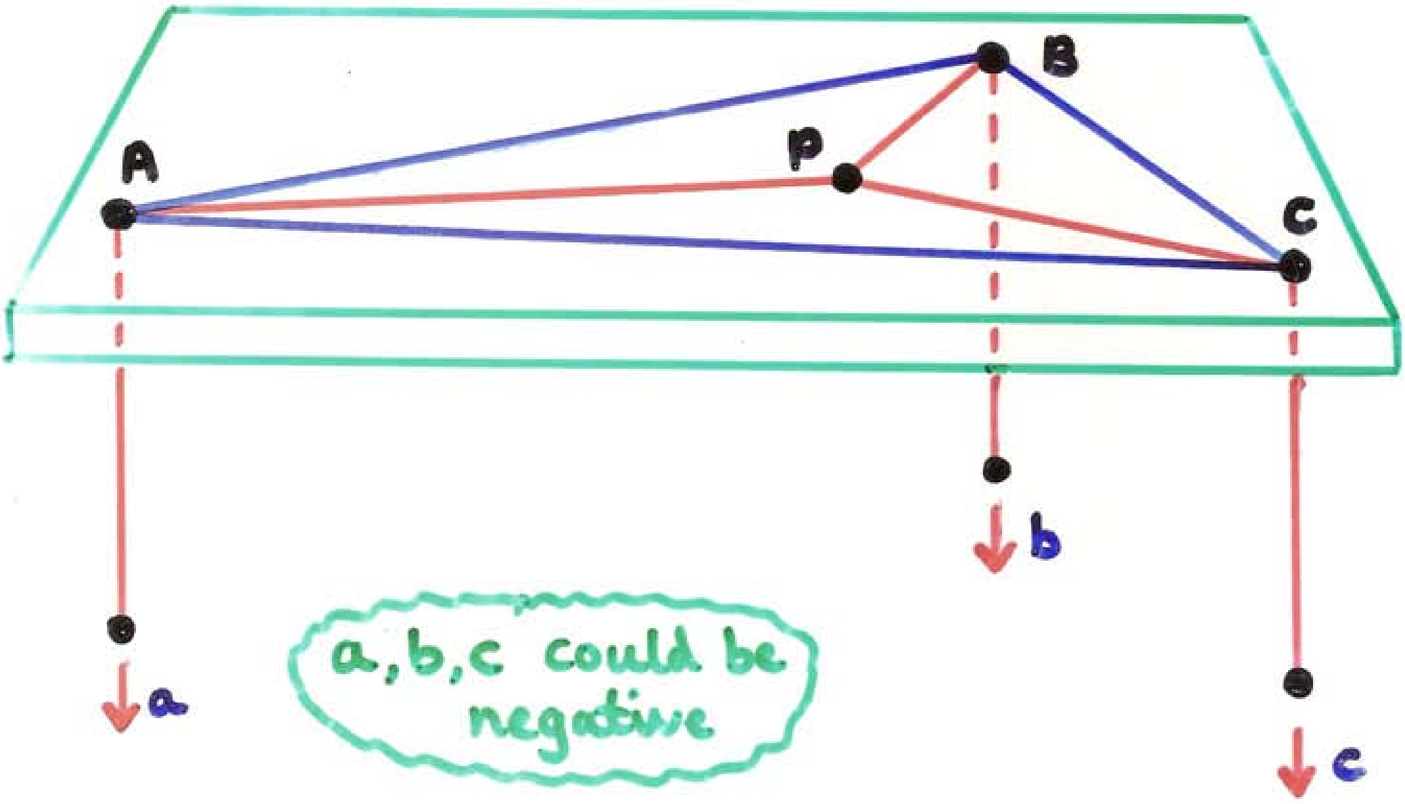
1. If there is a system of a certain number of lines meeting in a point, then in the other figure there will be a system of exactly as many points lying on a line.

Any two points determine a line.

Any two lines determine a point.

The Barycentric Calculus

(1827)



P has barycentric coordinates

[a, b, c]

note: [a, b, c] = [λa, λb, λc]

for any λ ≠ 0

'homogeneous coordinates'

DUALITY: points ⟷ lines

[a, b, c] ⟷ $ax+by+cz=0$

Solving Quadratic Equations

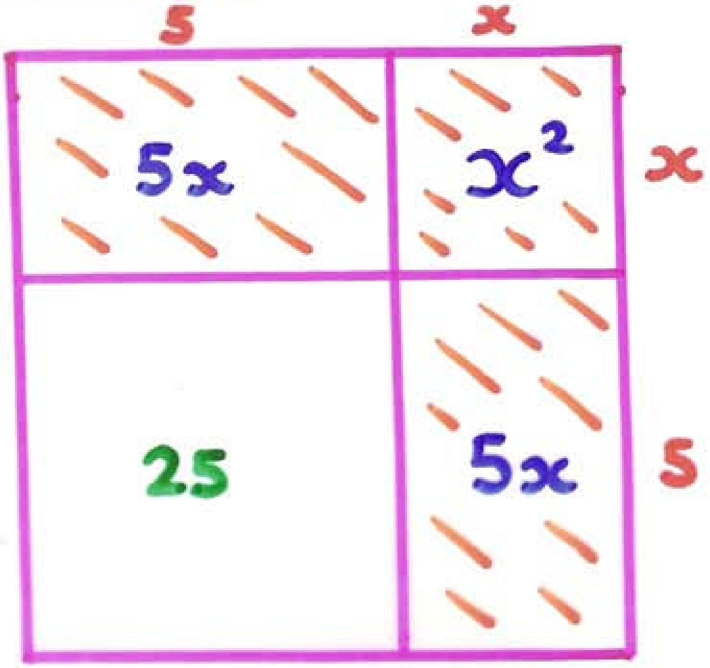
Six types: (a, b, c all positive)

$ax^2 = bx$, $ax^2 = b$, $ax = b$,

$ax^2 + bx = c$, $ax^2 + c = bx$, $ax^2 = bx + c$

'Roots and squares are equal to numbers'

$x^2 + 10x = 39$



$(x+5)^2 = 39 + 25 = 64$

so $x + 5 = 8$, $x = 3$

QVESITI,

ET INVENTIONI
DIVERSE

DE NICOLO TARTAGLIA,

Di nouo restampati con vna Gionta al sesto libro, nella quale si mostra duoi modi di redur vna Città inespugnabile.

La diuisione, & continentia di tutta l'opra nel seguente foglio si trouarà notata.

Solving a cubic equation

$$x^3 + 6x = 20$$

Find u and v so that

$$u - v = 20 \quad \text{and} \quad uv = (6/3)^3 = 8.$$

Since $v = u - 20$, we have

$$uv = u(u-20) = u^2 - 20u = 8.$$

Solving this quadratic equation:

$$u = \sqrt{108} + 10.$$

So $$v = u - 20 = \sqrt{108} - 10.$$

So

$$x = \sqrt[3]{u} - \sqrt[3]{v}$$

$$= \underline{\sqrt[3]{(\sqrt{108} + 10)} - \sqrt[3]{(\sqrt{108} - 10)}}$$

$$= 2.$$

Solving a quartic equation

$$x^4 = px^2 + qx + r$$

$$(x^2 + y)^2 = (p + 2y)x^2 + qx + (r + y^2)$$

For RHS a square we need:

$$(p + 2y)(r + y^2) = q^2/4 \qquad [\text{"}b^2 = 4ac\text{"}]$$

(*) $$2y^3 + py^2 + 2ry + (pr - q^2/4) = 0$$

(•) then reduces to two quadratics.

