Knot Just Numbers:
Mathematics and More in Andean Khipu Strings
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Khipus: A brief introduction

This next exploration of numbers in different languages and cultures will take us thousands of miles westward, as we set our sights on a South American context. However, refocusing here will require more than a geographic leap; it will also invite us to consider a distinctly three-dimensional way of recording information. The name for this as-of-yet undeciphered recording device, which will be the subject of my talk today, is the khipu—a knotted-string register that served as the functional analogue to writing during the Inka Empire, between 1400–1532 AD. Indeed, the Inkas, whose territories at their height stretched some 4,000 kilometres along the spine of the Andes Mountains, from Ecuador and Colombia in the north to Chile, Bolivia, and northern Argentina in the south, relied on khipus for the administration of a population often estimated at greater than 10 million. A class of cord-keepers called khipukamayuqs maintained accounts of relevant information, whether numerical or otherwise, for the exercise of Inka power, rendering the Empire, by all measures, the largest civilization of the pre-Columbian Americas.

However, among my goals for this talk will be to demonstrate that this is not the whole story; to dispel some common misconceptions about when khipus were used, what they looked like, and what type of information they recorded. This will go hand-in-hand with an assessment of where we stand with regard to our understanding of khipu numeracy. So, after introducing you to the canonical structure of a khipu and its myriad components, I will divide the remainder of the talk into two parts: the first focusing on mathematical aspects of some khipus with clear archaeological provenience, or what I will term archaeological khipus. By this I mean khipus for which we possess detailed information about their locations and contexts of discovery. In contrast, the second set of case studies will focus on what I call decontextualized khipus: examples often found in museums and private collections around the world that lack reliable context in the same sense. As we will see, each of these sources provides complementary information in our ongoing quest to decipher the knotted, primary sources of the Andes. By way of conclusion, I will offer some reflections on the path forward in khipu decipherment, focusing on a set of questions facing those who seek to better understand Andean mathematical practices.

First, on chronology: for how often khipus are associated with the Inkas, it may come as a surprise that khipus were not an Inka invention. The earlier, Wari polity of the south-central Andes produced the first khipus currently recognized by the broader academic community, leveraging them for their own state administration (Splitstoser 2020). Khipu use would also continue following the Spanish conquest of the Inkas from 1532 onward, as the early conquistadors (among other colonial actors) queried Andean cord keepers to obtain demographic and other information soon after their arrival. Recent research has shown just how long cord-keeping persisted from that time, yielding the current estimate of 1,000 years, or roughly from 950–1950 AD, for the minimum period of active khipu use in the Andes (Medrano 2021b). This grand period of three-dimensional inscription would witness many cord traditions, circumscribed by a variety of social, economic, and political factors.

In practice, these traditions have often been understood as numbering between four and five, with Wari khipus giving way to Inka khipus, which, following the conquest, gave way to both colonial khipus (between 1532 and 1821 AD) and the subsequent Republican, or “modern” khipus observed since 1821, the year of Peru’s declaration of independence. For the purposes of this talk, suffice it to say that, while khipu use reached its apex during Inka times, this represents only about 15% of the suspected functional lifespan of the Andean khipu.

Given the significant variation in khipus during this interval, here I will only focus on a specific subset of the many khipu traditions, which I will describe using the synonymous terms “canonical” and “Inka-style” khipu. The former, because khipus of the general structure shown in this diagram have become the prototypical image of a khipu in the popular imagination. And the latter, because among the khipus that share this structure, many have been radiocarbon dated to the Inka period. More specifically, an Inka-style khipu consists of a thick, horizontal primary cord, from which hang between one and over 1,000 thinner pendant cords. Vertical top cords may emanate in a direction opposite to the pendant cords, sometimes recording the sum of the numerical values registered in the knotted pendants with which they are associated. Both pendant and top cords can have attached subsidiary cords, which sometimes have their own sub-subsidiary cords, and so on (Ascher & Ascher 1981).

