

The story of i

Robin Wilson

George Airy: 'I have not the smallest confidence in any result which is essentially obtained by the use of imaginary symbols.'

Augustus De Morgan: 'We have shown the symbol $\sqrt{-1}$ to be void of meaning, or rather self-contradictory and absurd.'

George Airy

not the smallest confidence in any result which is essentially obtained by the use of imaginary symbols,

Augustus De Morgan

We have shown the symbol $\sqrt{-1}$ to be void of meaning, or rather self-contradictory and absurd.

Euler

All such expressions as $\sqrt{-1}$, $\sqrt{-2}$, etc., are consequently impossible or imaginary numbers, since they represent roots of negative quantities; and of such numbers we may truly assert that they are neither nothing, nor greater than nothing, nor less than nothing, which necessarily constitutes them imaginary or impossible.

Some Numbers

Natural numbers :

$$1, 2, 3, 4, 5, \dots$$

Integers :

$$\dots, -2, -1, 0, 1, 2, 3, \dots$$

Rational numbers :

$$\frac{5}{7}, \frac{11}{3}, -\frac{1}{7}, \dots$$

Real numbers :

$$\sqrt{2}, \sqrt[3]{7}, \sqrt{2} + \sqrt{3}, \dots$$

$$\pi, e, \dots$$

Complex numbers :

$$\sqrt{-1}, 3 - 4\sqrt{-1}, \dots$$

Complex Numbers

$a + b\sqrt{-1}$, or $a + bi$, $i^2 = -1$.

Addition

$$(2 + 3\sqrt{-1}) + (4 + 5\sqrt{-1})$$

$$= (2 + 4) + (3 + 5)\sqrt{-1} = 6 + 8\sqrt{-1};$$

$$\text{or: } (2 + 3i) + (4 + 5i) = 6 + 8i.$$

Multiplication

$$(2 + 3\sqrt{-1}) \times (4 + 5\sqrt{-1})$$

$$= (2 \times 4) + (3 \times 4)\sqrt{-1} + (2 \times 5)\sqrt{-1} + (3 \times 5)(-1)$$

$$= (8 - 15) + (12 + 10)\sqrt{-1} = -7 + 22\sqrt{-1};$$

$$\text{or: } (2 + 3i) \times (4 + 5i) = -7 + 22i.$$

Complex Numbers

$a + b\sqrt{-1}$, or $a + bi$, $i^2 = -1$.

Addition

$$x + 3 = 7$$

$$(2 + 3\sqrt{-1}) + (4 + 5\sqrt{-1})$$

$$= (2 + 4) + (3 + 5)\sqrt{-1} = 6 + 8\sqrt{-1}; \quad x + 7 = 3$$

$$\text{or: } (2 + 3i) + (4 + 5i) = 6 + 8i.$$

$$7x = 5$$

Multiplication

$$(2 + 3\sqrt{-1}) \times (4 + 5\sqrt{-1})$$

$$= (2 \times 4) + (3 \times 4)\sqrt{-1} + (2 \times 5)\sqrt{-1} + (3 \times 5)(-1)$$

$$x^3 = 7$$

$$= (8 - 15) + (12 + 10)\sqrt{-1} = -7 + 22\sqrt{-1};$$

$$\text{or: } (2 + 3i) \times (4 + 5i) = -7 + 22i.$$

$$x^2 = -1$$

Three Quadratic Equations

- $x^2 - 4x + 3 = 0$

$$x^2 - 4x + 3 = (x-3)(x-1) = 0,$$
$$\text{so } x = 3 \text{ or } 1.$$

- $x^2 - 4x + 4 = 0$

$$x^2 - 4x + 4 = (x-2)^2 = 0,$$
$$\text{so } x = 2 \text{ or } 2.$$

- $x^2 - 4x + 5 = 0$

$$x^2 - 4x + 5 = (x-2-\sqrt{-1})(x-2+\sqrt{-1}) = 0,$$
$$\text{so } x = 2 + \sqrt{-1} \text{ or } 2 - \sqrt{-1}.$$

Check: $(2+\sqrt{-1})^2 - 4(2+\sqrt{-1}) + 5$

$$= (4+4\sqrt{-1}-1) - (8+4\sqrt{-1}) + 5 = 0.$$

Solving Quadratic Equations

$ax^2 + bx + c = 0$ has solutions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

So $x^2 - 4x + c = 0$ has solutions

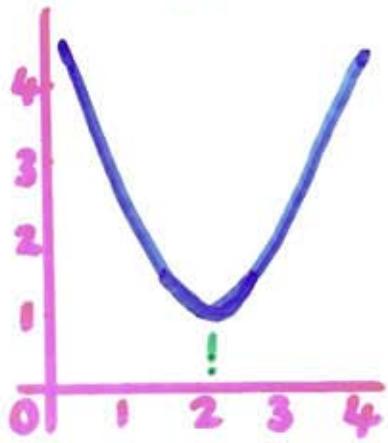
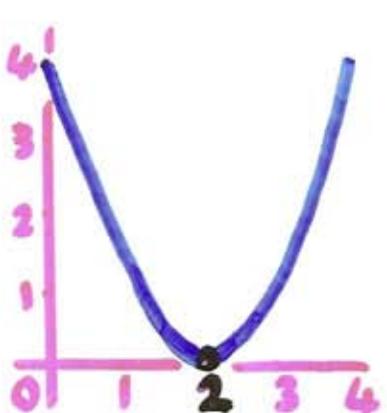
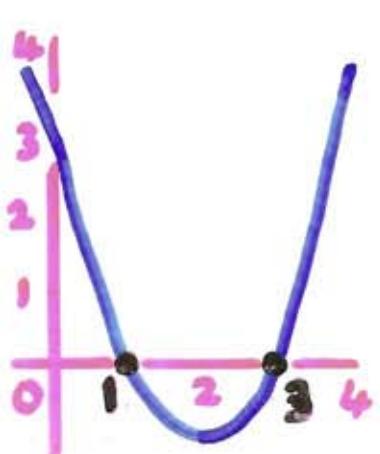
$$x = \frac{1}{2}(4 \pm \sqrt{16 - 4c}) = 2 \pm \sqrt{4 - c}.$$

$$c = 3 : x = 2 \pm 1 = 3 \text{ or } 1$$

$$c = 4 : x = 2 \pm 0 = 2 \text{ or } 2$$

$$c = 5 : x = 2 \pm \sqrt{-1}$$

$$= 2+i \text{ or } 2-i, \text{ where } i = \sqrt{-1}$$



A Sextic Polynomial

$$x^6 - 12x^5 + 60x^4 - 160x^3 + 239x^2 - 188x + 60 = 0$$

$$= (x^2 - 4x + 3)(x^2 - 4x + 4)(x^2 - 4x + 5)$$

↓

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$$= (x-1)(x-3) \times (x-2)^2 \times (x^2 - 4x + 5)$$

↓

↓

$$= (x-1)(x-3)(x-2)^2 \times (x-2-i)(x-2+i),$$

so $x = 1, 3, 2$ (twice) and $2 \pm i$.

Four scenarios

- We can solve *all* equations using only real and complex numbers;
- We may need to bring in some new ‘hyper-complex’ numbers to solve certain ~~some~~ equations;
- Some equations may have solutions that are not numbers and don’t behave like them;
- Some equations may not have solutions of any kind.

What is the square root of i ?

If $x^2 = i$ and $x = a + bi$,

then $(a + bi)^2 = i$,

$$\text{so } (a^2 - b^2) + 2abi = i.$$

$$\text{So } a^2 - b^2 = 0 \text{ and } 2ab = 1.$$

This has solutions $a = b = \pm \sqrt{\frac{1}{2}}$

$$\text{so } x = \pm \frac{1}{\sqrt{2}}(1+i).$$

The Fundamental Theorem of Algebra

- Every polynomial $p(x)$ can be factorized into linear and quadratic polynomials with real coefficients.
- Every polynomial $p(x)$ can be factorized completely into linear factors with complex coefficients.
- Every polynomial equation $p(x) = 0$ of degree n has at least one complex solution.
- Every polynomial equation $p(x) = 0$ of degree n has exactly n complex solutions (as long as we count them appropriately).

Solving a 'Quadratic Equation'

I have subtracted the side of my square from the area: 14,30.

You write down 1, the coefficient.

You break off half of 1. 0;30 and 0;30 you multiply. You add 0;15 to 14,30.

Result 14,30; 15. This is the square of 29;30.

You add 0;30, which you multiplied, to 29;30.