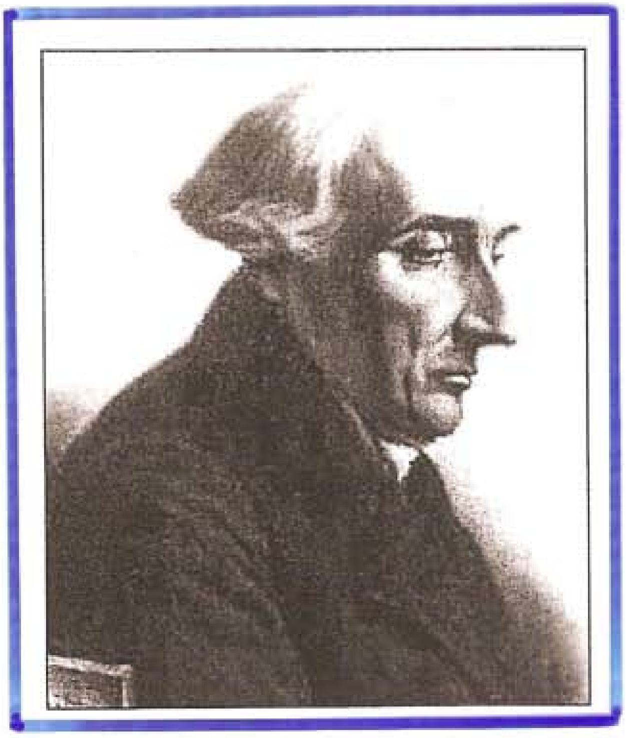
e.g. $$x^4 = x + 2$$

$$x = \sqrt{\sqrt[3]{\sqrt{\frac{2075}{442368}} + \frac{1}{128}} - \sqrt[3]{\sqrt{\frac{2075}{442368}} - \frac{1}{128}}}$$

$$+ \sqrt{\sqrt[3]{\frac{1051}{3456} + \sqrt{\frac{2075}{442368}}} + \sqrt[3]{\frac{1051}{3456} - \sqrt{\frac{2075}{442368}}} \ldots}$$

$$\overline{\ldots + \frac{2}{3} - \sqrt[3]{\sqrt{\frac{2075}{442368}} + \frac{1}{128}} + \sqrt[3]{\sqrt{\frac{2075}{442368}} - \frac{1}{128}}}$$

Lagrange's reduction methods



$x^3 + nx + p = 0$

put $x = y - \frac{n}{3y}$

$y^6 + py^3 - \frac{1}{27}n^3 = 0$

quadratic in y^3

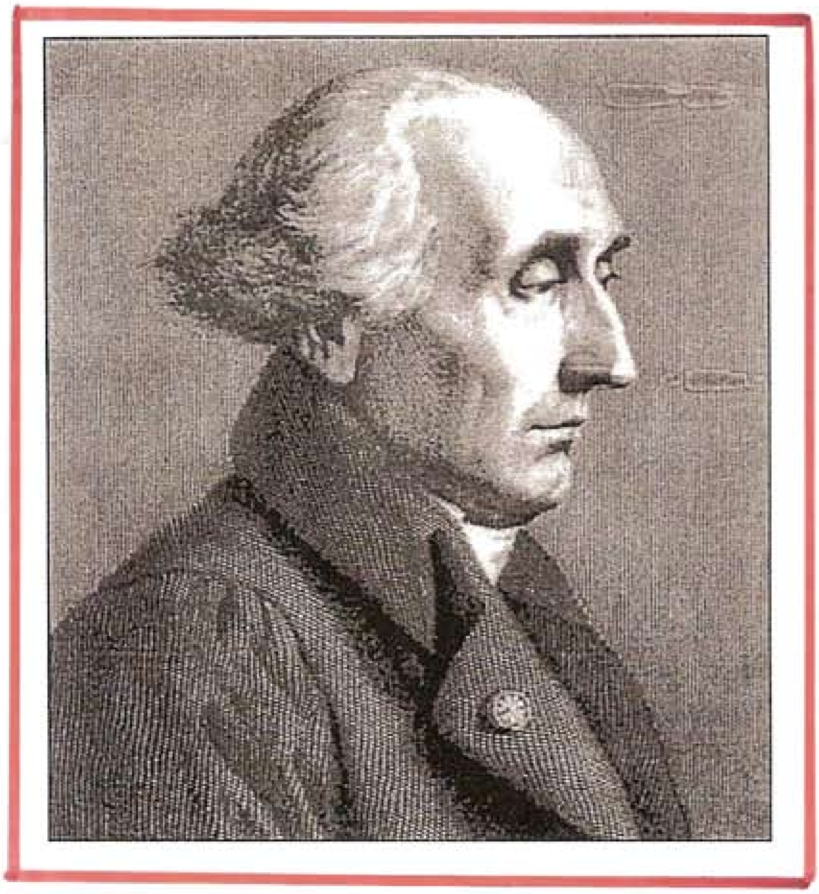
six $\underline{y}$s → three $\underline{x}$s

Quartic: $x^4 + nx^2 + px + q = 0$

reduces to

$$y^3 - \frac{n}{2}y^2 - qy + \frac{4nq - p^2}{8} = 0$$

cubic in y, ...



Permuting the solutions

$x^3 + ax^2 + bx + c = 0$

solutions p, q, r

$x^3 + ax^2 + bx + c = (x-p)(x-q)(x-r)$

$c = -pqr$, $b = pq + pr + qr$,

$a = -(p+q+r)$

Permute the solutions – no change

$pq + 2r$: $pr + 2q$ $qr + 2p$ (3 values)

$p + 2q + 5r$: $p + 2r + 5q, \ldots$ (6 values)

...

(always divides 6)

Paolo Ruffini

(1765 - 1822)

The algebraic solution of equations of degree greater than 4 is always impossible.

Behold a very important theorem which I believe I am able to assert ...

Niels Henrik Abel

(1802 - 1829)

proved impossibility of solving quintic equations

Evariste Galois

(1811 - 1832)

developed criteria for deciding which equations can be solved

Galois to Chevalier, 29 May 1832

My dear friend,

I have done several new things in analysis. Some concern the theory of equations; others, integral functions.

In the theory of equations I have found out in which cases the equations are solvable by radicals, which has given me the occasion to deepen the theory and to describe all the transformations admitted by an equation, even when it is not solvable by radicals.

. . .

You will publicly beg Jacobi or Gauss to give their opinion not of the truth but of the importance of the theorems.

After this, there will, I hope, be people who will find it to their advantage to decipher all this mess.