These together define the structural blueprint of a canonical khipu, but only one element gives a khipu its name: its knots. (Khipu means “knot” in Quechua, the Andean language family.) Arranged in horizontal levels like in the diagram, a khipu registered the number one with a so-called figure-eight knot, the values two through nine with a long knot of the corresponding number of turns, and place values of 10 or higher with simple, overhand knots. As far as we know, the Inkas lacked a unique sign for zero, other than to leave the corresponding cord or place value blank—that is, unknotted. In the canonical form, each hanging pendant cord registered a single numerical value, which would allow us to define the *lower bound* of an Inka-style khipu’s contents as a collection of numbers. So, of a pendant cord registering the number 2,035, we would expect to see two overhand knots in the thousands’ place (closest to the horizontal primary cord), followed by no knots in the hundreds place, three overhand knots in the tens’ place, and a long knot with five turns in the ones’ place. The canonical decimal hierarchy is as simple as that.

And yet, since the early 20th century, this is the only element of a khipu that can be reliably interpreted across most Inka-style examples. A Brooklyn-based historian of mathematics named L. Leland Locke, himself a collector of historical calculating instruments, deciphered the meaning of the three types of knots, publishing the result in a 1912 paper (Locke 1912). In studying the dozens of khipus held in the collections of the American Museum of Natural History (AMNH), in Manhattan, he highlighted one in which the top cords’ numerical values equalled the sum of the values registered on the individual pendant cords through which each was interlaced. Locke has been likened by some khipu researchers to Champollion, of hieroglyphic fame, for this reason (e.g., Radicati di Primeglio [1991] 1992: 191). But unlike the ancient Egyptian case, here much remains to be deciphered. For example, knots are the only canonical khipu element which, as of today, we are sure represents specific numerical values. As a result, to have the words “khipu” and “mathematics” in the same sentence cannot escape reference to a khipu’s knots.

The century of study since the work of Locke in large part represents the history of academic research on khipus. But in the time since his findings in the AMNH, and in keeping with the wordplay that gives this talk its title, it has become ever clearer that a khipu is “knot” just numbers. We have identified variation in the spin and ply of each pendant cord, which refers to the direction in which the component fibres are twisted and re-twisted, producing an oblique axis oriented either like the backbone of a capital “S” or “Z”; in long knot directionality, such that the backbone of each is Z- or S-oriented; in attachment knot direction, wherein each pendant cord is attached to the horizontal primary cord in either a recto or verso fashion; among others.

Further expanding this complexity is a khipu’s colours, which are often many. Carrie Brezine, an American mathematician and weaver, produced the colour chart displayed on the slide, which includes over 60 potential options for Inka-style cord colours. Notably, these colours can appear in various schemas, such that an individual pendant can be solid, mottled (in which two or more colours appear randomly interlaced), barber-pole (so-named for its visual similarity), or colour-change (in which, as is visible in the image, a pendant cord changes colour layout somewhere along its length, whether it be from solid to solid, mottled to solid, barber-pole to solid, or something else).

It seems undeniable, then, that khipus represented information through some combination of the elements I have surveyed, if not more. Khipu users would have been able to convey meaning from running their hands along the cords, aided by the khipu’s own internal spatial and tactile relationships, but exactly how they did so remains unclear. To conclude this introduction, then, we might safely call a canonical or Inka-style khipu a collection of elements that registered both numerical and nonnumerical information. Numerical, in the form of three types of hierarchically ordered knots, and nonnumerical, through some combination of categorical variables like colour, cord twist, attachment knot, knot directionality, and so on. Crucially, this is not to say that all these elements were in use at the same time on every khipu of this style. Nor is it to suggest that these are all genuine data-bearing loci—some variation in these elements between different khipus may ultimately be revealed as idiosyncratic. But it will hopefully serve us as we now move to discuss khipus with particularly notable properties—which, given our current state of knowledge, has before taken the form of identifying striking mathematical relationships. The first examples of these are to be found in what I have termed archaeological khipus, which will form the entry point to our exploration of mathematics in knotted strings.

I. Archaeological khipus

Recall that archaeological khipus are here taken to mean those for which we have clear provenience. The picture on the screen, for example, shows a cache of khipus recovered from a deerskin pouch at Pachacamac, a prehispanic coastal pilgrimage site about 30 kilometres south of Lima, in 1976. These khipus have been posited as potentially recording information relevant to local ritual offerings, given the details of the site to which they pertain.