Result: 30, the side of the square.

$$\underline{x^2 - x = 870}:$$

$$1 \rightarrow \frac{1}{2} \rightarrow \left(\frac{1}{2}\right)^2 = \frac{1}{4} \rightarrow 870\frac{1}{4} \rightarrow 29\frac{1}{2} \rightarrow 30.$$

$$\underline{x^2 - bx = c}:$$

$$b \rightarrow \frac{b}{2} \rightarrow \left(\frac{b}{2}\right)^2 \rightarrow \left(\frac{b}{2}\right)^2 + c \rightarrow \sqrt{\left(\frac{b}{2}\right)^2 + c}$$

$$\rightarrow \frac{b}{2} + \sqrt{\left(\frac{b}{2}\right)^2 + c}.$$



al-Khwarizmi (c. 825 AD)

Solving Quadratic Equations

Six types: (a, b, c all positive)

$$ax^2 = bx, ax^2 = b, ax = b,$$

$$ax^2 + bx = c, ax^2 + c = bx, ax^2 = bx + c$$

'Roots and squares are equal to numbers'

$$x^2 + 10x = 39$$

$$(x+5)^2 = 39+25=64$$

$$\text{so } x+5=8, \underline{x=3}$$

$$x^2 + 10x = 39$$

$$(x+2\frac{1}{2}+2\frac{1}{2})^2$$

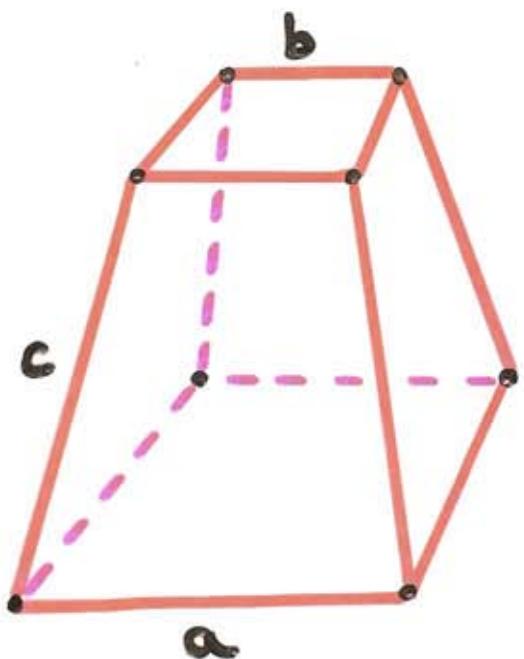
$$= 39 + (4 \times 6\frac{1}{4}) = 64$$

$$\text{so } x+5=8, \underline{x=3}$$

5	x	x
5x	x^2	x
25	5x	5

$6\frac{1}{4}$	$2\frac{1}{2}x$	$6\frac{1}{4}$	$2\frac{1}{2}$
$2\frac{1}{2}x$	x^2	$2\frac{1}{2}x$	x
$6\frac{1}{4}$	$2\frac{1}{2}x$	$6\frac{1}{4}$	$2\frac{1}{2}$

Heron becomes frustrated



frustrum
of a
pyramid

$$a = 28, b = 4, c = 15$$

$$\begin{aligned}\text{height} &= \sqrt{c^2 - 2\left(\frac{a-b}{2}\right)^2} \\ &= \sqrt{15^2 - 2\left(\frac{28-4}{2}\right)^2} \\ &= \sqrt{225 - (2 \times 144)} \\ &= \sqrt{225 - 288} = \sqrt{-63}\end{aligned}$$

- but it appears as $\sqrt{63}$!

HIERONYMI CAR
DANI, PRÆSTANTISSIMI MATHE-
MATICI, PHILOSOPHI, AC MEDICI,
ARTIS MAGNAE.
SIVE DE REGVLIS ALGEBRAICIS,
Lib. unus. Qui & totius operis de Arithmeticā, quod
OPVS P E R F E C T V M
inscripsit, est in ordine Decimus.



HAbes in hoc libro, studiose Lector, Regulas Algebraicas (Itali, de la Cos
sa uocant) nouis adiuventionibus, ac demonstrationibus ab Auctore ita
locupletatas, ut pro pauculis ante*a* uulgo tritis, iam septuaginta euaserint. Ne-
c quod solum, ubi unus numerus alteri, aut duo uni, uerum etiam, ubi duo duobus,
aut tres uni *æquales* fuerint, nodum explicant. Hunc aut librum ideo scor-
sim edere placuit, ut hoc abstrusissimo, & plane inexhausto totius Arithmeti-
ce thesauro in lucem eruto, & quasi in theatro quodam omnibus ad spectan-
dum exposito. Lectores incitarecetur, ut reliquos Operis Perfecti libros, qui per
Tomea edentur, tanto zuidius amplectantur, ac minore fastidio perdiscant.

Solving a cubic equation

$$x^3 + 6x = 20$$

Find u and v so that

$$u - v = 20 \text{ and } uv = (6/3)^3 = 8.$$

Since $v = u - 20$, we have

$$uv = u(u-20) = u^2 - 20u = 8.$$

Solving this quadratic equation:

$$u = \sqrt{108} + 10.$$

$$\text{So } v = u - 20 = \sqrt{108} - 10.$$

So

$$x = \sqrt[3]{u} - \sqrt[3]{v}$$

$$= \frac{\sqrt[3]{(\sqrt{108} + 10)} - \sqrt[3]{(\sqrt{108} - 10)}}{3}$$

$$= 2.$$

Cardano's problem

Divide 10 into two parts
whose product is 40.

If the parts are x and $10-x$,
then $x(10-x) = 40$.

Cardano: $x = 5 + \sqrt{-15}$ or $5 - \sqrt{-15}$

'Nevertheless we will operate, putting aside the mental tortures involved.'

$$\begin{aligned}x(10-x) &= (5 + \sqrt{-15})(5 - \sqrt{-15}) \\&= 25 - (-15) = 40.\end{aligned}$$

L'ALGEBRA OPERA

DI RAFael BOMBELLI da Bologna
Divisa in tre Libri.

Con la quale ciascuno da se potrà venire in perfetta cognizione della teorica dell'Aritmetica.

Con vna Tauola copiosa delle materie, che in essa si contengono.

Posta hora in luce à beneficio degli Studiosi di detta professione.



IN BOLOGNA,
Per Giovanni Rossi. MDLXXIX.
Con licenza de Superiori

Bombelli and complex numbers

$$x^3 = 15x + 4$$

Solutions: $4, -2+\sqrt{3}, -2-\sqrt{3}$

Cardano's method yields

$$x = \sqrt[3]{(2+\sqrt{-121})} - \sqrt[3]{(-2+\sqrt{-121})}$$

first appearance of complex numbers

Bombelli calculated that:

$$(2+\sqrt{-1})^3 = 2+\sqrt{-121},$$

$$(2-\sqrt{-1})^3 = 2-\sqrt{-121}.$$

$$\text{So } x = (2+\sqrt{-1}) - (-2+\sqrt{-1}) = 4.$$

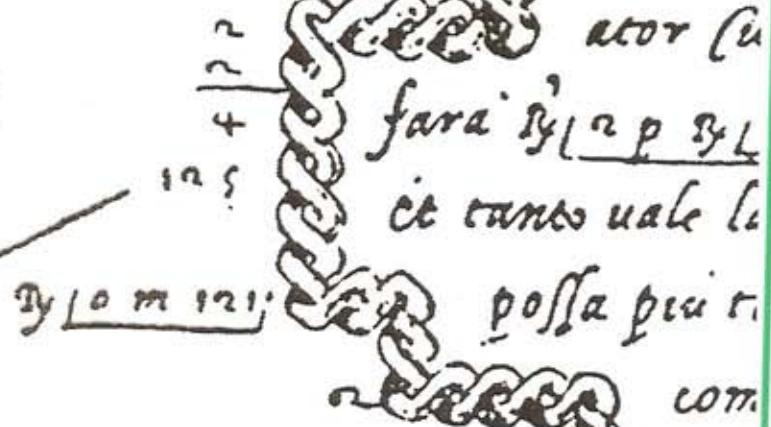
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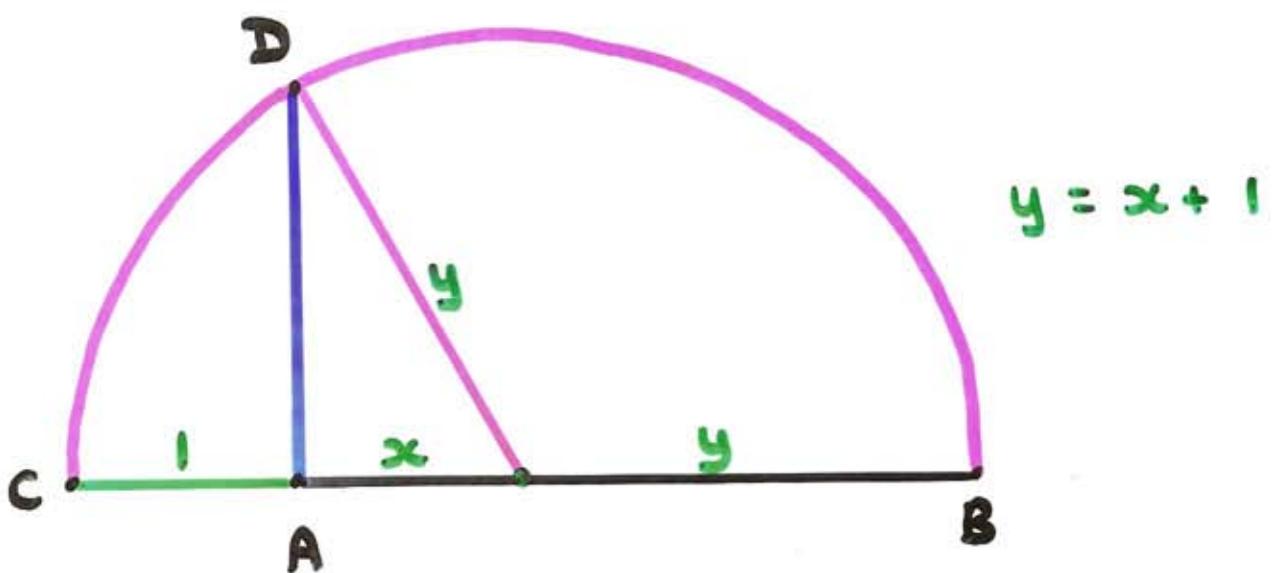
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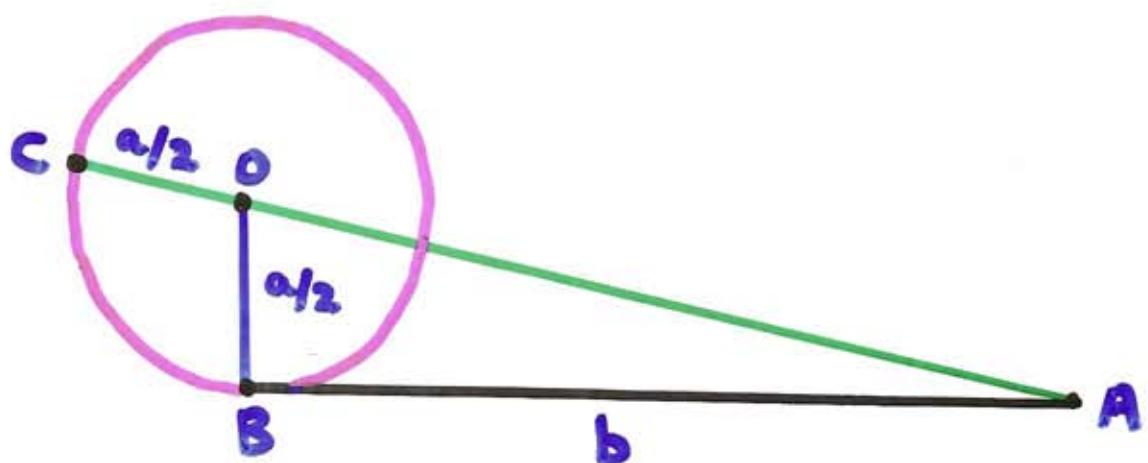
n m. 2 1 0 m. i / Le appurati uscime fanno q.



Descartes' constructions

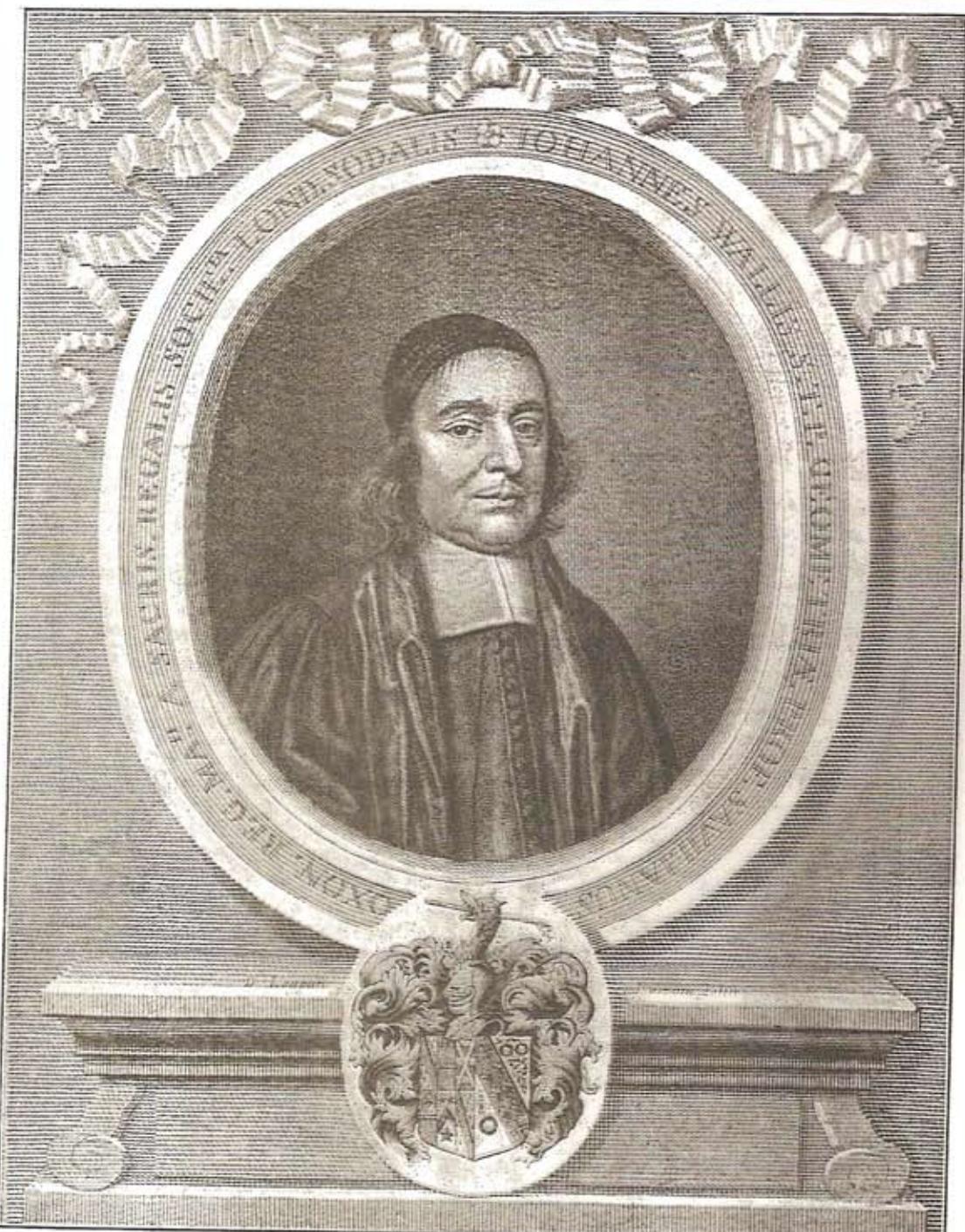


The square root of AB is AD.



AC is the positive root of

$$x^2 = ax + b^2.$$





John Wallis

A Treatise on Algebra, 1685

C H A P. LXVII.

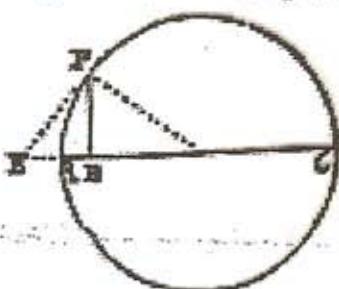
The same Exemplified in Geometry.

W HAT hath been already said of $\sqrt{+bc}$ in Algebra, (as a Mean Proportional between a Positive and a Negative Quantity:) may be thus Exemplified in Geometry.