Euclid's Postulates

Let the following be postulated:

1. To draw a straight line from any point to any point.

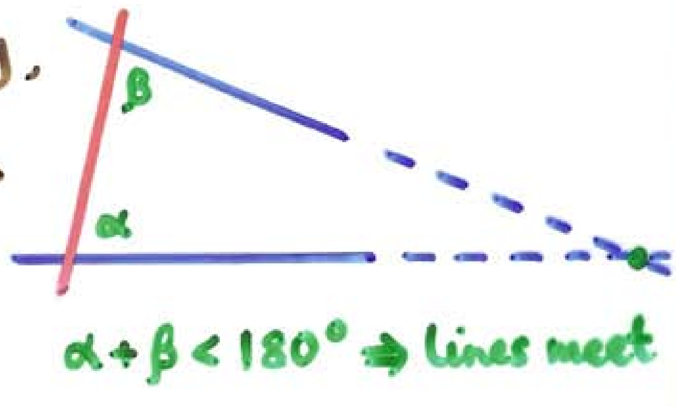
2. To produce a finite straight line continuously in a straight line.

3. To describe a circle with any centre and distance.

4. That all right angles are equal to one another.

---

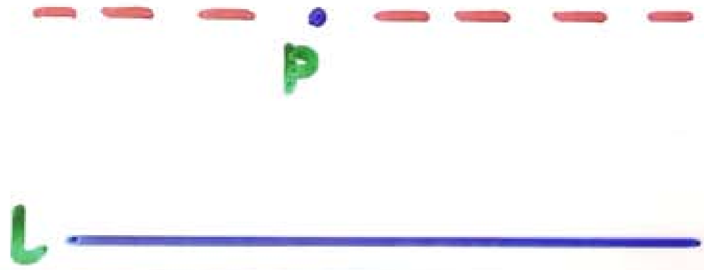
5. That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.



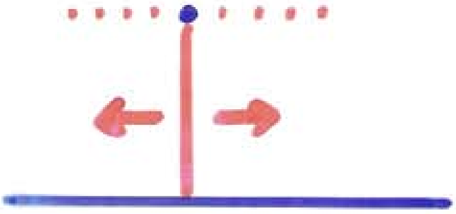
Euclid's Fifth Postulate

$\alpha + \beta < 180° \Rightarrow$ lines meet

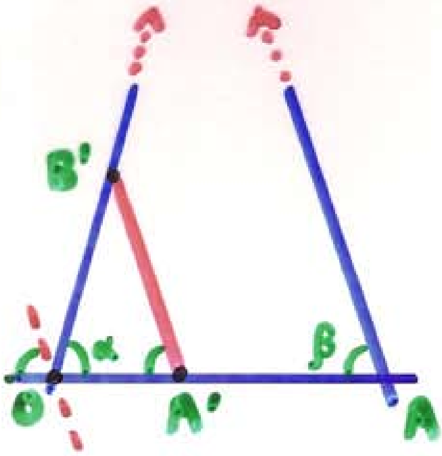
Euclid I.29:



'parallel postulate'

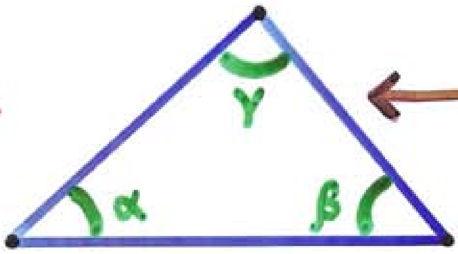


Alhazen (1000AD)



J. Wallis (1663)

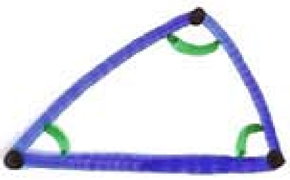
G. Saccheri (1733)



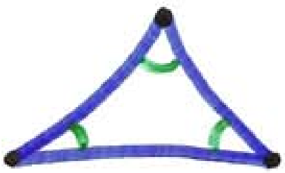
← (Nasir Eddin: c.1250)

← Euclidean

Sum = 180°



Sum > 180° (impossible)



Sum < 180° ?

Non-Euclidean Geometry (c. 1830)

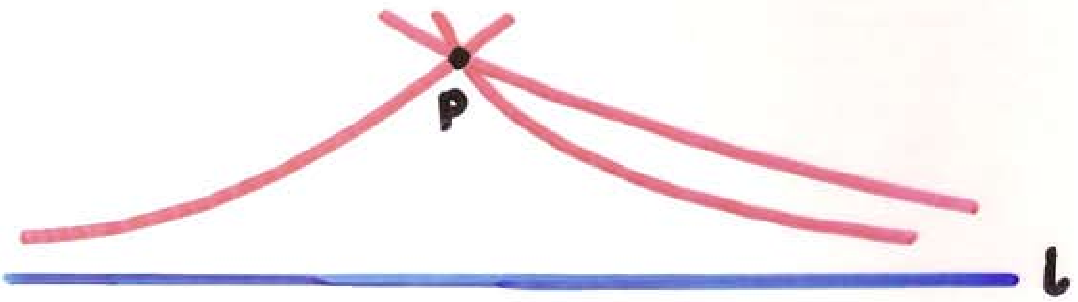
Nikolai Lobachevskii

János Bolyai

(C. F. Gauss)

(Wolfgang Bolyai)

- The angle sum of every triangle < 180°.
- Given any line l and any point p not on l, there are infinitely many lines through p parallel to l.
- Any two similar figures must be congruent.



Letter from Gauss:

Another theme which is almost 40 years old with me that I have been thinking about now and again in a few free hours; I mean the foundations of geometry. …

