

The story of π

Robin Wilson

3 · 14159265358979...

'Tis a favourite project of mine

A new value of π to assign.

I would fix it at 3

For it's simpler, you see,

Than 3 · 14159...

What is π ?

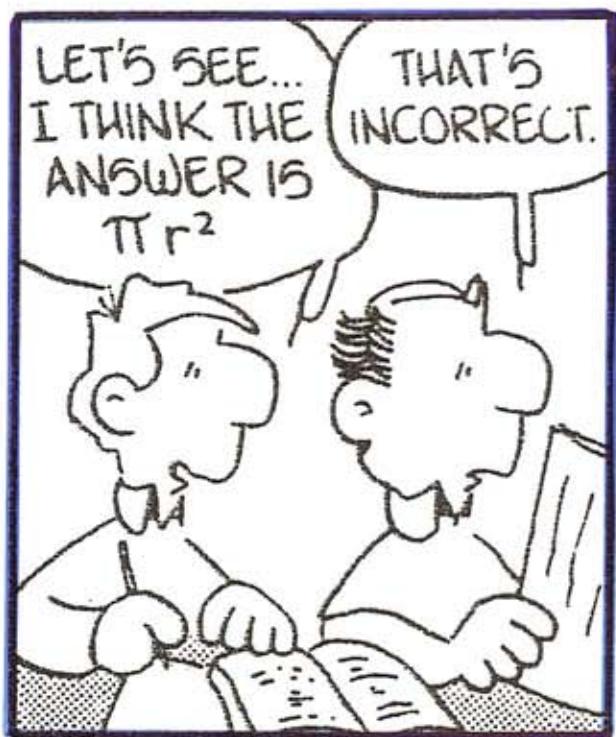
π is :

- the ratio of the circumference of a circle to its diameter

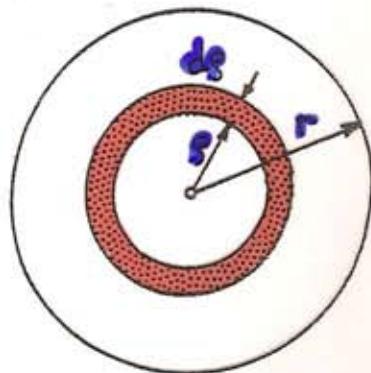
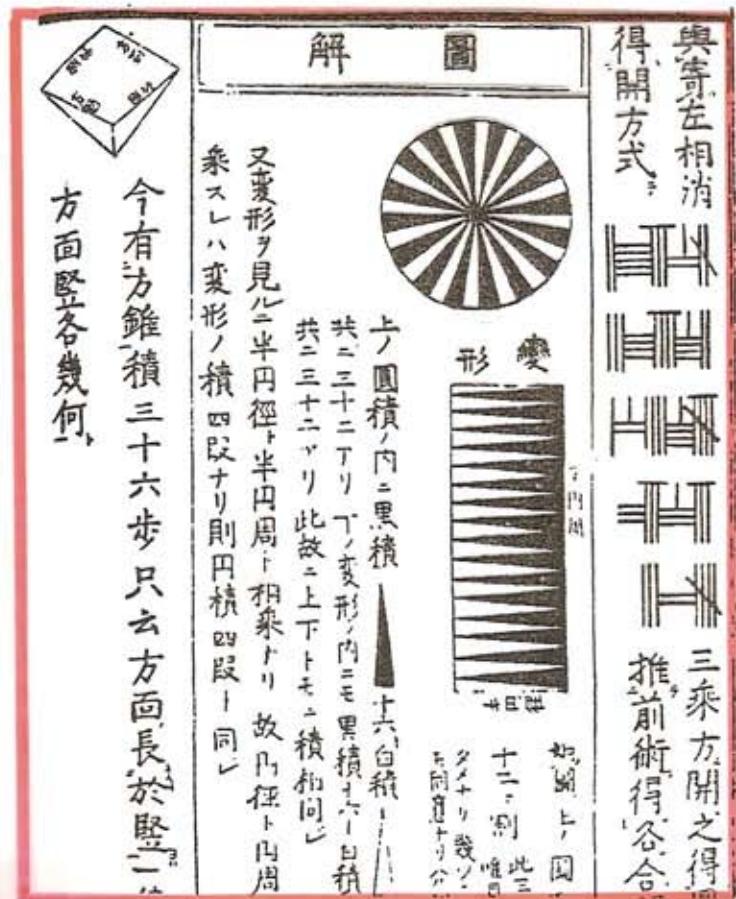
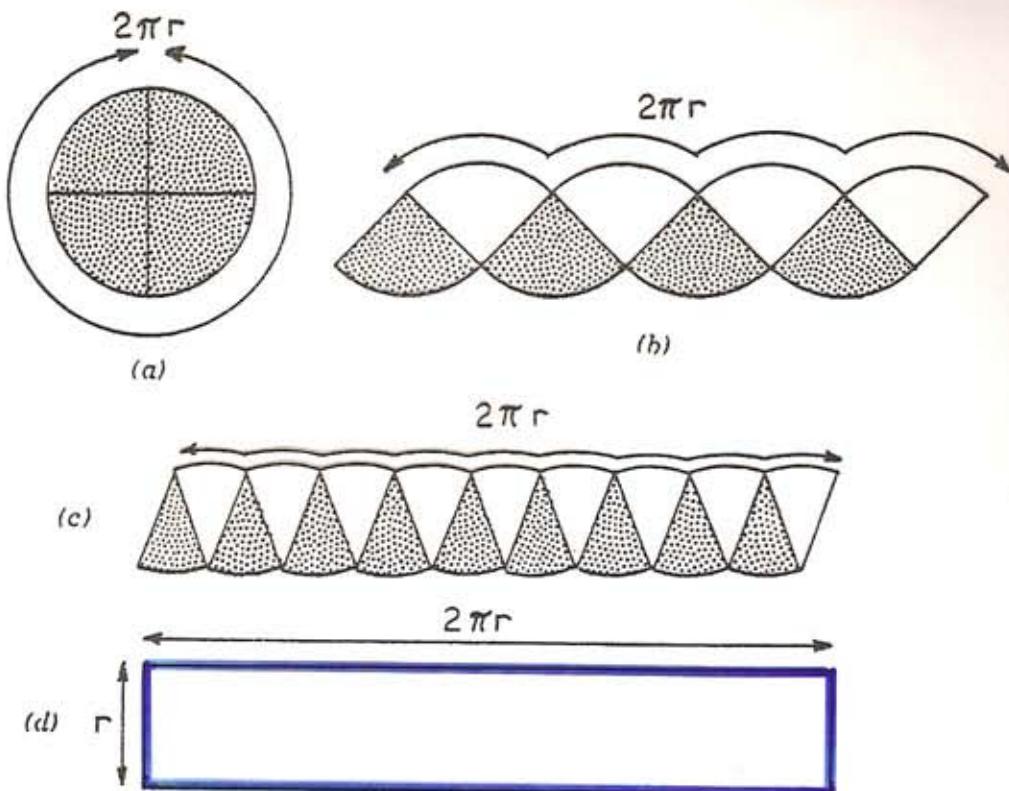
$$C = \pi d = 2\pi r \quad (r = \text{radius})$$