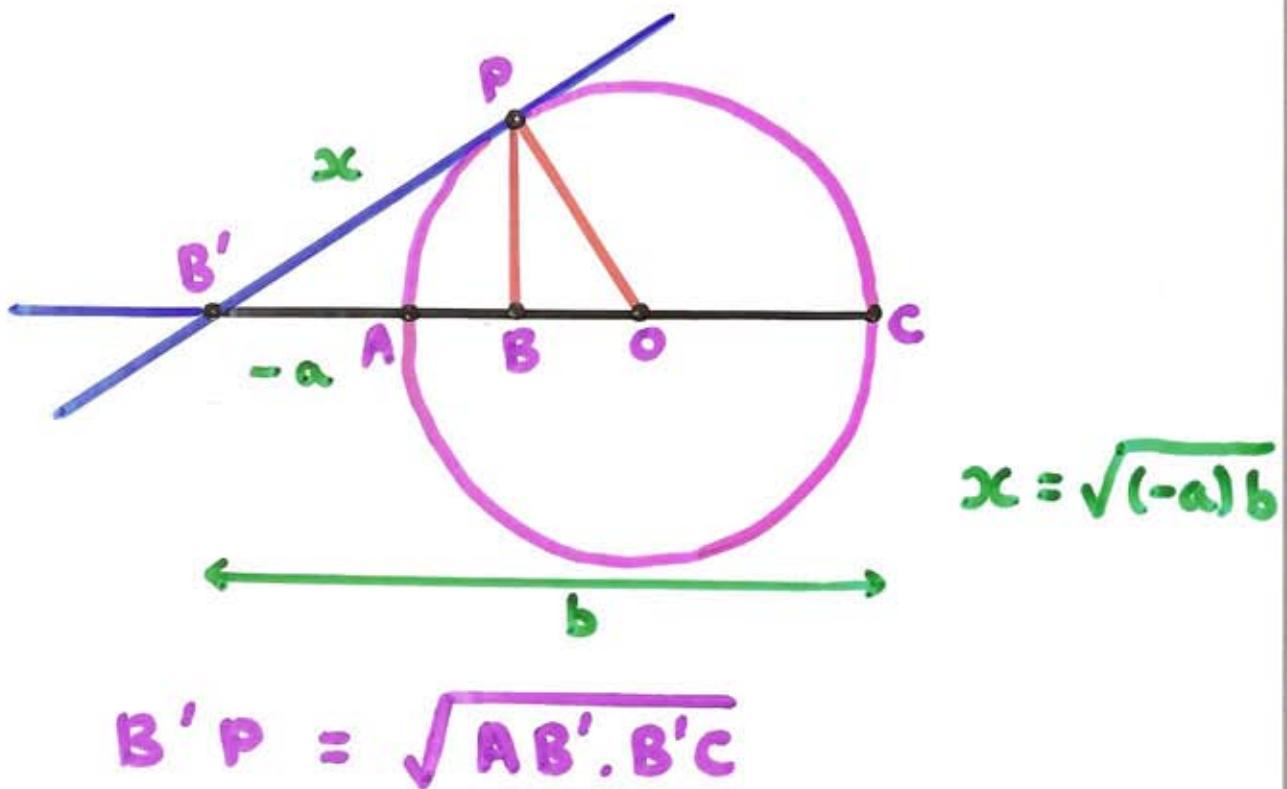
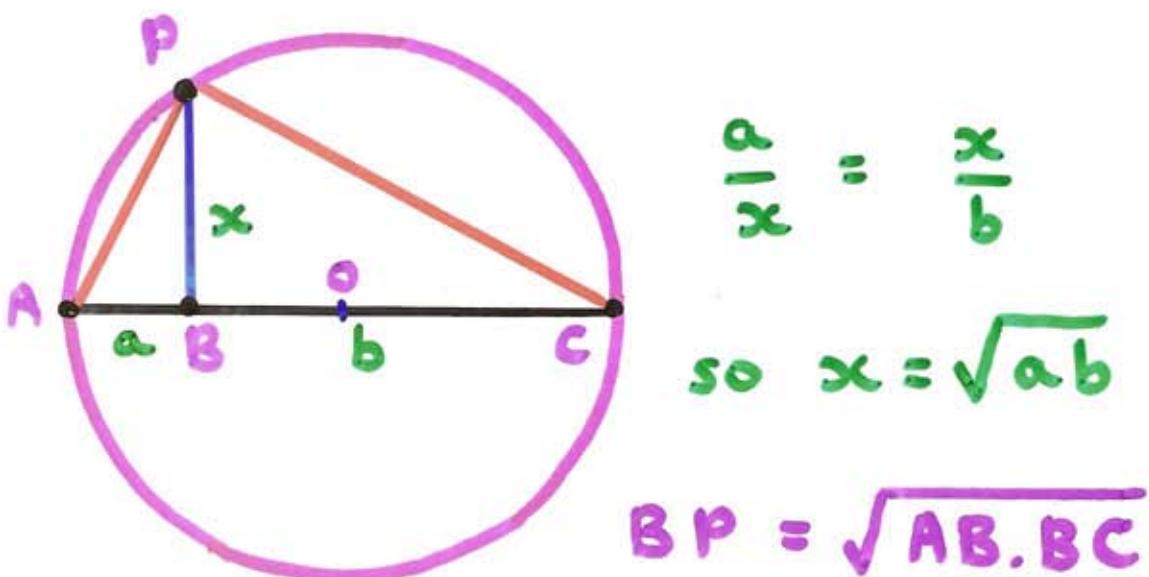
If (for instance,) Forward from A, I take AB = $+b$; and Forward from thence, BC = $+c$; (making AC = $+AB + BC = +b + c$, the Diameter of a Circle:) Then is the Sine, or Mean Proportional BP = $\sqrt{+bc}$.

But if Backward from A, I take AB = $-b$; and then Forward from that B, BC = $+c$; (making AC = $-AB + BC = -b + c$, the Diameter of the Circle:) Then is the Tangent or Mean Proportional BP = $\sqrt{-bc}$.

So that where $\sqrt{+bc}$ signifies a Sine; $\sqrt{-bc}$ shall signify a Tangent, to the same Arch (of the same Circle,) AP, from the same Point P, to the same Diameter AC.



Wallis's constructions



Om
Directionens analytiske Betegning.

et Forsøg,
anvendt fornemmelig
til
plane og sphæriske Polygoners Oplosning.

af
Caspar Wessel, Landmaaler.

Nærværende Forsøg angaaer det Spørgsmål, hvordan Directionens analytisk betegnes, eller hvordan dette bindeburde udtrykkes, naat af en eneste Ligning mellem een ukendt og andres givne Linier skalde kunne findes et Udtysk, der forestillede både den ukendtes værdie og dens Direction.

Før nogensledes at kunne besvare dette Spørgsmål, lægger jeg til Grundvold to Sætninger, der synes mig unegelige. Den første er: at den Directionens Forandring, der ved algebraiske Operationer kan foretages, også kan ved deres Tegn at forestilles. Den anden: at Direction er ingen Stiens staad for Algebra, uden for saavært den ved algebraiske Operationer kan forandres. Men da den ved disse si kan forandres (i det mindste efter den sædvanlige Forklaring), uden til den modsatte, eller fra positiv til privativ, og omvendt; saa skalde disse to Directioner alene kunne betegnes paa den bekendte Maade, og i Hensigt til de øvrige Problemet være uoploseligt. Dette er vel,

horizont cylinderne ved et betragt af y fra Fig 3, saa at det fore-
faller ab, der forefaller ved etc, saa vi i følge §2 ved, at y etc
med denne form af den første er b^1 , den anden $= b^2$, den tredje $= b^3$, og den
fjerde $= b^{3+1}$; da den sidste bortrives indexen ved $B^1 \cdot B^{3+1} = b^{3+1}$

2) Ifølge § 3 nr 6 nr 26, 45, af de viselbunn- og graderne an-
tal i den første og $\lambda^1/m B'$, i den anden $\lambda^2/m B''$, og i almindeligst
parabolens bue i bredden B'' og af det antal grader λ^n blomster i pla-
net gennemført ved en viselbunn af sammeantal grader som $\lambda^1/m B'$

7) D. medianser, der går igennem et parallelt bane $\mathcal{B}^{(j)}$ og
 - et par fomfiller i planet med rette linier, der smider i banens
 centrum, og en par større som før tangenten i parallelt bane-
 ns bane ($53^{\circ} 5,6$), således at $a_b = T = 6h$, $s_m = T'' = \frac{1}{2}m$ etc
 og da tangenterne i banen B'' først baneindex, og i følge § 3. nr
 6 er radii til projektorien af parallelen i samme bane, men
 projections bane af parallelen i banen B'' midfalds gradene
 $\pi_{\perp m B''}$ ($54^{\circ} 0,2$), saa bliver den mindre som i planet ind-
 skrænket af de yar tangentter, fodes index er n , saa først
 $\pi_{\perp m B''}$, og dernæst faar man samme index som tangenterne.
 Den mindre som indskrænket af $2h/0,2$ af yar tangentter er altsaa
 $\pi_{\perp m B''}$

A) centrum (m) af den forgaante bue (4 ω), og centrum (n) af den
følgende bue (nf) vil ligge i samme meridian (ϑ_m), og centrum (n)
af buen (nf) i den følgende bue B' vil ligge bort (nf), (dvs.) over-
-over mod det andet (na) i samme bue. I almindelighed
omrigaars tangenten (ϑ_m) i en mindre bue i d. Fig. 2 tan-
genten (en) i en større bue og over mod den større bue
se omrigaar den mindre den; thi de vinkelne m og en
som foruden producerede axis nem gior mod tangenterne
 ϑ_m , en, og som med/altres vertikale ^{linjer} fra buevinkles ym
størrelse ϑ og α , og alle hider (prid), følgelig nu en om obter,
obter $m > n$, da subtilaterne fra bogen $\vartheta_m = \vartheta_n$, fordi
 $ym > en$, men $ym = dm - dv - vy$, alltsaa $dm - dv - vy > en$, dvs.
 $vy > dw$, følgelig $dm - dw > en$, og $dm - en > dw$