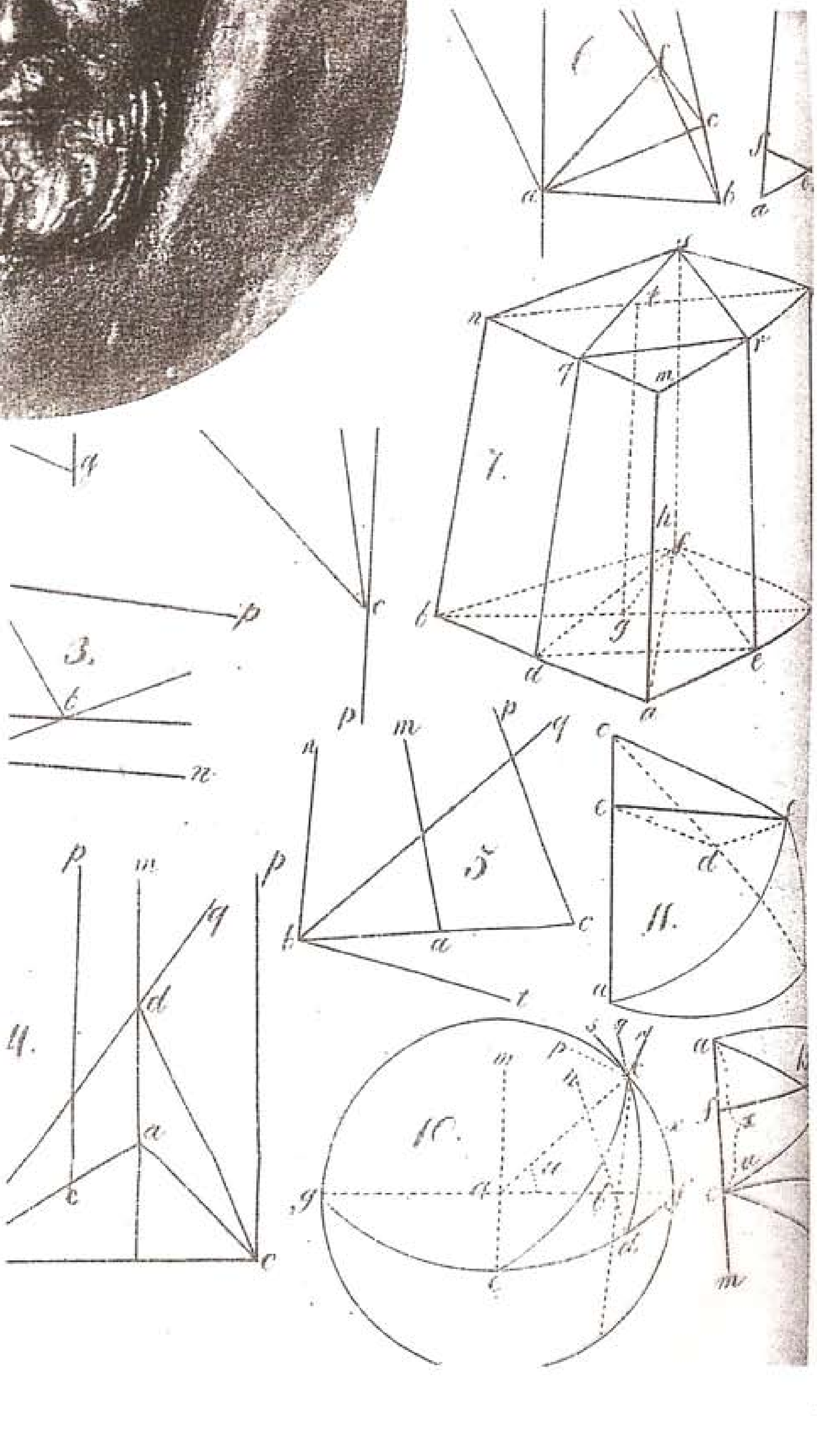
My opinion that we cannot establish geometry completely *a priori* is, if possible, much firmer. Meanwhile I will still not get round to it for some time and work up my very extensive researches for publication, and perhaps they will never appear in my lifetime, for I fear the howl of the Boeotians if I speak my opinion out loud.

Abel, from Berlin in 1825:

Crelle says that all Gauss writes is gruel since it is so obscure that it is almost impossible to understand.

János Bolyai

(1802-1860)

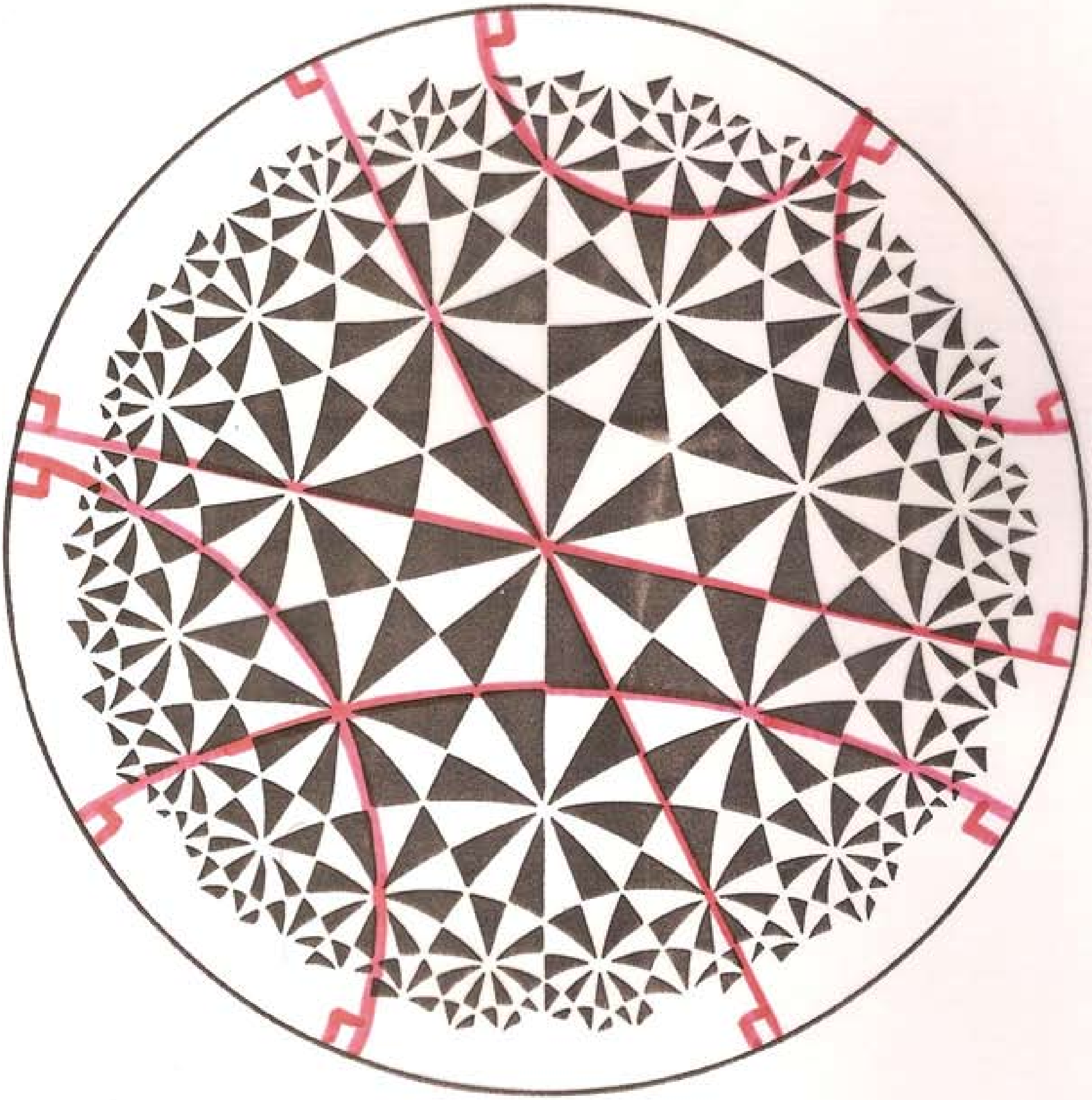


Letter from Gauss to F. Bolyai, 1831:

If I commenced by saying that I am unable to praise this work, you would certainly be surprised for a moment. But I cannot say otherwise. To praise it would be to praise myself.

Indeed the whole contents of the work, the path taken by your son, the results to which he is led, coincide almost entirely with my meditations, which have occupied my mind partly for the last thirty or thirty-five years.