- the area of a circle of radius 1

$$A = \pi r^2 = \pi \cdot 1^2 = \pi$$

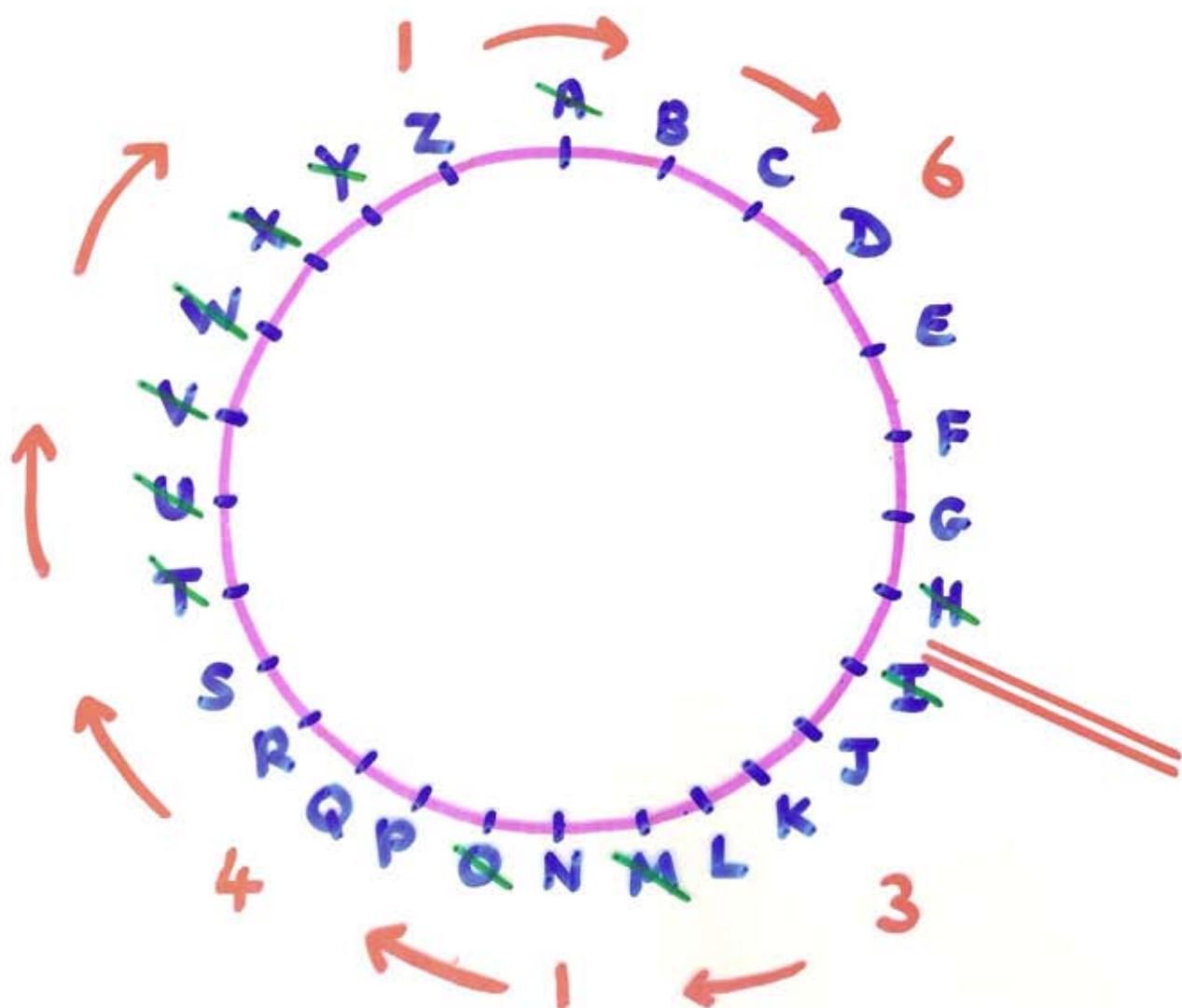


From $2\pi r$ to πr^2



$$\text{Area} = \int_0^R 2\pi r dr = \pi r^2$$

An alphabetical method



How I wish I could calculate pi !

May I have a large container of coffee.

3 . 1 4 1 5 9 2 6 ...

How I need a drink, alcoholic of course, after all these lectures informing Gresham audiences...

3 . 1 4 1 5 9 2 6 5 3 5 8 9 7 9 ...

Ἄει ο Θεος ο Μεγας γεωμετρει /
το κυκλου μηκος ινα ὄριση
διαμετρω / παρηγαγεν ἀριθμον
ἀπεραντον / και ον φευ ούδεποτε
όλον / θυητοι θα εύρωσι.

3 . 1 4 1 5 9 2 6 5 3 5 8 9 7 9 3 2 3 8 4 6 2 6 ...

For

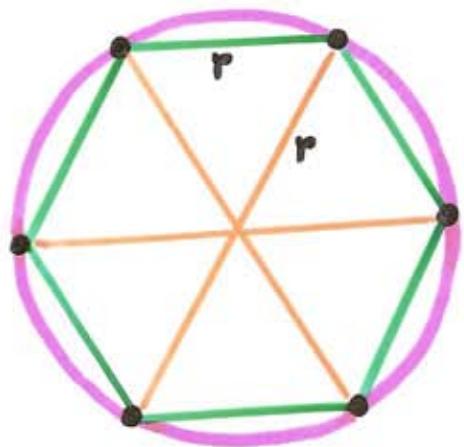
a time I stood pondering on circle sizes. The large computer mainframe quietly processed all of its assembly code.

Inside my entire hope lay for figuring out an elusive expansion. Value: pi.

Decimals expected soon. I nervously entered a format procedure. The mainframe processed the request. Error. I, again entering it, carefully retyped. This iteration gave zero error printouts in all-success. Intently I waited. Soon, roused by thoughts within me, appeared narrative mnemonics relating digits to verbiage! The idea appeared to exist but only in abbreviated fashion—little phrases typically. Pressing on I then resolved, deciding firmly about a sum of decimals to use—likely around four hundred, presuming the computer code soon halted! Pondering these ideas, words appealed to me. But a problem of zeros did exist. Pondering more, solution subsequently appeared. Zero suggests a punctuation element. Very novel! My thoughts were culminated. No periods, I concluded. All residual marks of punctuation = zeros. First digit expansion answer then came before me. On examining some problems unhappily arose. That imbecilic bug! The printout I possessed showed four nine as foremost decimals. Manifestly troubling. Totally every number looked wrong. Repairing the bug took much effort. A pi mnemonic with letters truly seemed good. Counting of all the letters probably should suffice. Reaching for a record would be helpful. Consequently, I continued, expecting a good final answer from computer. First number slowly displayed on the flat screen—3. Good. Trailing digits apparently were right also. Now my memory scheme must probably be implementable. The technique was chosen, elegant in scheme: by self reference a tale mnemonically helpful was ensured. An able title suddenly existed—"Circle Digits." Taking pen I began. Words emanated uneasily. I desired more synonyms. Speedily I found my (alongside me) Thesaurus. Rogets is probably an essential in doing this, instantly I decided. I wrote and erased more. The Rogets clearly assisted immensely. My story proceeded (how lovely!) faultlessly. The end, above all, would soon joyfully overtake. So, this memory helper story is incontestably complete. Soon I will locate publisher. There a narrative will I trust immediately appear, producing fame.