5) foregår nuværlig at alle tangenterne T_1, T_2, \dots, T_n i en rektion og θ -vinklen i overensstemmelse med den fra førne yndstikke b som yndstikke er meridian eller tangenten $T = b + \alpha$ er det højst
muligt, at sammen af de n foregående vinkler, fjældes ind i en θ -vinkel
af begge sammenhæng gennem tangenten, og da det som den vinkel findes tangent af højst ynd
gröbne med yndstikke er meridian eller end den foregående tangent T' , og højstens man θ' den
vinkel som den vedligeholder tangents af det rette ynd grøbne med yndstikke er meridian, da er
 $\theta' = \theta_1^* B + \theta_2^* B + \theta_3^* B + \dots + \theta_n^* B$, hvoraf den yndstikke er meridian højstens
tangenter fra b til a , da at $b + a = T'$, men vinklen θ' holder positionen fra T' mod foruden, og
negativer mod foruden, daa exprimerer $T'' = (\cos \theta' + f \cdot 1 \cdot \sin \theta')$ direktionen og θ' -vinklen
af den foregående tangent i det ynd, som har en b til index, og $T'' = (\cos \theta' + f \cdot 1 \cdot \sin \theta')$ er den
-vinkel og direktionen af den foregående tangent i det ynd, som har en $n+1$ til index; højstens θ' -
tangenter, som i yndet som deres direktion

Fig. 1

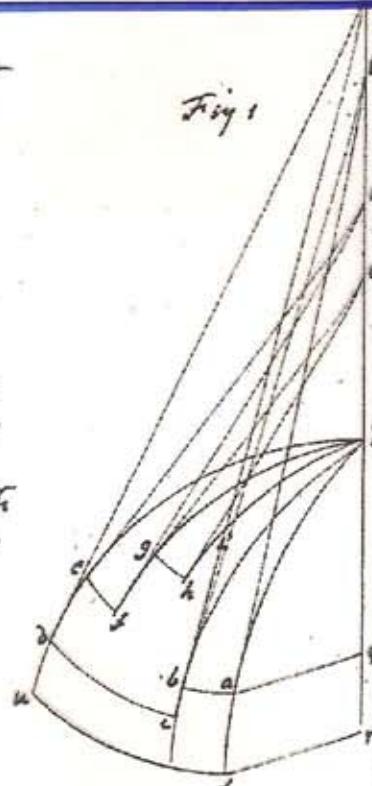
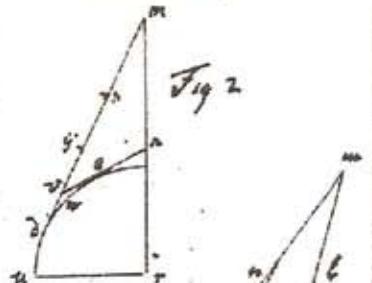
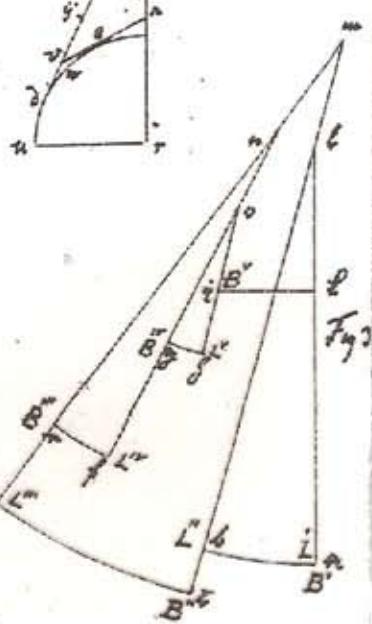


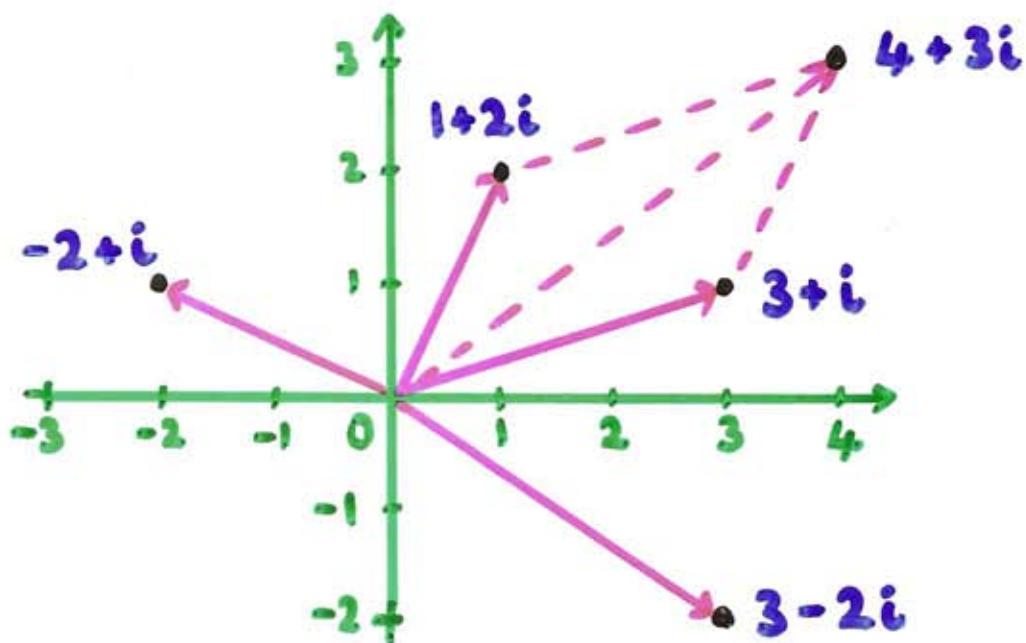
Fig. 2



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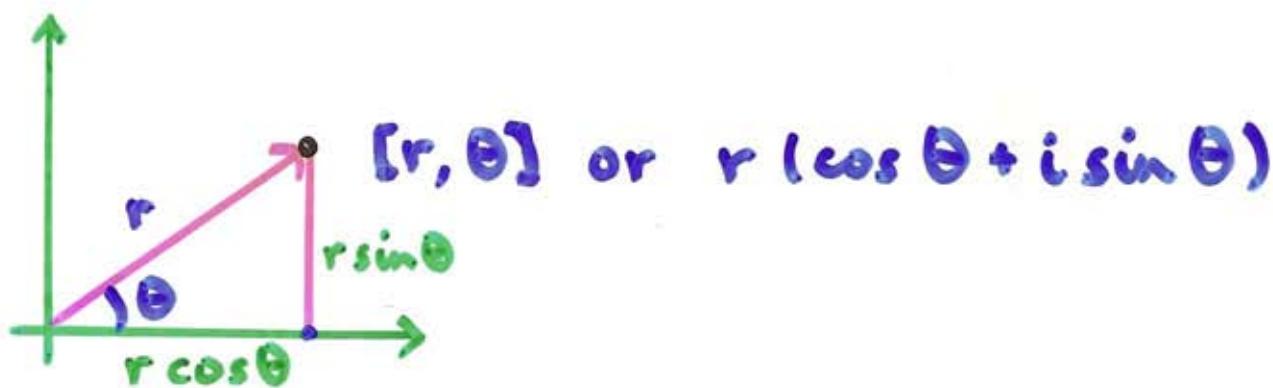


The complex plane



represent the point $a+bi$ by (a, b)

Addition: $(3+i) + (1+2i) = 4+3i$



Multiplication: $[r, \theta] \times [s, \varphi] = [rs, \theta + \varphi]$

So, to multiply by i , rotate ↗ through 90°

Do so twice: $i \times i = -1$.

Wessel and De Moivre's Theorem

$$[r, \theta] \times [s, \varphi] = [rs, \theta + \varphi]$$

Take $r = s = 1$, $\varphi = \theta$:

$$(\cos \theta + i \sin \theta)^2 = (\cos 2\theta + i \sin 2\theta)$$

$$(\cos \theta + i \sin \theta)^3 = (\cos 3\theta + i \sin 3\theta)$$

...

$$(\cos \theta + i \sin \theta)^{1/3} = (\cos \theta/3 + i \sin \theta/3)$$

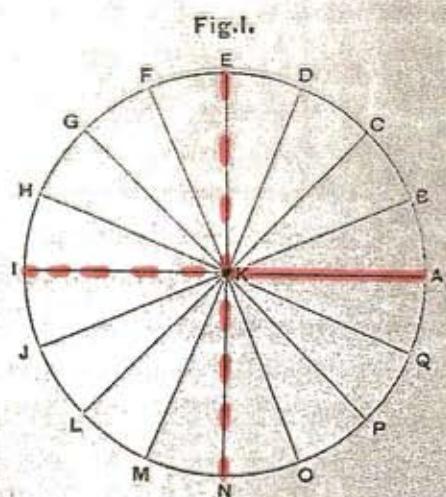
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Argand on Complex Numbers, 1806

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may apply the idea of direction, so that having chosen two opposite directions, one for positive and one for negative values, there shall exist a third—such that the positive direction shall stand in the same relation to it that the latter does to the negative.