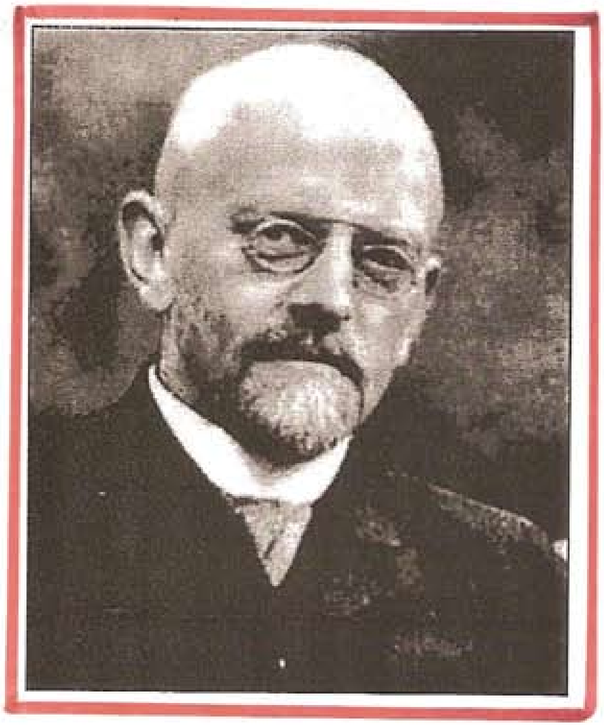
Poincaré's non-Euclidean geometry



Kleinian View of Geometry (1872)

Erlanger Programm

1. A geometry is a space and a group of transformations on that space.
2. We are concerned with geometrical properties: which are unchanged by elements of the group?
3. The larger the group, the fewer geometrical properties are unchanged.



David Hilbert:

'Foundations of Geometry'

axiomatised Euclidean and projective geometry (1899)

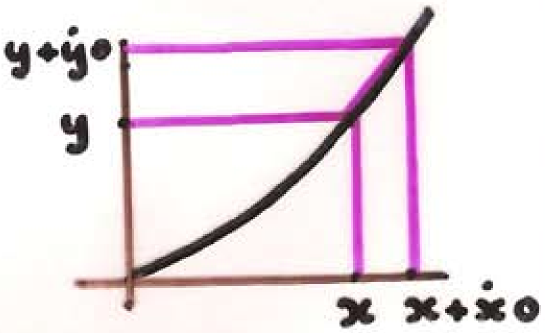
2nd edition (1903) axiomatised non-Euclidean geometry

Newton's calculus

Variables: changing with time
– 'flowing'

Derivatives: based on velocity
(tangents) – notation $\dot{x}$, $\dot{y}$

Example: $y = x^2$



Substitute $x+\dot{x}o$ for x
$y+\dot{y}o$ for y:

$$\cancel{y} + \dot{y}o = \cancel{x^2} + 2x\dot{x}o + \dot{x}^2o^2$$

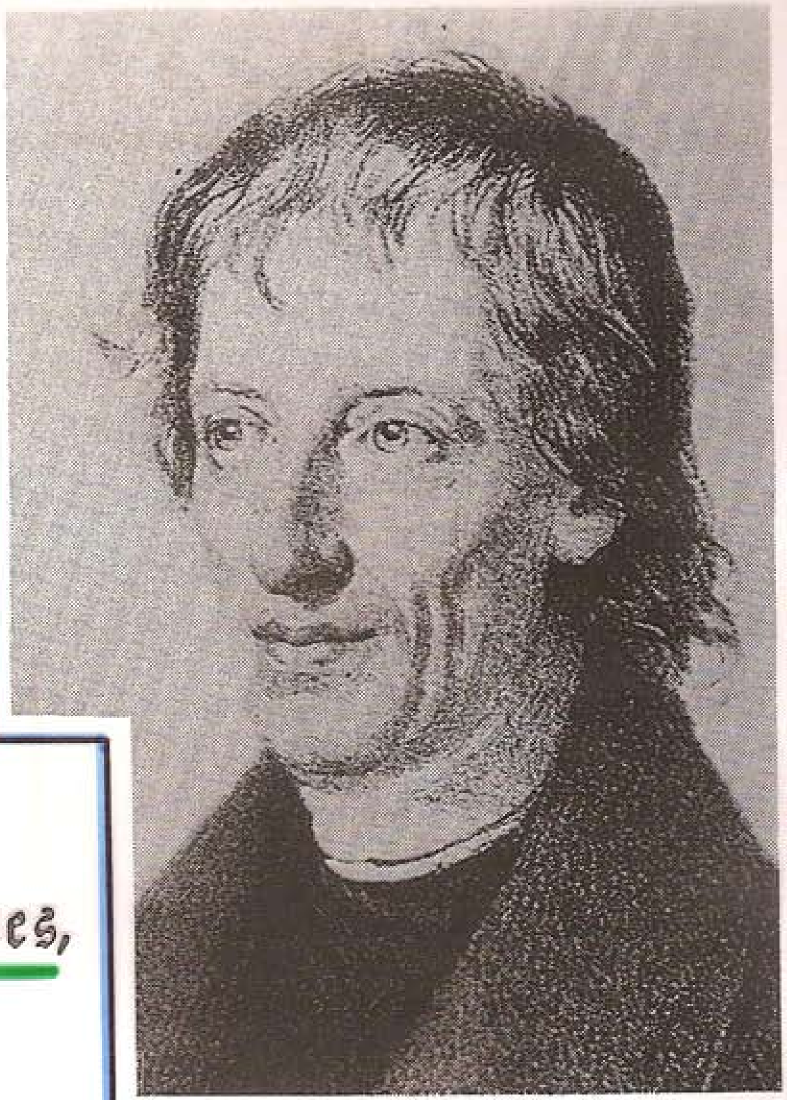
Cancel o: $\dot{y} = 2x\dot{x} + \dot{x}^2 o$

Ignore o: $\dot{y} = 2x\dot{x}$, or $\dot{y}/\dot{x} = 2x$

Integrals: find anti-derivatives
(areas) (fundamental theorem)

Bernard Bolzano

(1781-1848)



Rein analytischer

Beweis des Lehrsatzes,

daß

zwischen je zwey Werthen, die ein entgegengesetztes Resultat gewähren, wenigstens eine reelle Wurzel der Gleichung liege;

von

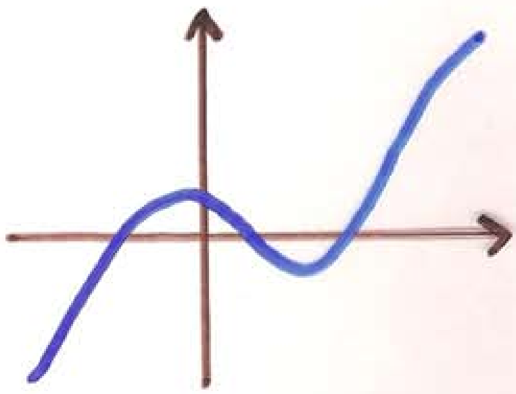
Bernard Bolzano,

Weltpriester, Doctor der Philosophie, k. k. Professor der Religionswissenschaft, und ordentlichem Mitgliede der k. Gesellschaft der Wissenschaften zu Prag.

Für die Abhandlungen der k. Gesellschaft der Wissenschaften.