The end

Mesopotamian Value of π (c.1800 BC)

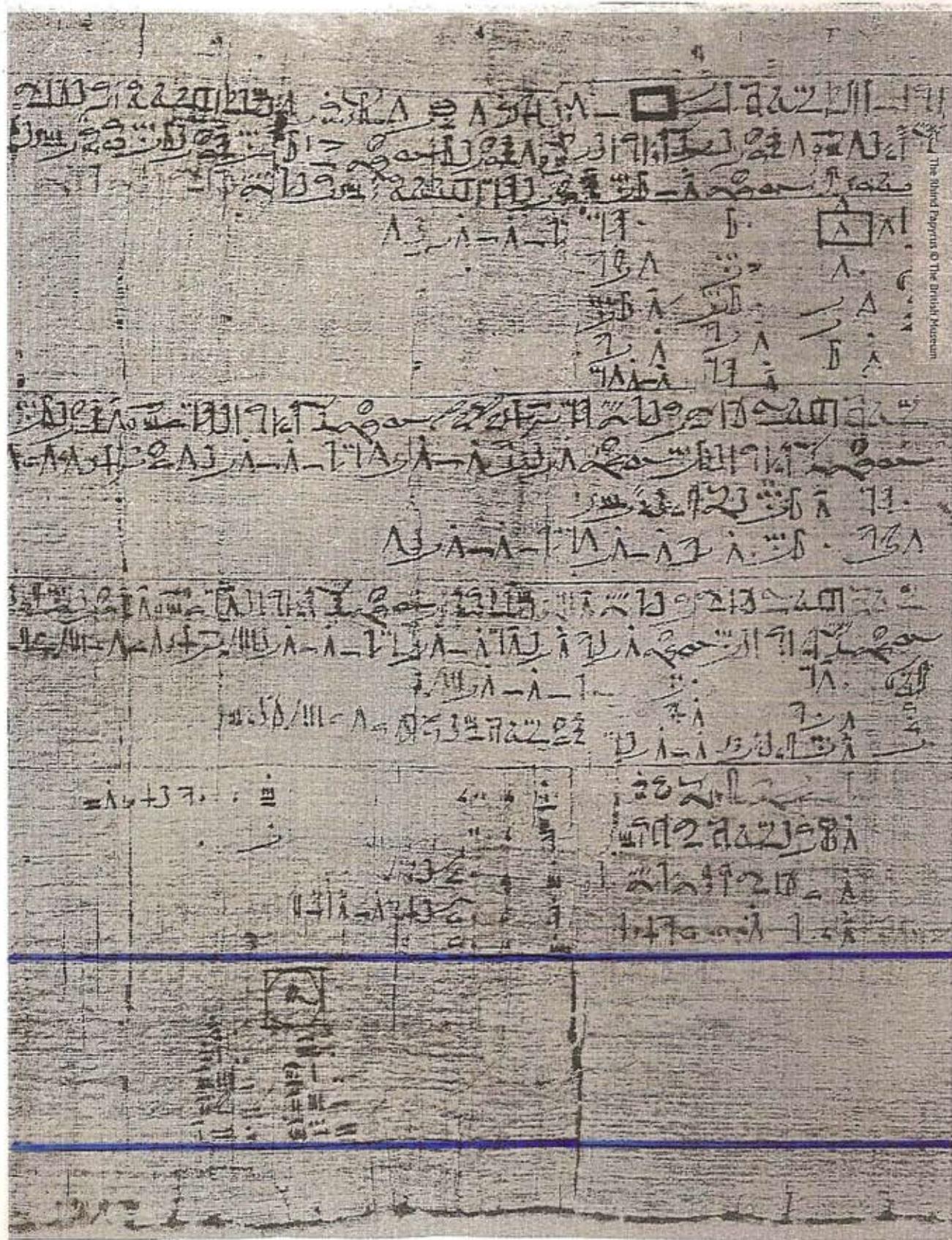


ratio of perimeter of hexagon to circumference of circle = $0; 57, 36 :$

$$\frac{6r}{2\pi r} = \frac{3}{\pi} = \frac{57}{60} + \frac{36}{(60)^2}$$

$$\text{so } \underline{\pi = 3\frac{1}{8} = 3.125}$$

Egyptian Rhind Papyrus



The Rhind Papyrus © The British Museum

A Problem in Geometry (c. 1650 BC)

Problem 48. Compare the area of a circle and its circumscribing square.

The circle of diameter 9 The square of side 9

1 8 setat

2 16 setat

4 32 setat

8 64 setat

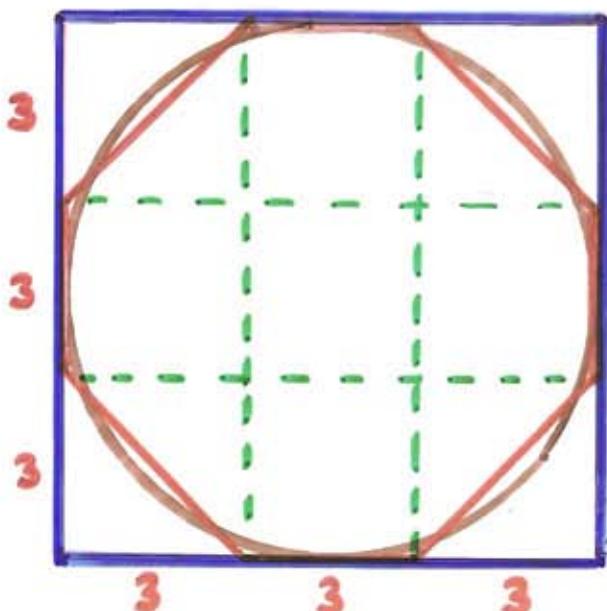
1 9 setat

2 18 setat

4 36 setat

8 72 setat

Total 81 setat



$$\text{Area} = \left(d - \frac{d}{9}\right)^2$$

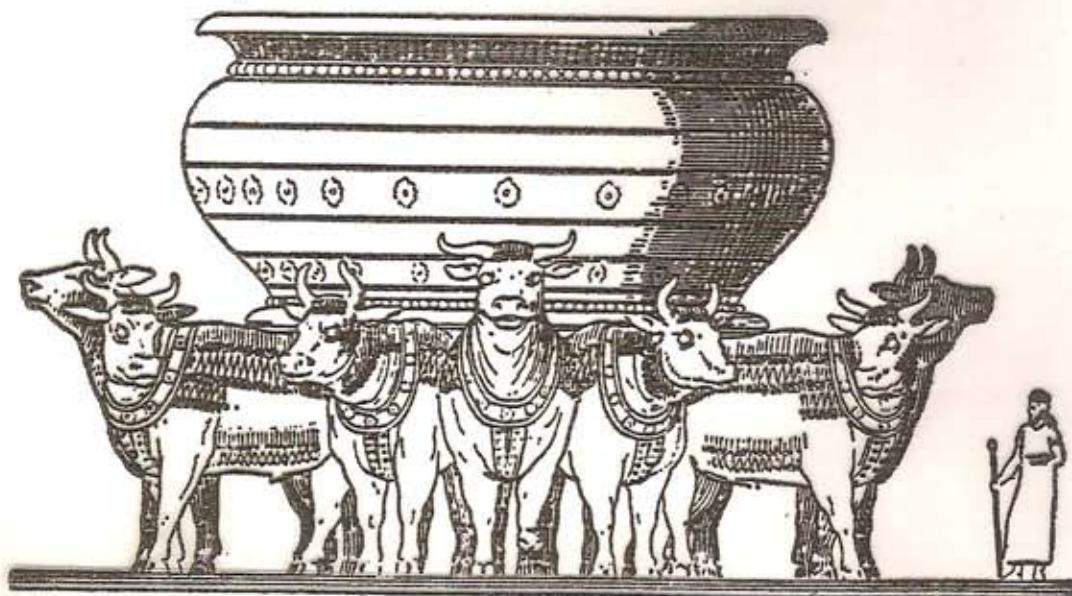
$$= \frac{256}{81} r^2 \approx 3.16 r^2$$