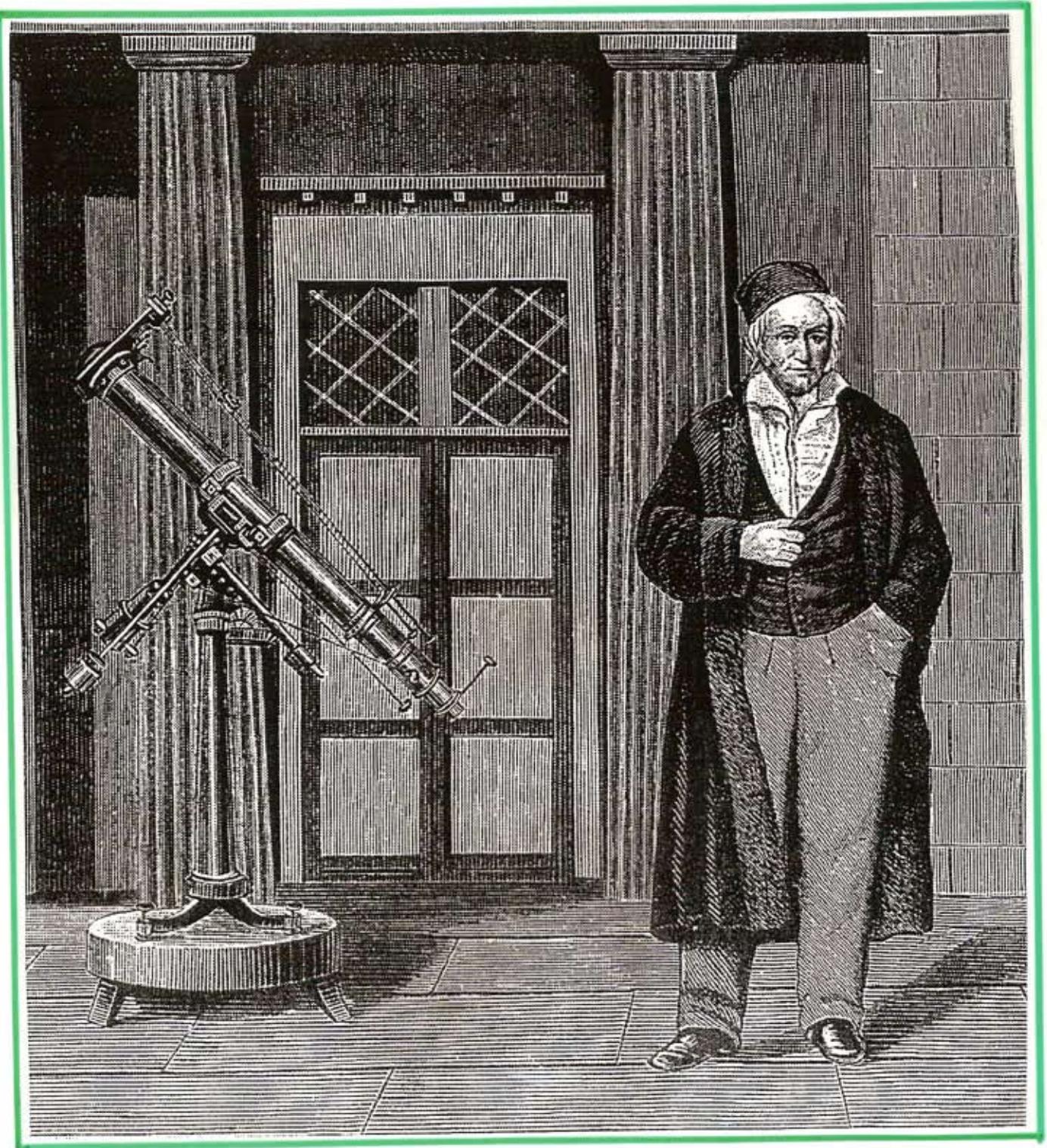
4. If now we assume a fixed point K (Fig. 1) and the line KA be taken as



positive unity, and we also regard its direction, from K to A, and write \overline{KA} to distinguish it from the line KA as simply an absolute distance, negative

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unity will be \overline{KI} , the vinculum having the same meaning as before, and the condition to be satisfied will be met by \overline{KE} , perpendicular to the above and with a direction from K to E, expressed in like manner by \overline{KE} . For the direction of \overline{KA} is to that of \overline{KE} as is the latter to that of \overline{KI} . Moreover we see that this same condition is equally met by \overline{KN} , as well as by \overline{KE} , these two last quantities being related to each other as $+1$ and -1 . They are, therefore, what is ordinarily expressed by $+\sqrt{-1}$, and $-\sqrt{-1}$. In an analogous manner we may insert other mean proportionals between the quantities just considered. Thus to construct the mean proportional between \overline{KA} and \overline{KE} , the line \overline{CKL} must be drawn so as to bisect the angle AKE , and the required mean will be \overline{KC} or \overline{KL} . So the line \overline{GKP} gives in like manner the means between \overline{KE} and \overline{KI} , or between \overline{KA} and \overline{KN} . We shall obtain in the same way \overline{KB} , \overline{KD} , \overline{KF} , \overline{KH} , \overline{KJ} , \overline{KM} , \overline{KO} , \overline{KQ} , as means between \overline{KA} and \overline{KC} , \overline{KC} and \overline{KE} . . .



biquadraticorum sset, quam ab omni parte perfectam reddere in continuatione subsequentie suscipiemus *).

31.

Ante omnia quasdam denominations praemittimus, per quarum introductionem breuitati et perspicuitati consuletur.

Campus numerorum complexorum $a + bi$ continet

I. numeros reales, vbi $b=0$, et, inter hos, pro indole ipsius a

- 1) cifram
- 2) numeros positiuos
- 3) numeros negatiuos

II. numeros imaginarios, vbi b cifrae inaequalis. Hic iterum distinguuntur

- 1) numeri imaginarii absque parte reali, i.e. vbi $a=0$
- 2) numeri imaginarii cum parte reali, vbi neque b neque $a=0$.

Priores si placet numeri imaginarii puri, posteriores numeri imaginarii mixti vocari possunt.

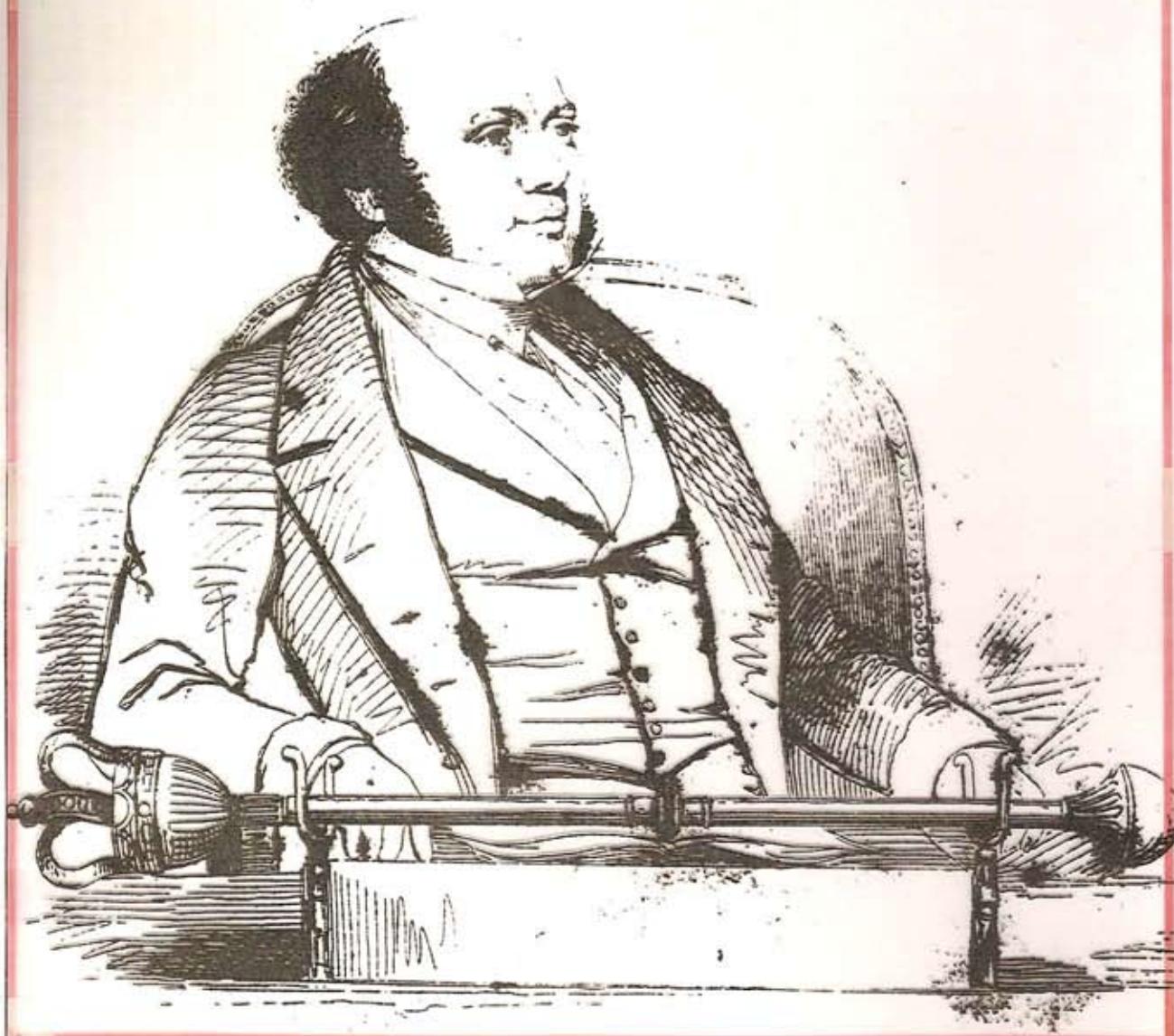
Vnitatibus in hac doctrina vtimur quaternis, $+1, -1, +i, -i$, quae simpliciter positua, negatiua, positua imaginaria, negatiua imaginaria audient.