Similarly, in the following two case studies, knowledge of the location of discovery has aided efforts in interpreting the meaning of the khipus' specific mathematical properties. But what subset of mathematical practice is reflected in each? Two types will be discussed here: hierarchically ranked sums across khipus, which I term vertical summation, as well as arithmetic relevant to accounting.

A. Vertical summation at Puruchuco

This first example constitutes one of the most striking instances of numerical interaction between khipus discovered in close physical association. Just to the northeast of Lima, on the bank of the Rímac River, lies the archaeological site of Puruchuco, which is thought to have served as a centre of local Inka administration. Some 60 years ago, the team excavating the site discovered an urn under the floor of a surrounding building, which enclosed within it over 20 relatively well-preserved khipus of various sizes (Brezine & Villacorta 2020). Six of these in particular have informed an analysis, published most recently by Carrie Brezine and Luis Felipe Villacorta (2020), that demonstrates a type of vertical aggregation and partition that organizes the khipus into a three-level hierarchy. In addition, the khipus grouped at each horizontal level of the hierarchy demonstrate identical or closely matching numerical values and colour patterning, which may have served as a possible checks-and-balances mechanism.

It is apparent that what we are seeing here is the summation and partitioning of numerical values in a vertically ordered way, which has before been hypothesized to be evidence of how information was transmitted within the Inka-era administrative hierarchy. In short, numerical information from local accounts (Level I, at the bottom of the screen) would have been summed within Level II khipus, which themselves would have been synthesized in the higher-ordered Level III khipus that were transmitted regionally—that is, potentially outside of Puruchuco itself. Certain knots toward the left-hand sides of the Level II and III khipus have even been described as possible nonnumerical place identifiers, signing "Puruchuco," that may have allowed for the identification, by Inka administrators, of the khipus' destination or origin (Brezine & Villacorta 2020: 66).

As for the more specific conditions of their use, these are not fully clear. The American researcher Carol Mackey (1970: 66) hypothesized in her initial study of the cache that the cords belonged to a khipu master who lived in the small dwelling outside of the main palace at the site. If so, such an individual may have interfaced with higher-ranking Inka overseers. However, it seems that only the study of other "archives" of khipus discovered at known Inka sites will allow for more definitive conclusions in this regard.

B. Accounting arithmetic at Inkawasi

The second example takes us further south along the Peruvian coast, to the Cañete Valley. It is here that we find the site of Inkawasi, which, per the Spanish chroniclers, is said to have been a staging area for Inka military campaigns on the southern coast. Its associated storehouse facilities have also been the source of dozens of the most recently excavated archaeological khipus known today.

Specifically, in the 2013–2014 field season, the Peruvian archaeologist Alejandro Chu recovered a total of 34 khipus from the storage area designated "Sector A." Subsequent analysis of the available samples has informed the hypothesis that the khipus discovered in this sector were registers of the reception and storage of goods arriving at the site. This interpretation is supported by the contexts of discovery: several of the excavated khipus were found in direct association with food crops, including chili peppers, black beans, and peanuts (Chu 2018). I will focus briefly on two of these, called “UR267A” and “UR267B,” a picture of which is in the upper-right corner of the screen. In both of these khipus, though a numerical excerpt only from UR267A is used here for reference, a repeating sequence of apparently arithmetic operations is registered in three-cord sequences of the form $y – n = x$, where n is the constant value 15. So, for example, we see in cords 11, 12, and 13 of UR267A the equation 206 – 15 = 191. In addition, two other khipus found in close proximity register some of the same numerical values, but instead using the inverted form $y – x = n$, with the fixed n recorded as a repeating result (Clindaniel 2019: chap. 4).

Two phenomena seem to be at play. The first is the recording of arithmetic relevant to an Inka accounting context. There has been disagreement regarding what the fixed values of 15 mean, for example, with proposals of everything from a type of “tax” on incoming products to support Inkawasi’s managing personnel, to a fixed amount of seed set aside for planting (Urton & Chu 2019). But suffice it to say that these khipus provide contextual evidence for the types of numerical operations that would have been relevant to the maintenance of the facility’s storehouse.