The Biblical Value (c. 550 BC)

I Kings VII, 23 and II Chronicles IV, 2 :

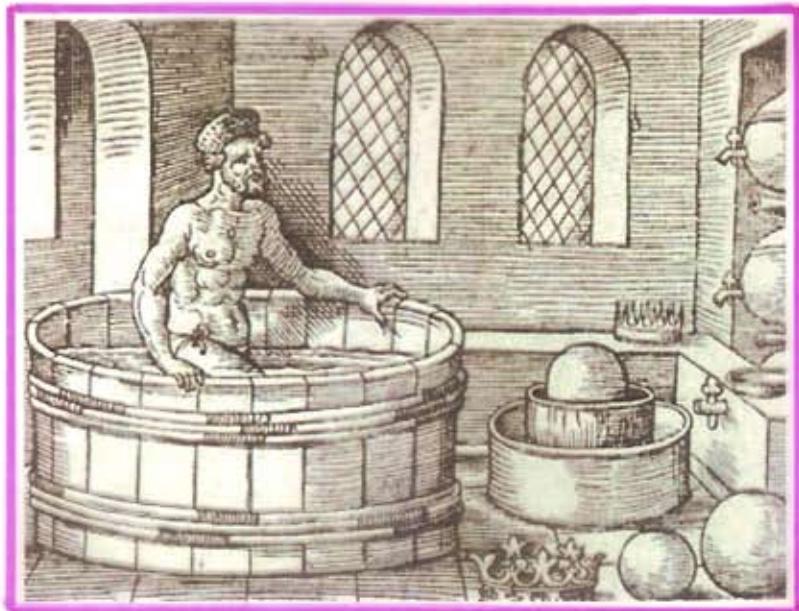
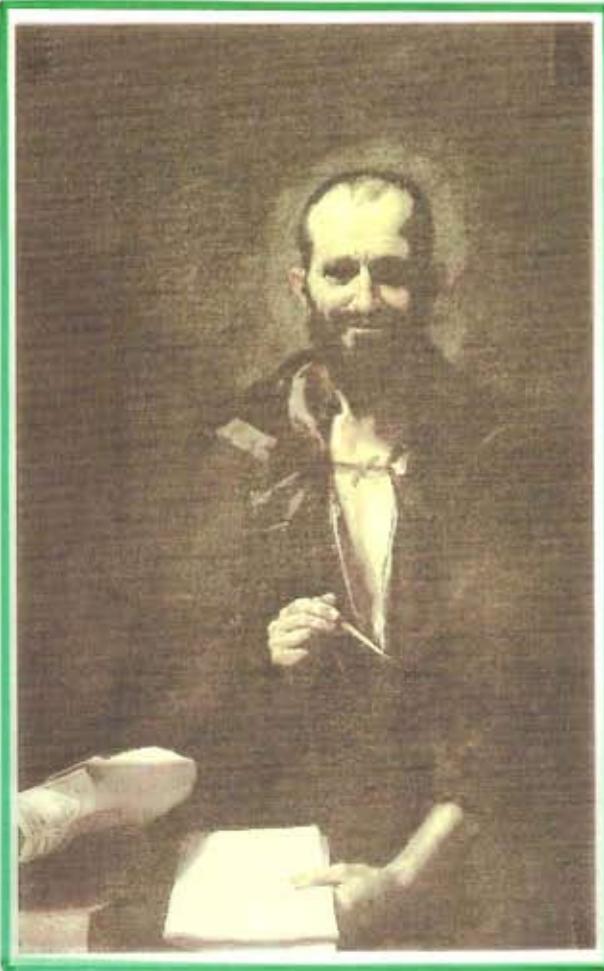
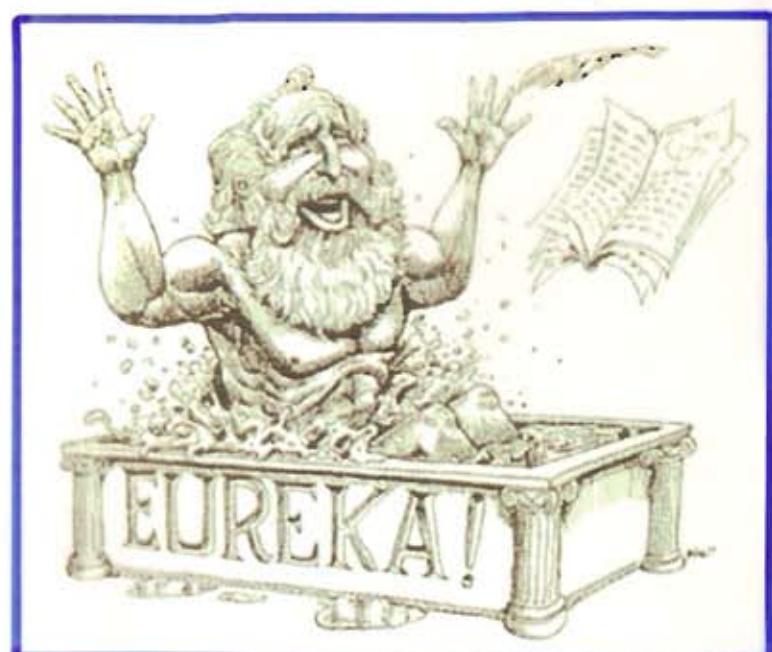
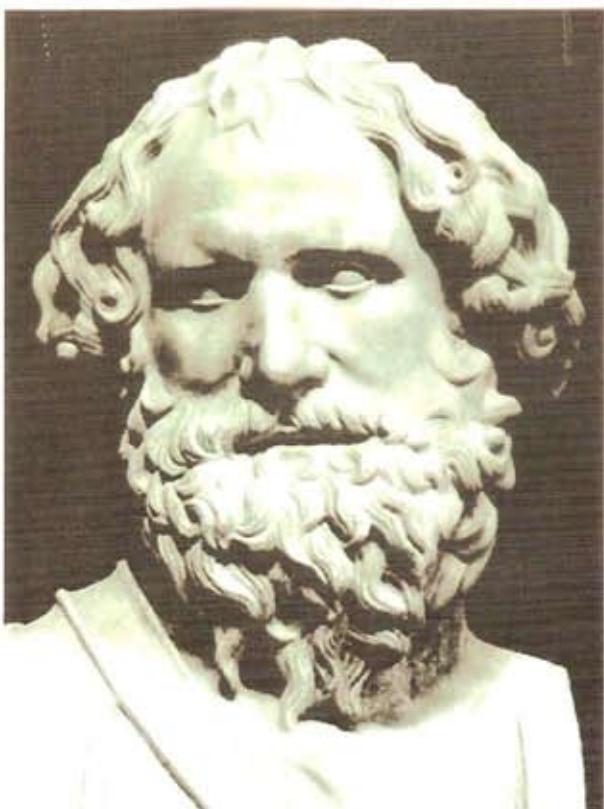
Also, he made a molten sea of ten cubits from brim to brim, round in compass, and five cubits the height thereof ; and a line of thirty cubits did compass it round about.

So $\pi = 30/10 = 3$.