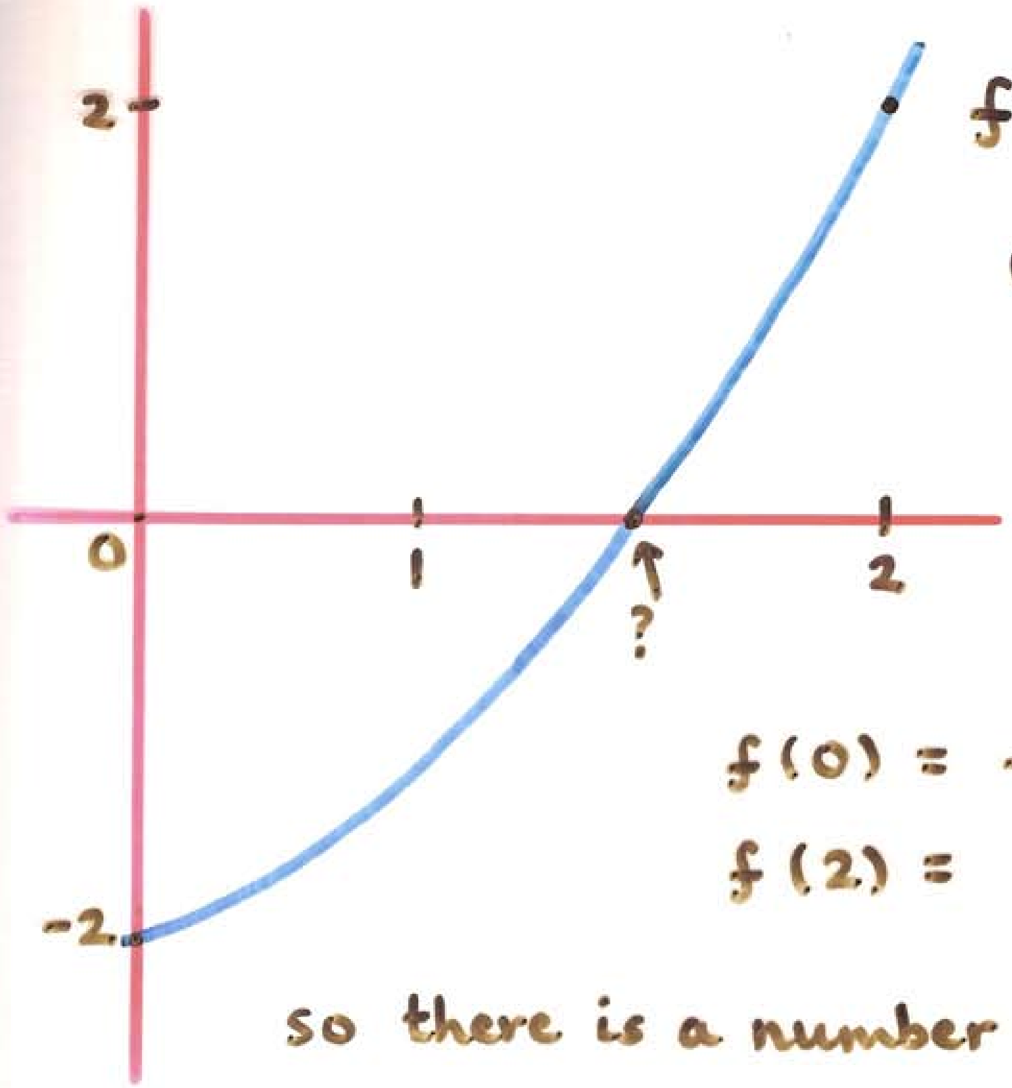
Prag. 1817,
gedruckt bey Gottlieb Haase.

Bolzano's 1817 pamphlet



Intermediate Value Theorem

The 'number' $\sqrt{2}$



$f(x) = x^2 - 2$

on the interval

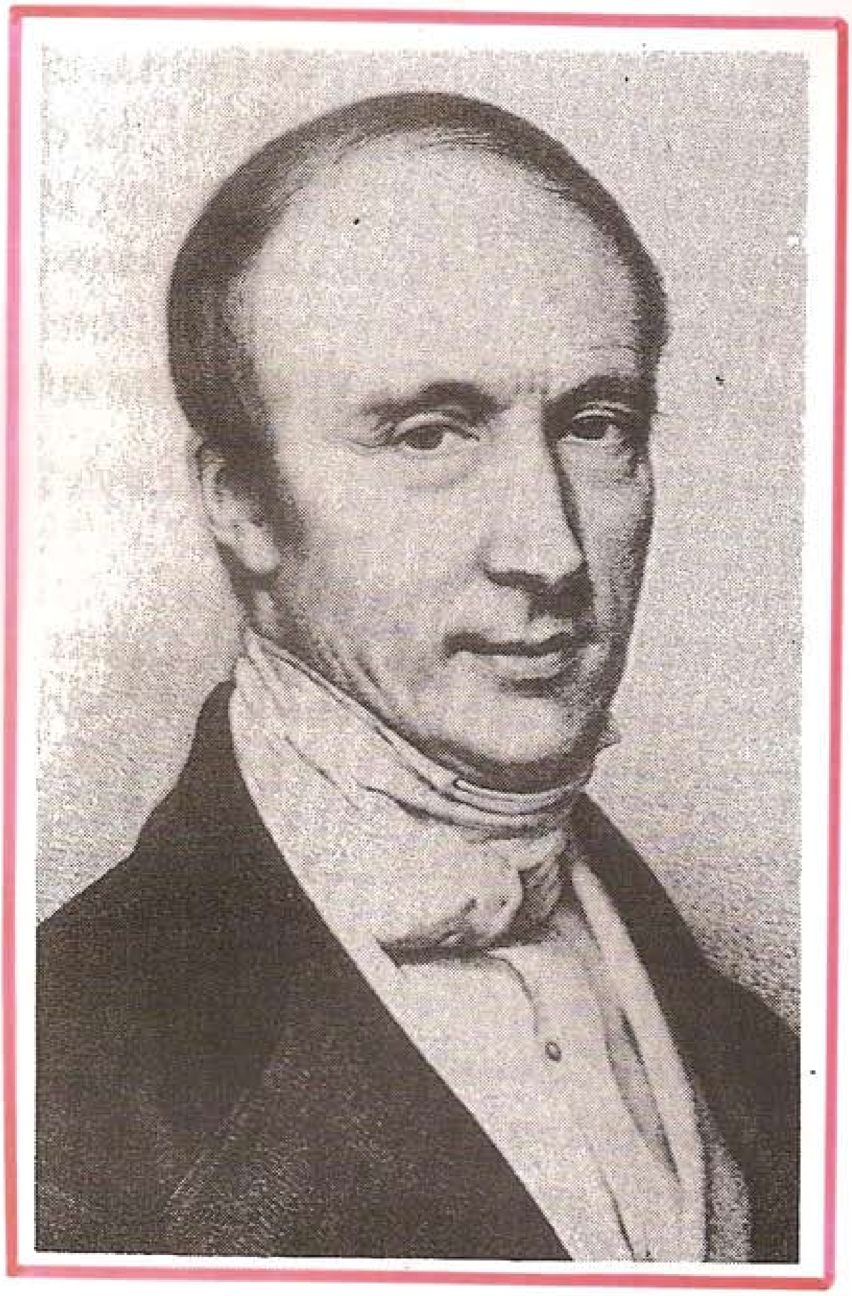
$[0, 2]$

$f(0) = -2$

$f(2) = 2$

so there is a number x such that

$x^2 - 2 = 0.$



Augustin-Louis Cauchy

(1789 - 1857)

Cauchy's 1821 book

COURS D'ANALYSE

DE

L'ÉCOLE ROYALE POLYTECHNIQUE;

PAR M. AUGUSTIN-LOUIS CAUCHY,

Ingénieur des Ponts et Chaussées, Professeur d'Analyse à l'École polytechnique, Membre de l'Académie des sciences, Chevalier de la Légion d'honneur.