The second property is that of the “paired” or “matching” khipus. The repetition of values in complementary arrangements—that is, the recording of $y – n = x$ on some khipus but $y – x = n$ on others—has been explored by Jon Clindaniel in his doctoral research (2019), and in a couple conference talks since then (e.g., 2021a), as a physical manifestation of net credit and net debit calculations. And a new set of 23 khipus discovered in “Sector B” by Clindaniel and Chu in 2016, which sometimes record much larger numerical values, have been pointed to as possible registers of more permanent, archival style repositories of data relevant to the site’s operation (Clindaniel 2019: 54). In any case, further work on this extraordinary corpus will someday allow for more specific statements about accounting practices at Inkawasi.

II. “Decontextualized” khipus

The second set of case studies I will cover in this section pertains to what I have termed decontextualized khipus—here referring to examples for which we lack specific information about provenience. Our ability to discuss these khipus, many of which today rest silently in museums and private collections, is in large part thanks to the work of previous scholars who called for and implemented the large-scale cataloguing of such examples. A brief survey of these activities will help in contextualizing what follows.

In 1897, the German archaeologist Max Uhle, a foundational figure in the development of Peruvian archaeology, wrote in his report on a modern khipu used near Lake Titicaca that museums possessing khipus ought to publish them. “We would then clearly see what details remain to be explained,” Uhle (1897: 63) suggested, such that “further inquiries could be proceeded with accordingly.”

The ensuing decades would witness a moderate, albeit patchy, response to Uhle’s call. Locke would follow his 1912 article on the khipu in the American Museum of Natural History with a monograph in 1923, which included, among over a dozen images of khipus, a preliminary inventory listing the locations of over 40 of them in Peru, England, France, Germany, and the United States (Locke 1923: 8). Erland Nördenskiold, a Swedish archaeologist, published two studies in 1925 claiming that repeated appearances of the number “7” in a small sample of khipus were proof of its magical nature for Andean peoples, the khipus, for him, reduced to defences against evil spirits and registers of the synodic orbits of various celestial bodies (Nördenskiold 1925a, 1925b). However, more serious investigations would also emerge, including those of khipus in private collections. They would often yield photographs, like the one taken by an Argentinian researcher in 1937 that is shown on the screen (Altieri 1937: 5), that are the last known images of individual specimens, which have since been lost or otherwise consigned to obscurity.

However, to do justice to the history of the large-scale cataloguing of khipus will require paying homage to Marcia Ascher and Robert Ascher, she an Ithaca College mathematician and he a Cornell anthropologist, who throughout the 1970s and 1980s, grew the known inventory of khipus in worldwide collections from just over 70 to about 400 examples (Ascher & Ascher 1972: 28; M. Ascher 2005: 99). In the photographs on the screen, you can see them both holding one of the largest khipus ever found, originally discovered by a Chilean archaeologist named Percy Dauelsberg in the country’s dry north.

The fruits of the Aschers’ labour in this regard would be two massive databooks, originally stored on microfiche at the Cornell University Library Archives, with cord-by-cord descriptions of over 200 khipus using the general form shown on this slide. Careful measurements were taken of everything from the khipu’s component colours to the groupings of pendant cords along the main cord, to information about the lengths, numerical values, and subsidiaries of individual pendant strings. Each khipu’s entry was supplemented with observations about its notable mathematical properties, if any, which were written using abstract mathematical notation. These mathematical “relations,” as the Aschers called them, will be revisited later in this talk. In any case, thanks to digitization efforts at Cornell, both Ascher databooks, as well as many of their photographs, are today available for download at https://courses.cit.cornell.edu/quipu/.