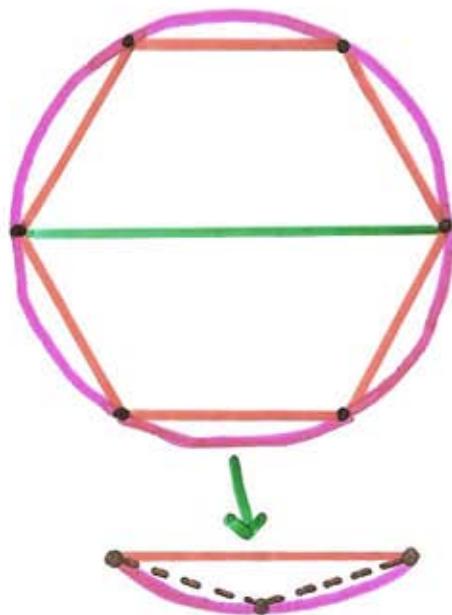


Archimedes

(c. 287 - 212 BC)



The Value of π



perimeter of inscribed 6-gon

< circumference of circle

< perimeter of exscribed 6-gon

double the number of sides:

6, 12, 24, 48, 96.

Archimedes obtained the estimates:

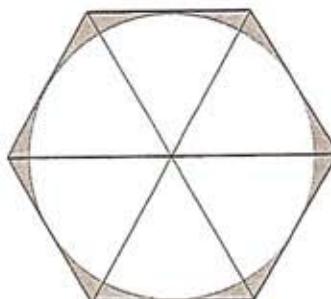
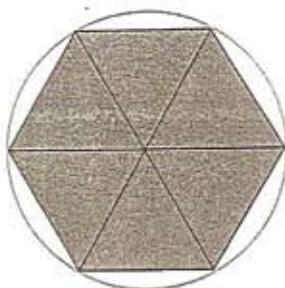
$$3\frac{10}{71} < \pi < 3\frac{1}{7}$$

3.14084

3.14286

Archimedes' polygons

$n = 6$

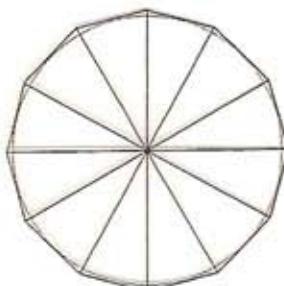
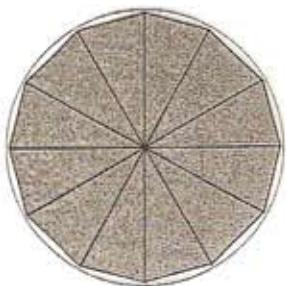


Semi-perimeter

$$l = 3$$

$$L = 3 \cdot 464$$

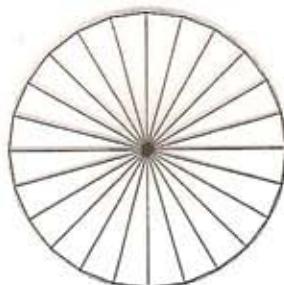
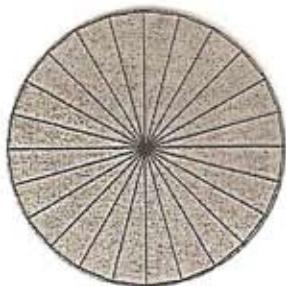
$n = 12$



$$l = 3 \cdot 105$$

$$L = 3 \cdot 215$$

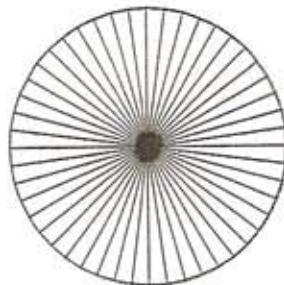
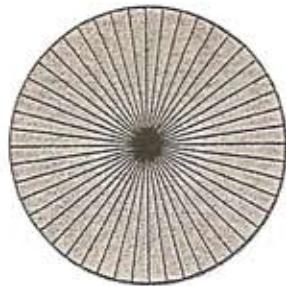
$n = 24$



$$l = 3 \cdot 133$$

$$L = 3 \cdot 160$$

$n = 48$



$$l = 3 \cdot 139$$

$$L = 3 \cdot 146$$

Indian values for π

380 AD : $3 \frac{177}{1250} = 3.1416\dots$

Anyabhata (499 AD) :

Add 4 to 100, multiply by 8, and add 62,000.
The result is approximately the circumference of
a circle with diameter 20,000.

$$\text{So } \pi = 62832 \div 20000 = 3.1416.$$

Brahmagupta (6th c.) :

$$\pi \sim \sqrt{10} = 3.162\dots$$

Chinese Values for π



Zhang Heng (100 AD) : $\pi = \sqrt{10}$

Liu Hui (263 AD) : $\pi = 3.14159$ 3072

Zu Changzhi (500 AD) : $\pi = 3.1415926$ 24576
and $\pi \approx 355/113$

Using polygons

Fibonacci $\pi = 3.141818$ 96 sides
(Italy, 1200)

al - Kashi 14 decimal places
(Samarkand, 1430)

A. Anthoniszoon $335/113$
(Dutch, 16c.) $(377/120 > \pi > 333/106)$

A. van Roomen 15 decimal places 100 million sides
(Dutch, 1593)

François Viète 9 decimal places 393216 sides

Ludolph van Ceulen 20 d.p. 60×2^{29}
(Dutch 1596, 1615) 35 d.p. sides

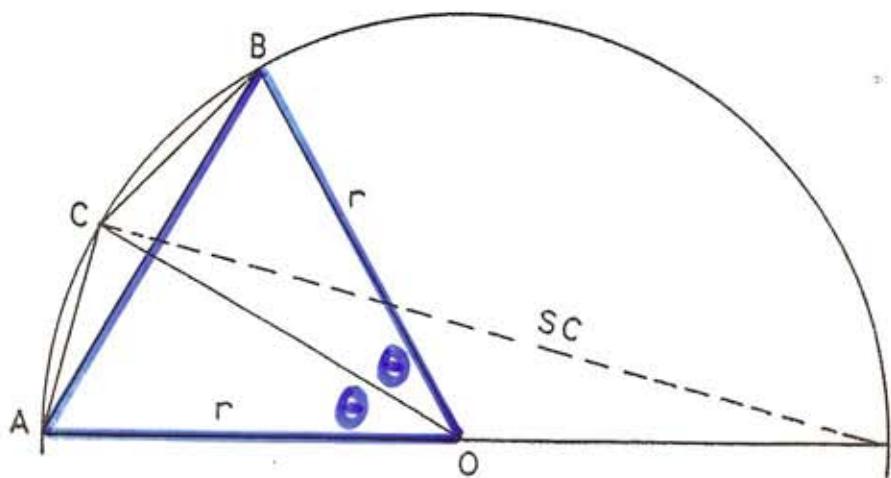
FRANÇOIS VIÈTE

(1540 - 1603)

- pioneered improvement in notation - use of letters for unknowns
- insisted on 'dimension': e.g. cannot add lines to areas
- computed π to 9 decimal places
- $\frac{2}{\pi} = \cos \frac{\pi}{4} \cdot \cos \frac{\pi}{8} \cdot \cos \frac{\pi}{16} \cdot \cos \frac{\pi}{32} \dots$
 $= \frac{1}{2}\sqrt{2} \times \frac{1}{2}\sqrt{2+\sqrt{2}} \times \frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2}}} \times \dots$