Producta terna cuiuslibet numeri complexi per $-1, +i, -i$ illius *socios* vel *numeros illi associatos* appellabimus. Excepta itaque cifra (quae sibi ipsa associata est), semper quaterni numeri *inaequales* associati sunt.

Contra numero complexo *coniunctum* vocamus eum, qui per permutationem ipsius i cum $-i$ inde oritur. Inter numeros imaginarios itaque bini *inaequales* semper coniuncti sunt, dum numeri reales sibi ipsi sunt coniuncti, siquidem denominationem ad hos extendere placet.

*) Obiter saltem hic adhuc monere conuenit, campum ita definitum imprimis theoriae residuorum biquadraticorum accommodatum esse. Theoria residuorum cubicorum simili modo superstruenda est considerationi numerorum formae $a+bh$, vbi h est radix imaginaria aequationis $h^3-1=0$, puta $h=-\frac{1}{2}+\frac{\sqrt{3}}{2}i$; et perinde theoria residuorum potestatum altiorum introductionem aliarum quantitatum imaginariarum postulabit.

2. POLYHEDRA



Sir William Rowan Hamilton
(1805 - 1865)

Hamilton explains imaginaries

Define complex numbers as pairs

$$a+bi \rightarrow (a, b)$$

which are combined as follows:

Addition:

$$(a, b) + (c, d) = (a+c, b+d)$$

$$[(a+bi) + (c+di) = (a+c) + (b+d)i]$$

Multiplication

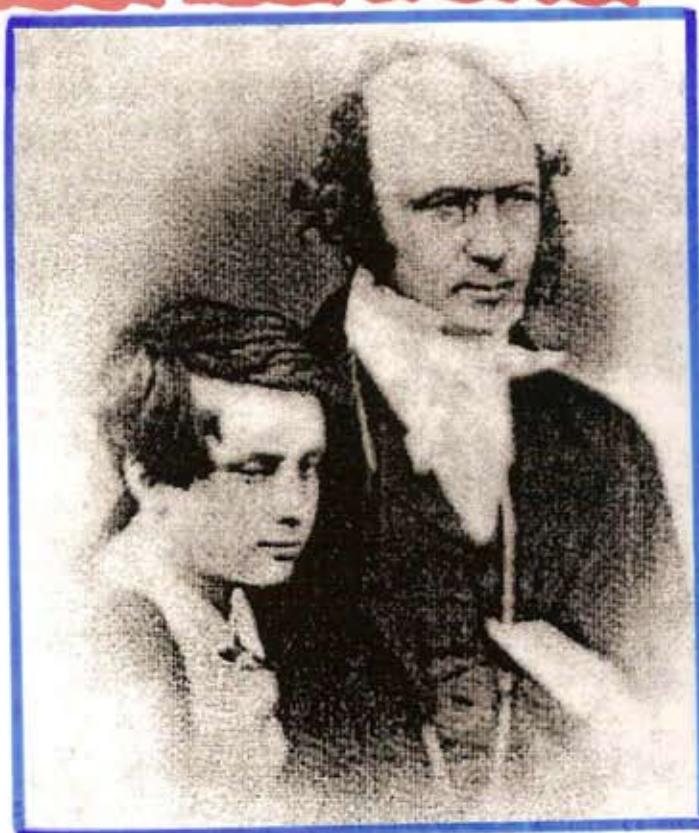
$$(a, b) \times (c, d) = (ac-bd, ad+bc)$$

$$[(a+bi) \times (c+di) = (ac-bd) + (ad+bc)i]$$

Note that $(a, 0)$ corresponds to a (real)
and $(0, 1)$ corresponds to i ,

$$\text{and } (0, 1) \times (0, 1) = (-1, 0) \quad [i^2 = -1]$$

Hamilton writes to his son



Every morning, on my coming down to breakfast, your little brother William Edwin and yourself used to ask me, 'Well Papa, can you multiply triples? Whereto I was obliged to reply, with a shake of the head: 'No, I can only add and subtract them'.

Hamilton takes a walk

As I was walking with Lady Hamilton to Dublin, and came up to Brougham Bridge, I then and there felt the galvanic circuit of thought close; and the sparks which fell from it were the fundamental equations exactly as I have used them ever since.

I pulled out on the spot a pocket book and made an entry ... it is fair to say that this was because I felt a problem to have been at that moment solved — an intellectual want relieved which had haunted me for at least fifteen years since.

Hamilton's Quaternions

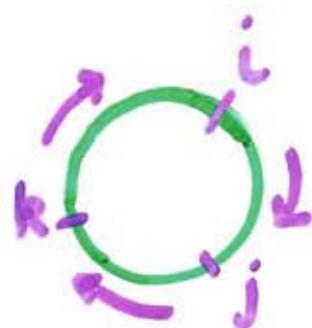
$$a + bi + cj + dk, \quad i^2 = j^2 = k^2 = -1:$$

Addition: ✓

Multiplication:

$$ij = k, \quad jk = i, \quad ki = j$$

$$ji = -k, \quad kj = -i, \quad ik = -j$$



More concisely:

$$i^2 = j^2 = k^2 = ijk = -1.$$

non-commutative system

Pauli matrices:

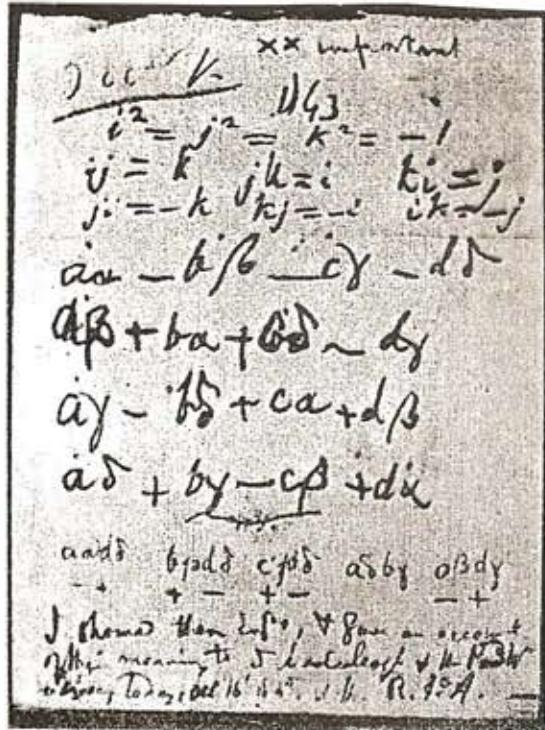
$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