Without a doubt, the move toward the digital cataloguing of khipus that has taken hold in the ensuing decades has drawn inspiration from the Aschers' own persistence. Some of these khipus have since been visited and studied by other khipu researchers, including myself, with the resulting data often digitized and entered into one of the various database projects that are today hosted in both Peru and the United States (e.g., Adawi Schreiber et al. 2011). The largest of these, which was designed and implemented by Carrie Brezine at Harvard in 2002, is today maintained at the University of Chicago (Clindaniel 2021b) by the American archaeologist Jon Clindaniel. In the example on the screen, the first 13 pendant cords of the left-hand khipu are reproduced in an Excel spreadsheet, with detailed information about each's spin, ply, attachment knot direction, numerical knots, length, and colour registered in additional columns. Of course, not all khipus in museum storage depots are complete or prepared for viewing. Detached or otherwise fragmentary cordage often accompanies a museum's khipus, which, given its often tangled and fragile state, precludes its cataloguing and digitization.

However, for hundreds of khipus more, researchers have been able to log their locations. As part of a new book on the state of khipu studies that has come out just last week (Medrano 2021b), I have prepared a comprehensive survey of khipu cataloguing efforts over the last 100 years, which is summarized in the graph on the screen. The solid line represents the aggregate number of khipus inventoried in museums and private collections over this time, while the dashed line shows the number of those khipus that have been digitized and entered into an existing khipu database project. This re-evaluation of dozens of published and unpublished sources has raised the known total of khipus in global collections to at least 1,386, with 866 of these maintained digitally in an existing database. This new estimate represents a 50% increase in the global inventory of khipus since the last attempt, published a few years ago (Urton 2017: 261–64). These updated data can then serve as inputs to different types of visualizations, including dynamic maps of historical cataloguing efforts, as well as static maps showing where we stand today.

In any case, having a clear record of khipu quantities and locations is essential, not least because many of these examples are decontextualized and merit closer examination. In the following three short case studies, I will survey some of the work that my predecessors and I have been conducting on this subset of the global khipu corpus, including the identification of abstract mathematical relationships, seemingly geometric information, and possible “Rosetta” khipus—that is, khipus that correspond to a surviving written source.

A. Ascher mathematical relations

The first of these has been introduced previously: what Marcia and Robert Ascher called a khipu's mathematical “relations.” In their databook entries for the hundreds of examples that they studied in the Americas and Europe, the Aschers appended notes describing a variety of striking numerical properties that they had observed. They described them using a variety of terms, including *cross-categorization and summation*, *sub-charts*, and *hierarchical categorization*, among others. Take, for just one brief example, the following khipu in the collections of the Berlin Ethnological Museum. I will focus here on its first six groups, each of which has 10 pendant cords (yielding 60 total pendant cords).

Three properties are apparent, as noted by the Aschers. First, the 60 pendant cords consist of repetitions of only four numerical values: 11, 12, 20, and 21. Second, the sum of the 60 pendant cords, correcting for one missing pendant, is exactly 1,000. And third, when the pendants are added by their index—e.g., pendant 1 (group 1) + pendant 1 (group 2) + pendant 1 (group 3) … + pendant 1 (group 6)—each sum, within a range of plus or minus one, corresponds to a simple fractional part of 1,000. The list of those 10 fractions, each corresponding to one index in the 10-pendant sized cord groups, is shown on the screen. Of course, and as the Aschers pointed out, any indexed sums like these will yield fractional parts of 1,000, when 1,000 is the sum of all the cords in question. But the simplicity of the fractions is notable, and, in the Aschers' view, can be plausibly interpreted as a trace of intentional decisions by khipu makers (Ascher & Ascher 1981: 138; see also 1978: 828).

The Aschers would go on to identify even more complex properties, some of which involving apparently multiplicative relationships knotted within individual khipus. Toward the end of the 1981 book in which they reported their findings, the Aschers (1981: 151–52) concluded the following with respect to the “body of arithmetic ideas used by the Incas”: in their view, it included, “at a minimum, addition, division into equal parts, division into simple unequal fractional parts, division into proportional parts, multiplication of integers by integers, and multiplication of integers by fractions." The years to come will be important in evaluating the plausibility of each of these claims. But suffice it to say that observations like these place the onus on researchers to investigate the range of possible contexts in which such varieties of mathematical practice would have been useful.