Viète's method



$A(n)$ = area of n -sided polygon

$$A(n) = n \times \text{area}(OAB) = n \cdot \frac{1}{2} r^2 \sin 2\theta$$

$2 \sin \theta \cos \theta$

$$A(2n) = 2n \cdot \frac{1}{2} r^2 \sin \theta$$

$$\text{So } A(n) / A(2n) = \cos \theta$$

$$\begin{aligned} \text{Also } A(n) / A(4n) &= \frac{A(n)}{A(2n)} \times \frac{A(2n)}{A(4n)} \\ &= \cos \theta \cdot \cos \theta/2 \end{aligned}$$

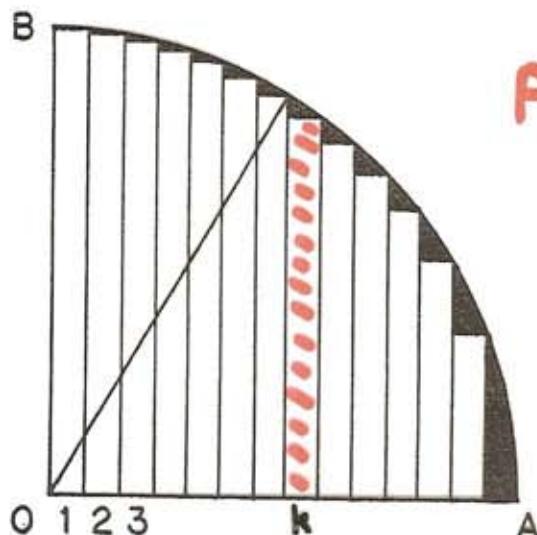
$$\frac{A(n)}{A(2^k n)} = \cos \theta \cdot \cos \theta/2 \cdots \cos \theta/2^k$$

$$\frac{\downarrow}{\pi r^2}$$

$$n=4 \quad \theta=45^\circ, \cos \theta = 1/\sqrt{2}$$

$$A(4) = 2r^2 \dots$$

John Wallis's formula (1655)



Area of k th strip is

$$\frac{1}{n} \sqrt{1 - \left(\frac{k}{n}\right)^2}$$

$$\text{So } \frac{\pi}{4} = \lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{1}{n} \sqrt{1 - \left(\frac{k}{n}\right)^2}$$

$$\frac{\pi}{4} = \int_0^1 \sqrt{1-x^2} dx$$

Leading to :

$$\frac{\pi}{4} = \frac{2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \cdots}{3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \cdots}$$

Continued fractions

$$\frac{4}{\pi} = 1 + \frac{1^2}{2 + \frac{3^2}{2 + \frac{5^2}{2 + \frac{7^2}{2 + \frac{9^2}{2 + \dots}}}}}$$

$$1, \frac{3}{2}, \frac{15}{13}, \frac{105}{76}, \frac{945}{789}, \dots$$

W. Brouncker
(1660s)

$$\pi = 3 + \frac{1}{7 + \frac{1}{15 + \frac{1}{1 + \frac{1}{292 + \frac{1}{1 + \dots}}}}}$$

$$3, \frac{22}{7}, \frac{333}{106}, \frac{355}{113}, \frac{103993}{33102}, \dots$$

J. Lambert
(1767)

π is irrational (not a fraction)

Johann Heinrich Lambert (1767)

If x is a rational number (other than 0),
then $\tan x$ cannot be rational.

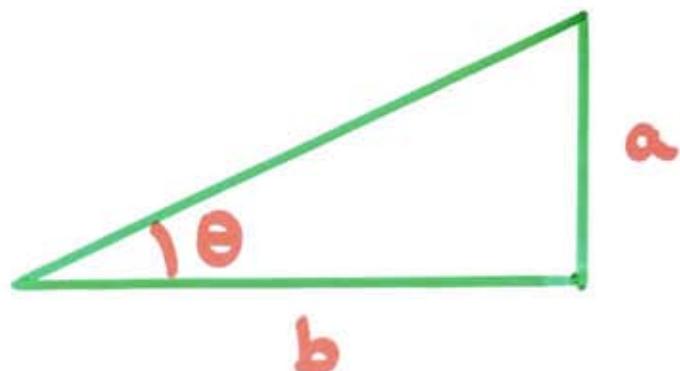
So:

If $\tan x$ is rational, then x must be
irrational (or 0).

But $\tan(\pi/4) = 1$, which is rational:

so $\pi/4$ (and hence π) is irrational.

The \tan^{-1} function

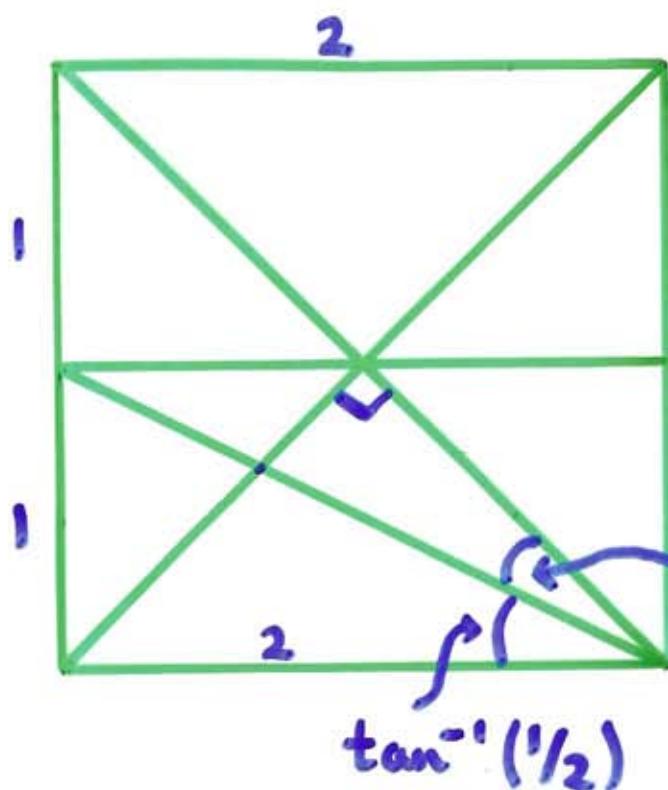


$$\tan \theta = a/b$$

$$\theta = \tan^{-1}(a/b)$$

$$\tan \pi/4 = 1, \text{ so } \tan^{-1}(1) = \pi/4$$

$$\tan \pi/6 = 1/\sqrt{3}, \text{ so } \tan^{-1}(1/\sqrt{3}) = \pi/6$$



$$\begin{aligned} \tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right) \\ = \pi/4 \end{aligned}$$

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1} \left(\frac{x+y}{1-xy} \right)$$

The series for $\tan^{-1} x$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

(J. Gregory)

so, with $x = 1$:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots \quad (\text{Leibniz})$$

This converges very slowly : 300 terms for two decimal digits of π .

$$\begin{aligned}\frac{\pi}{4} &= \tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right) \\ &= \left\{ \frac{1}{2} - \frac{(1/2)^3}{3} + \frac{(1/2)^5}{5} - \frac{(1/2)^7}{7} + \dots \right\} \\ &\quad + \left\{ \frac{1}{3} - \frac{(1/3)^3}{3} + \frac{(1/3)^5}{5} - \frac{(1/3)^7}{7} + \dots \right\}\end{aligned}$$

which converges much faster.