I i j k

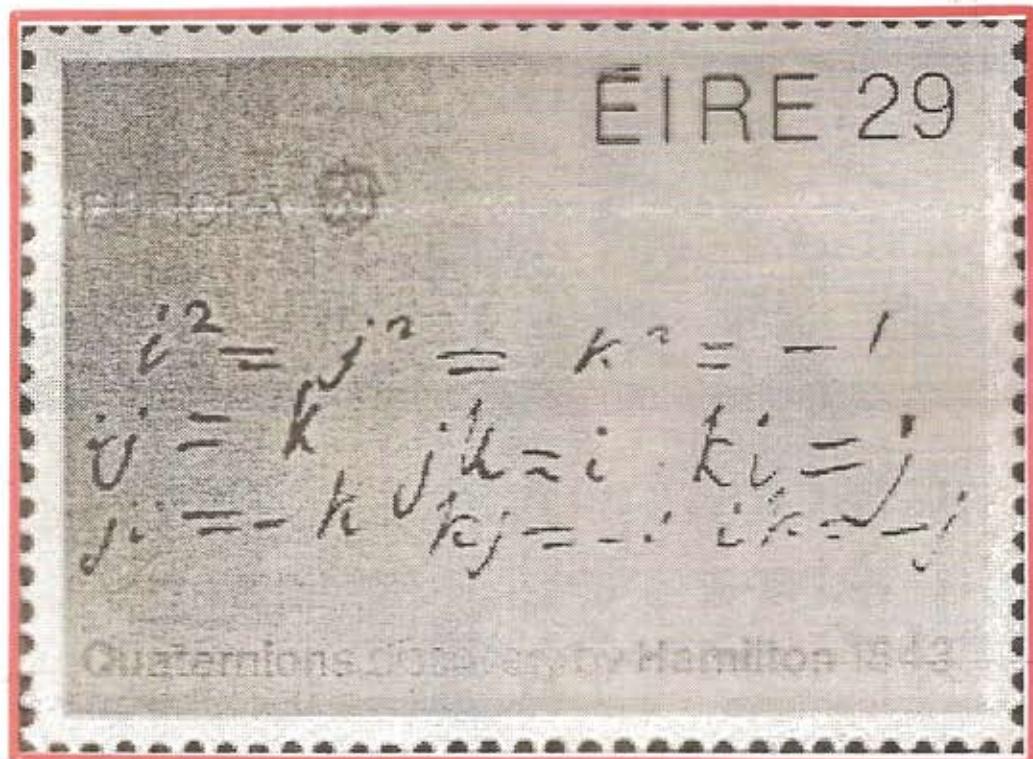
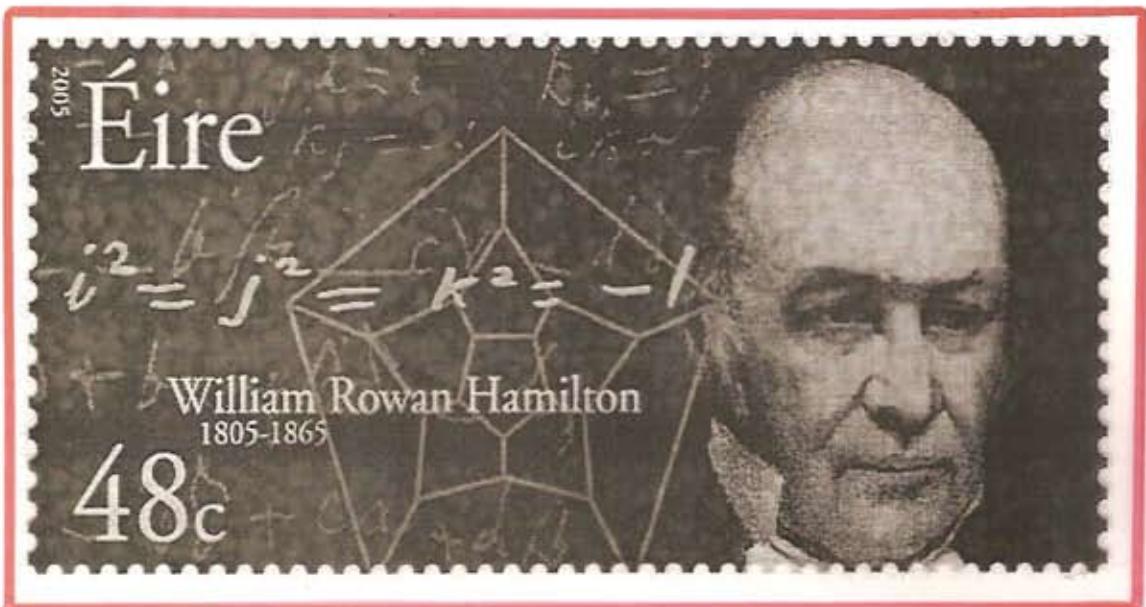
Hamilton at Brougham Bridge

Here as he walked by
on the 16th of October 1843
Sir William Rowan Hamilton
in a flash of genius discovered
the fundamental formula for
quaternion multiplication
 $i^2 = j^2 = k^2 = ijk = -1$
& cut it on a stone of this bridge

plaque on
bridge



pocket book



Octonions

$$\alpha + \beta i + \gamma j + \delta k + \varepsilon l + \zeta m + \eta n + \theta o,$$

where $i^2 = j^2 = k^2 = \dots = o^2 = -1$.

Addition as before

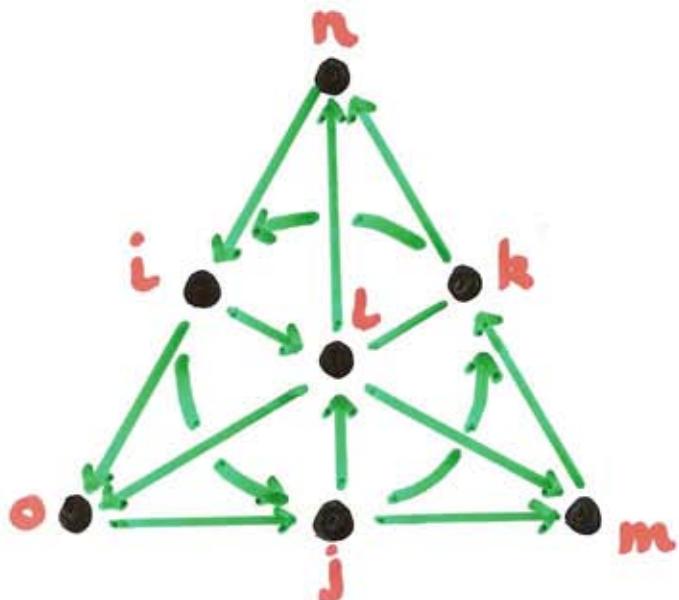
Multiplication:

$$ni = 0$$

$$mk = n$$

$$in = -o$$

$$mj = -o, \text{ etc}$$



Three systems : $xy = yx$ $(xy)z = x(yz)$

complexes : commutative, associative

quaternions : not commutative, associative

octonions : neither

Hamilton 1856

icosian calculus:

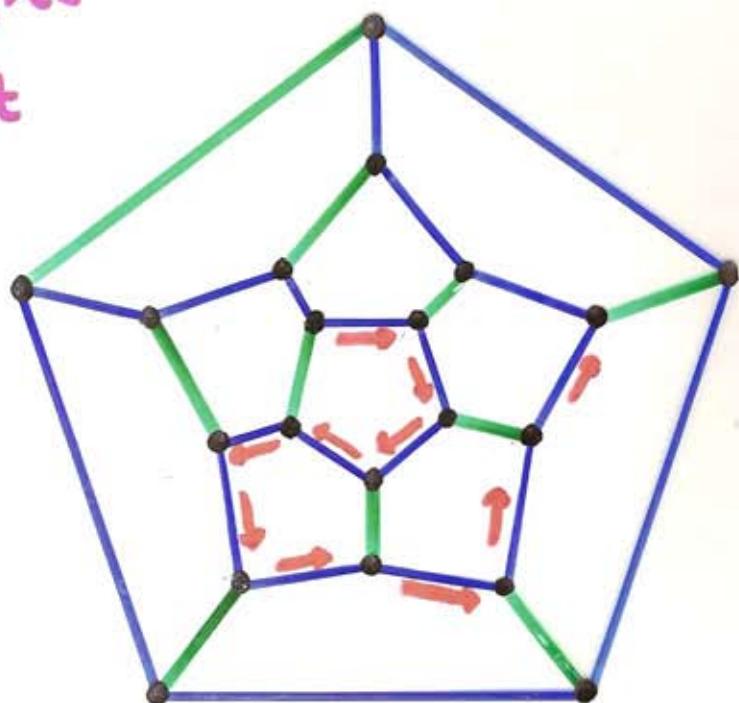
$$l^2 = k^3 = l^5 = 1, \text{ where } l = ik$$

Let $m = ik^2 = lk$: then

$$\underline{l^3 m^3 L m L m l^3 m^3 L m L m = 1}$$

l = right

m = left



Hamiltonian cycle on a dodecahedron

G. H. Hardy's only geometrical result

If a rectangular hyperbola is a parabola,
then it is also an equiangular spiral.

The curve

$$(x + iy)^2 = \lambda (x - iy)$$

is (i) a parabola,
(ii) a rectangular hyperbola,
and (iii) an equiangular spiral.

The first two statements are evidently true. The polar equation is

$$r = \lambda e^{-3i\theta},$$

the equation of an equiangular spiral. ...