I.re PARTIE. *ANALYSE ALGÉBRIQUE.*

DE L'IMPRIMERIE ROYALE.

Chez DEBURE frères, Libraires du Roi et de la Bibliothèque du Roi, rue Serpente, n.° 7.

1821.

Limits :

Continuity

Derivatives

Integrals.

Lorsque les valeurs successivement attribuées à une même variable s'approchent indéfiniment d'une valeur fixe, de manière à finir par en différer aussi peu que l'on voudra, cette dernière est appelée la *limite* de toutes les autres. Ainsi, par exemple, un nombre irrationnel est la limite des diverses fractions qui en fournissent des valeurs de plus en plus approchées. En géométrie, la surface du cercle est la limite vers laquelle convergent les surfaces des polygones inscrits, tandis que le nombre de leurs côtés croît de plus en plus; &c. . . .

Limits and Continuity

Bolzano (1817):

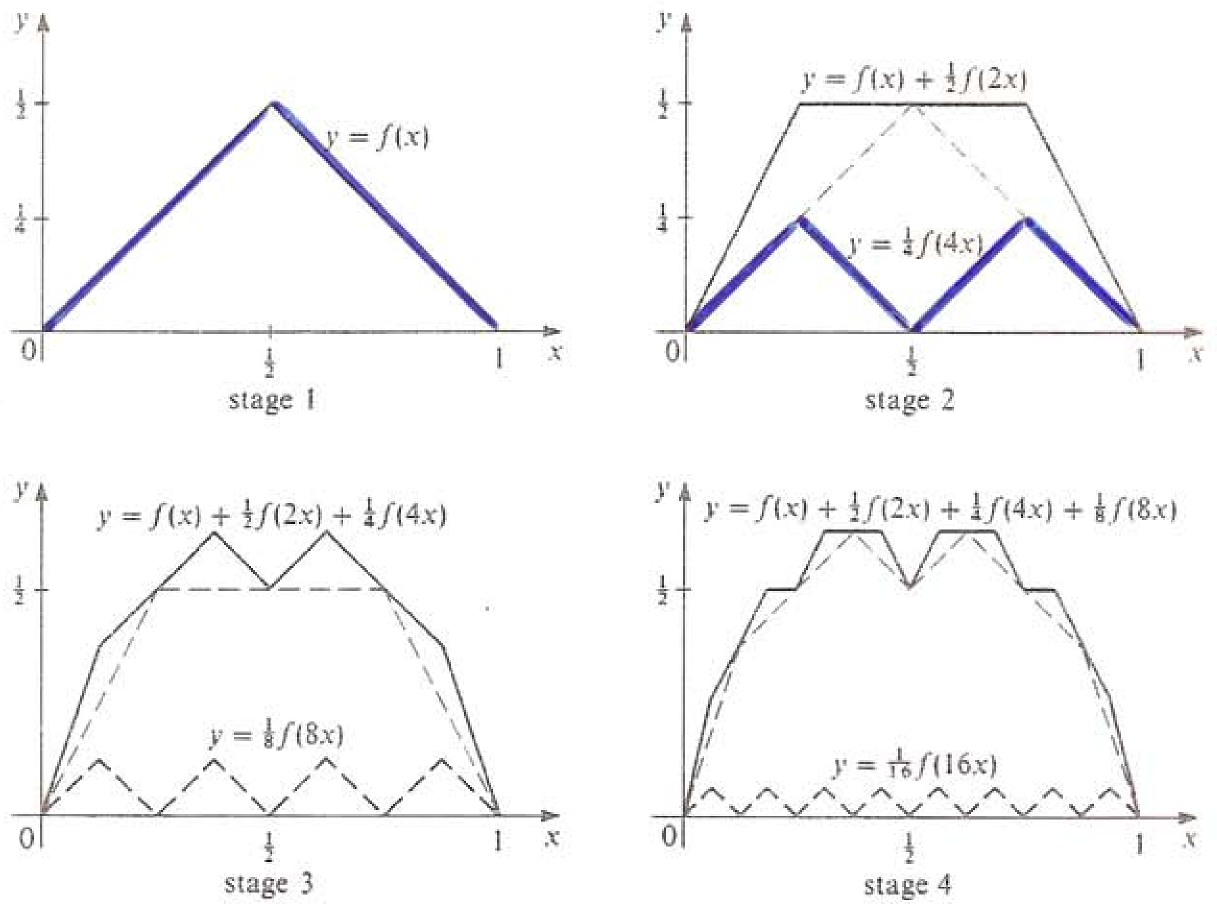
a function fx varies according to the law of continuity ... : if x has some such value, the difference $f(x+\omega) - fx$ can be made smaller than any given quantity provided ω can be taken as small as we please.

Cauchy (1821):

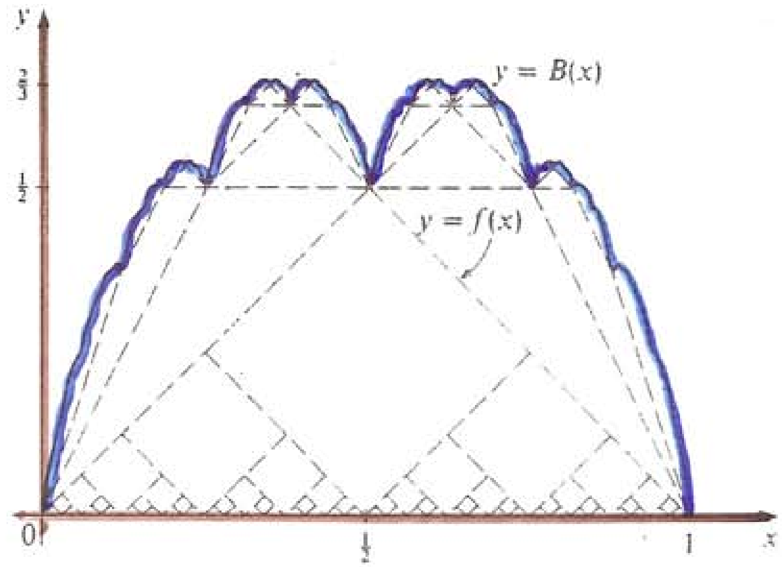
When the values successively attributed to the same variable approach a fixed value indefinitely, in such a way as to end up by differing from it as little as we could wish, this last value is called the limit of all the others ...