Better series ...

$$\frac{\pi}{4} = 4 \tan^{-1} \left(\frac{1}{5} \right) - \tan^{-1} \left(\frac{1}{239} \right)$$

John Machin
(1706)

$$= 4 \left\{ \frac{1}{5} - \frac{(1/5)^3}{3} + \frac{(1/5)^5}{5} - \dots \right\}$$

$$- \left\{ \frac{1}{239} - \frac{(1/239)^3}{3} + \frac{(1/239)^5}{5} - \dots \right\}$$

100
places

Euler (1764)

$$\frac{\pi}{4} = 4 \tan^{-1} \left(\frac{1}{5} \right) - \tan^{-1} \left(\frac{1}{70} \right) + \tan^{-1} \left(\frac{1}{99} \right)$$

Leray (1893)

200

$$\frac{\pi}{4} = 3 \tan^{-1} \left(\frac{1}{4} \right) + \tan^{-1} \left(\frac{1}{20} \right) + \tan^{-1} \left(\frac{1}{1985} \right)$$

528

Gauss

$$\frac{\pi}{4} = 12 \tan^{-1} \left(\frac{1}{18} \right) + 8 \tan^{-1} \left(\frac{1}{57} \right) - 5 \tan^{-1} \left(\frac{1}{239} \right)$$

1000000

New values for π

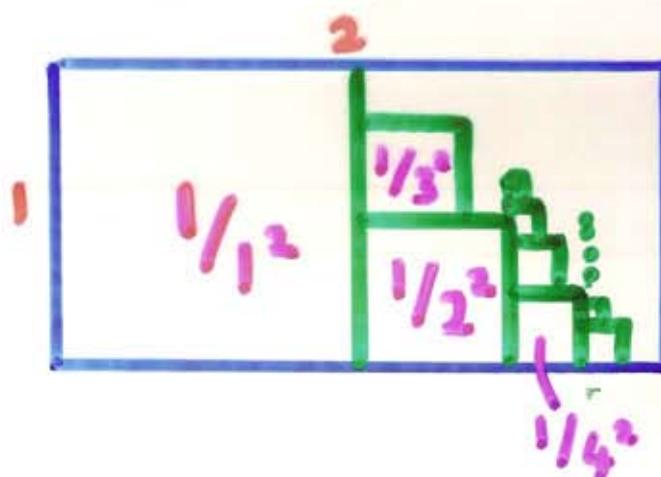
1706	Machin	100 places
1719	De Lagny	127 places
1794	Vega	140 places
1855	Richter	500 places
1874	Shanks	707 places <i>(wrong from 527th...)</i>
1946/7	Ferguson	808 places

Euler's Basel Problem (1730)

What is the value of :

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots ?$$

$$\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots \right]$$



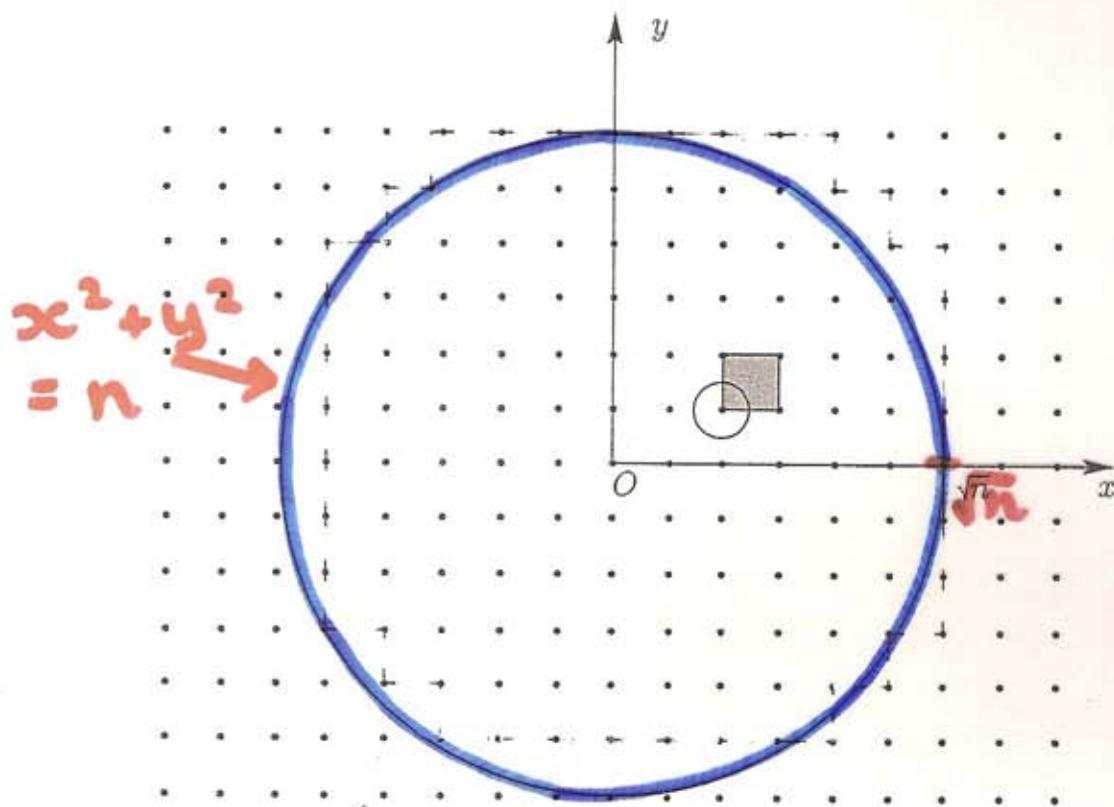
answer is
less than 2

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \frac{\pi^2}{6}$$

$$\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \dots = \frac{\pi^4}{90}$$

$$\frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} + \dots = \frac{\pi^6}{945} \dots$$

Gauss's Circle Problem



Let $r(n)$ be the number of ways of writing n as the sum of two squares:

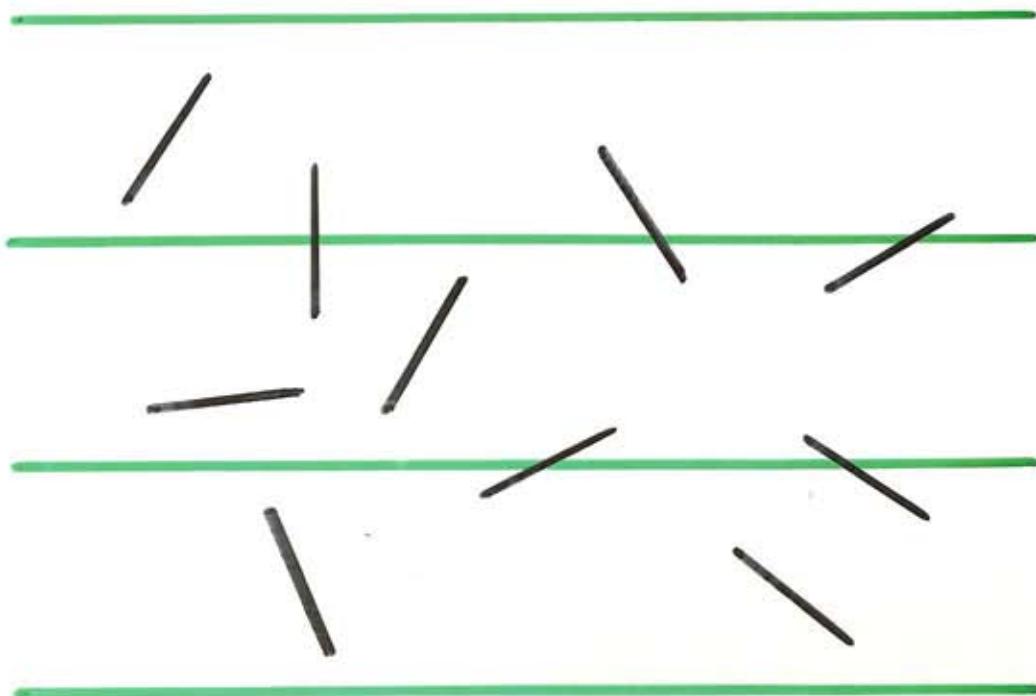
$$r(5) = 8 : \quad 5 = (\pm 2)^2 + (\pm 1)^2 = (\pm 1)^2 + (\pm 2)^2$$

$$r(7) = 0 \qquad r(250) = 16 \qquad r(251) = 0$$

$r(n)$ is very erratic: look at averages

$$\frac{r(1) + r(2) + \dots + r(n)}{n} \rightarrow \pi \text{ as } n \rightarrow \infty$$