Examples: irrationals (limit of fractions)
circles (limit of polygons)

The blancmange function



Eventually we obtain the following graph of B:



Some Numbers

Natural numbers:

$1, 2, 3, 4, 5, \cdots$ $\qquad x + 3 = 7$

Integers:

$\cdots, -2, -1, 0, 1, 2, 3, \cdots$ $\qquad x + 7 = 3$

Rational numbers:

$5/7, 11/3, -1/7, \cdots$ $\qquad 7x = 5$

Real numbers:

$\sqrt{2}, \sqrt[3]{7}, \sqrt{2} + \sqrt{3}, \cdots$ $\qquad x^3 = 7$

$\pi, e, \cdots$

Complex numbers:

$\sqrt{-1}, 3 - 4\sqrt{-1}, \cdots$ $\qquad x^2 = -1$

Rationals and Irrationals

$3/4 = 0 \cdot 75$

$1/3 = 0 \cdot 3333 \cdots$

$2/7 = 0 \cdot \underline{285714}28571428 \cdots$

Add:

$^{115}/_{84} = 1 \cdot 369\underline{047619}0476190 \cdots$

Every rational number has a finite or recurring decimal, and every finite or recurring decimal can be written as a fraction.

$$\left.\begin{aligned} x &= 0 \cdot 242424 \cdots \\ 100x &= 24 \cdot 242424 \cdots \end{aligned}\right\} \quad 99x = 24,$$

$$\text{so } x = \frac{24}{99} = \frac{8}{33}$$

Defining real numbers

Define a real number as an infinite decimal – for example:

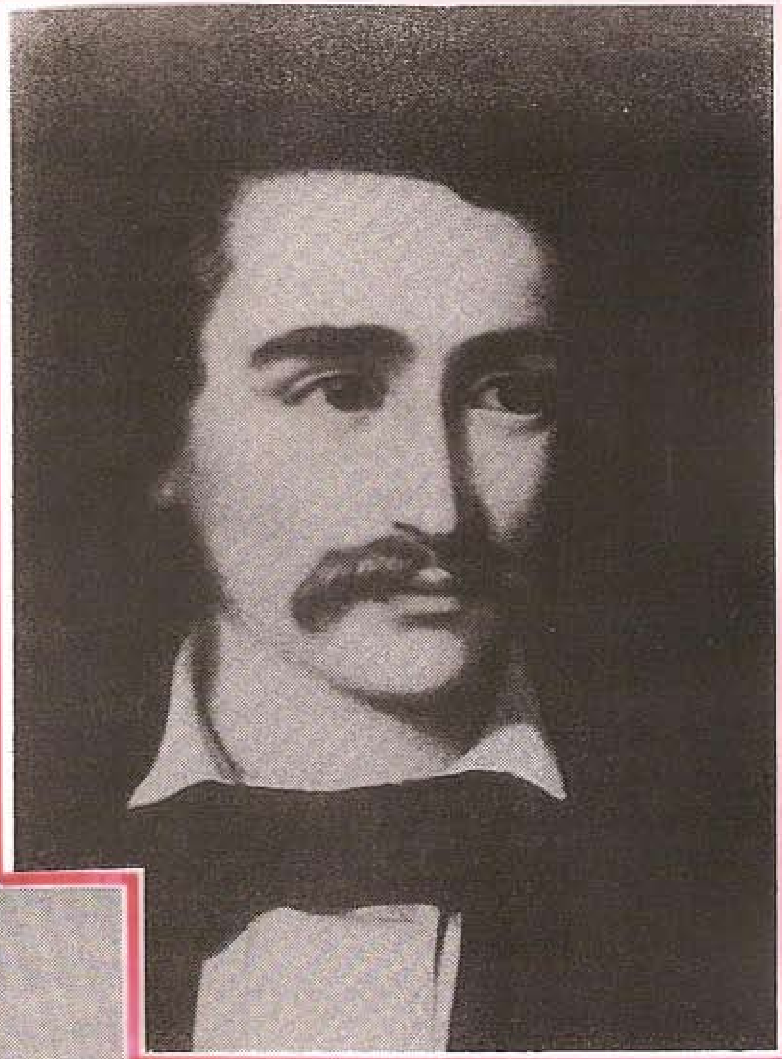
$$\sqrt{2} = 1{\cdot}414213\cdots$$

$$\pi = 3{\cdot}1415926\cdots$$

Problem: how do we prove that

$$\sqrt{2} \times \sqrt{2} = 2 \,?$$

Richard Dedekind
(1831-1916)



Georg Cantor
(1845-1918)

Dedekind cuts

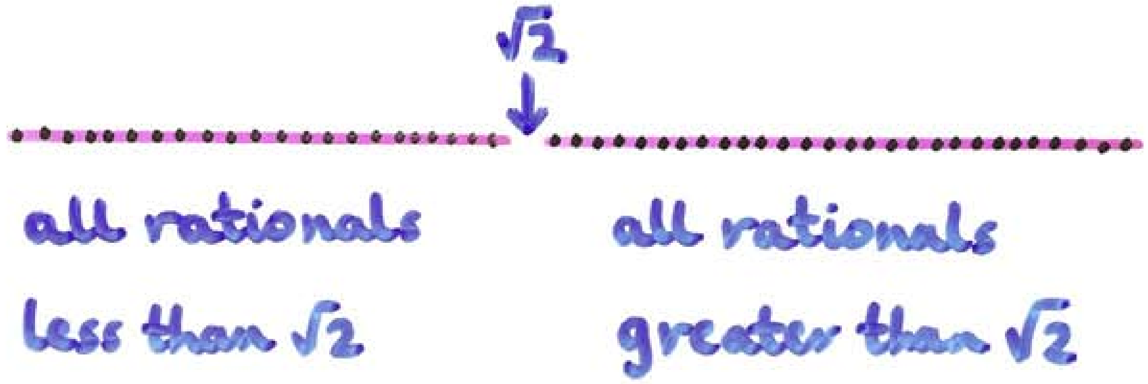
1858, 1872: Dedekind cut: definition of a real number

Basic idea: the real line and the rationals differ – the latter has gaps (e.g. at $\sqrt{2}, \pi, \dots$)

Aim: fill the gaps with numbers

Dedekind: each gap is a number

It is defined by all rationals less than it, and greater than it:



How big is a set?

- $\{100, 101\}$ $\{1, 2, 3, 4\}$
- $\{1, 2, 3, 4, 5, 6, \dots\}$

  $\{1, 4, 9, 16, 25, 36, \dots\}$
- $\{\frac{1}{2}, 1, 1\frac{1}{2}, 2, 2\frac{1}{2}, 3, \dots\}$
- $\{0, 1, -1, 2, -2, 3, \dots\}$

A set is countable if we can list them all – for example, the rationals:

1	2	3	4	5	6 ...
1/2	~~2/2~~	3/2	~~4/2~~	5/2	...
1/3	2/3	~~3/3~~	4/3	5/3	...
1/4	~~2/4~~	3/4	4/4	5/4	...
1/5	2/5	3/5	4/5	5/5	...

The real numbers are not countable

Suppose that the real numbers between 0 and 1 are countable: let's list them all –

$0 \cdot a_1 a_2 a_3 a_4 a_5 \cdots$

$0 \cdot b_1 b_2 b_3 b_4 b_5 \cdots$

$0 \cdot c_1 c_2 c_3 c_4 c_5 \cdots$

$0 \cdot d_1 d_2 d_3 d_4 d_5 \cdots$

$\cdots$

Now choose numbers $x_1, x_2, x_3, \cdots$ so that

$x_1 \neq a_1,\ x_2 \neq b_2,\ x_3 \neq c_3,\ x_4 \neq d_4, \cdots$

Then the number $0 \cdot x_1 x_2 x_3 x_4 \cdots$ is not in the above list – contradiction

So they are NOT countable.