Buffon's needle experiment (1777)



L = length of needle

a = distance between lines

$$\text{probability of } \} = \frac{2L}{\pi a}$$

crossing a line }

(Here, $\frac{1}{2} = \frac{2 \times 4}{\pi \times 5}$, so $\pi = \frac{16}{5} = 3.2$.)

Legislating for π

State legislature of Indiana (1897)

Edwin J. Goodman M.D. proposed
a bill:

A bill for an act introducing a new mathematical truth and offered as a contribution to education to be used only by the State of Indiana free of cost by paying any royalties whatever on the same, provided it is accepted and adopted by the official action of the legislature in 1897.

- House Committee on Swamp Lands
- Committee on Education: bill passed

Be it enacted by the General Assembly of the State of Indiana: It has been found that the circular area is to the quadrant of the circumference, as the area of an equilateral rectangle is to the square on one side. The diameter employed as the linear unit according to the present rule in computing the circle's area is entirely wrong...

- Committee on temperance

Mathematics professor C. A. Waldo (Purdue)

Enter the Computer

1949	ENIAC	2037	70 hours
1955	NORC	3089	13 mins
1957	Pegasus	10021	33 hours
1958	IBM 704	10000	1 hr, 40 mins
1961	IBM 7090	20000	39 mins
1961	Shanks/Wrench	100265	~8 hours
1967	CDC 6600	500000	28 hours
1989	Chudnovsky's	1 billion	
2002	Kanada	1 trillion	

π to 5000 decimal places

3 01415926535 8979323846 2643383279 5028841971 6939937510 5820974934 5923078164 0628620899 8628034825 3421170679
 8214808651 3282306647 0938446095 5058223172 5359408128 4811174502 8410270193 8521105559 6446229489 5493038196
 4428810975 6659334461 287564823 3786783165 2712019091 4564856692 3460348610 4543266482 1339360726 0249141273
 7245870066 0631558817 4881520920 962829250 9171536436 7892590360 0113305305 4882046652 1384146951 9415116094
 3305727036 5759591953 0921861173 8193261179 3105118548 0744623799 6274956735 1885752724 8912279381 8301195912
 9833673362 4406566430 8602139494 6395224737 1907021798 6094370277 0539217176 2931767523 8467481846 7669405132
 0005681271 4526356082 7785771342 7577896091 7363717872 1468440901 2249534301 4654958537 1050792279 6892589235
 4201995611 2129021960 8640344181 5981362977 4771309960 5187072113 499999837 2978049951 0597317328 1609631859
 5024459455 3469083026 4252230825 3344685035 2619311881 7101000313 7838752886 5875332083 8142061717 7669147303
 5982534904 2875546873 1159562863 8823537875 9375195778 1857780532 1712268066 1300192787 6611195909 2164201989
 3809525720 1065485863 2788659361 5338182796 8230301952 0353018529 6899577362 2599413891 2497217752 8347913151
 5574857242 4541506959 5082953311 6861727855 8890750983 8175463746 4939319255 0604009277 0167113900 9848824012
 8583616035 6370766010 4710181942 9555961989 4676783744 9448255379 7747268471 0404753464 6208046684 2590694912
 9331367702 8989152104 7521620569 6602405803 8150193511 2533824300 3558764024 7496473263 9141992726 0426992279
 6782354781 6360093449 2164121992 4586315030 2861829745 5570674983 8505494588 5869269956 9092721079 7509302955
 3211653449 8720275596 0236480665 4991198818 3479775356 6369807426 5425278625 5181841757 4672890977 7727938000
 81646706001 6145249192 1732172147 7235014144 1973568548 1613611573 5255213347 5751849468 4385233239 0739414333
 4547762416 8625189835 6948556209 9219222184 2725502542 5688767179 0494601653 4668049886 2723279178 6085784383
 8279679766 8145410095 3883786360 9506800642 2512520511 7392984896 0841284886 2694560424 1965285022 2106611863
 0674427862 2039194945 0471237137 8696095636 4371917287 4677646575 7396241389 0865832645 9958133904 7802759009
 9465764078 9512694683 9835259570 9825822620 5224894077 2671967826 8482601476 9909026401 3639443745 5305068203
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Some bizarre results

Ramanujan (1914) $\frac{1}{\pi} = \sum_{n=0}^{\infty} \binom{2n}{n}^3 \frac{42n+5}{2^{12n+4}}$

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{n=0}^{\infty} \frac{(4n)!}{(n!)^4} \times \frac{(1103 + 26390n)}{(4 \times 99)^{4n}}$$

Borweins and Chudnovskys:

$$\frac{1}{\pi} = 12 \sum_{n=0}^{\infty} \frac{(-1)^n (6n)! (13591409 + 545160134n)}{(n!)^3 (3n)! (640320^3)^{n+1/2}}$$

$$\pi = \sum_{n=0}^{\infty} \frac{1}{16^n} \left\{ \frac{4}{8n+1} - \frac{2}{8n+4} - \frac{1}{8n+5} - \frac{1}{8n+6} \right\}$$

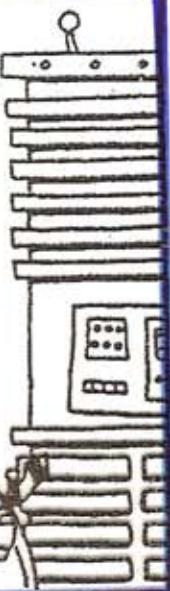
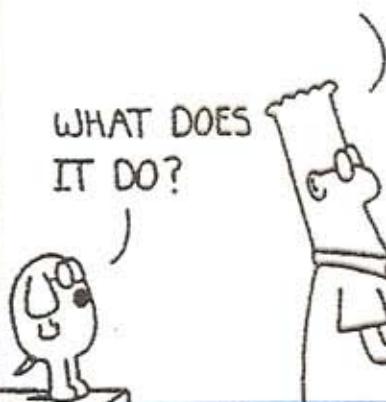
Why calculate so many digits?

- to enable us to look for patterns
- to test the accuracy of a new computer

The first trillion digits of π

<u>digit</u>	<u>occurrences</u>
0	999999485134
1	999999945664
2	100000480057
3	99999787805
4	100000357857
5	99999671008
6	99999807503
7	99999818723
8	100000791469
9	99999854780
<hr/>	
10000000000000	

I SPENT MY ENTIRE
FORTUNE TO BUY THIS
SUPERCOMPUTER.



IT CAN CALCULATE THE
VALUE OF PI TO ABOUT
A JILLION DECIMAL
PLACES...



7-19

A LOT OF PEOPLE TALK
ABOUT THE AREAS OF
CIRCLES, BUT I'M DOING
SOMETHING ABOUT IT.