B. Pereyra's "geometric" khipus

A second instance of mathematical properties that I will cover here will take us back 25 years, to the research of a Peruvian engineer and statistician named Hugo Pereyra Sánchez. In 1996, Pereyra analysed two khipus, supposedly from Peru's southern coast, but deposited in the collections of the Berlin Ethnological Museum in 1907. The Aschers had examined these two khipus in the 1970s and had assigned them the labels "AS120" and "AS143," with the "AS" prefix standing for "Ascher." In their initial study of these two khipus, the Aschers noted three striking properties: the first being that in both khipus, each pendant cord in its first group registers the sum of the corresponding cord index in several subsequent groups. That is, pendant 1 in group 1 is equal to the sum of the first pendants in groups 2, 3, 4, etc. Pendant 2 in group 1 is equal to the sum of the second pendants in groups 2, 3, 4, etc. In addition, in AS120 specifically, the ratios of the associated values were consistent within about 1% of deviation. That is, cord 2 in group 2 divided by cord 2 in group 1 equals about 0.34; cord 3 in group 2 divided by cord 3 in group 1 equals about 0.43; etc. Finally, one pendant cord in AS143, recording the value 97,357, was posited to be the largest single value on any of the first 200 khipus the Aschers had studied in global collections (Ascher & Ascher 1978: 799, 923–24).

Let us focus on AS120 for a moment. The numerical values for each of its four groups of cords are shown in the table, with each row written in the order of the cords in each group. What Pereyra noticed was that the constant ratios between pendant cords are not just limited to the pairs of groups the Aschers had identified, but, in fact, the quotient of any two groups of cords in this khipu produces a constant ratio. Most strikingly, the correlation coefficient for any pair of cord groups in this khipu is almost exactly one—they produce a near-perfect linear relationship (Pereyra Sánchez 1996: 196). Pereyra even produced a geometric model in which the slopes of the regression lines could implicitly define the interior angles of similar right triangles, with the lengths of the legs defined by (1) the values of each pendant and (2) the corresponding sum of the indexed pendant values.

What are we to think of this property? Did Andean khipu users in the pre-Columbian and/or colonial Andes utilize khipus for the measurement of geometric or angular information? How are we to explain the proportionality of these khipus' values, if the standard decimal knot system does not appear to leave room for registering fractions? Pereyra's (1996: 202) conclusion was that "the Inkas had the capacity to enact calculations in which these types of values played part." This seems to be a good start. And at least one other scholar of whom I am aware has suggested the possibility that khipus recorded proportional ratios relevant to building construction (Lee 1997). But furthering our understanding of mathematical khipus, to the extent such a label is even a valid one, will ultimately require analysing a larger sample to search for numerical properties on a grand scale.

C. Comparisons with written sources

A final usage of numerical khipu information that I will discuss here is the possibility of finding correspondences between existing khipus and surviving written sources. The identification of a "match" would raise the possibility of a so-called "Rosetta" khipu—one or more specimens that allow us to identify how specific information was registered in their cords, if not those of other khipus as well. In theory, any Inka-style khipu (recall that Inka-style khipus are those for which we know how to interpret numerical knots) might be a plausible candidate for correlation with a written source. However, which documents might serve as a missing link? Given our state of knowledge with regard to canonical khipus, written numerical records remain our clearest entry point to searching for exact correspondences. Fortunately, many early colonial documents are rife with numbers, and, in a subset of these, we know the written information to have been derived directly from one or more khipus.

Such khipu transcriptions, or "paper khipus," are among the most striking creations of intercultural encounter in the early-colonial Americas. For from the early years of the conquest, Spanish overseers elicited information from native Andean cord-keepers, which informed the allocation of the first *encomiendas* or native tributary grants; the writing of histories of the Inkas; the setting of Spanish tribute levels in line with Inka-era standards; and the adjudication of various legal and administrative disputes (Brokaw 2010: 200). During the associated khipu "readings," an Andean cord-keeper would vocalize the contents of one or more khipus, an interpreter would translate the rendering to Spanish, and a scribe would record the result. The surviving khipu transcriptions, most of which pertain to the legal sphere, are often extremely complex. Perhaps the most famous khipu transcriptions, registered as part of a 1560s proof-of-merit claim lodged by the central Andean Huanca ethnic group, enumerate over a decade of transfers to various Spanish visitors to the Mantaro Valley. These included everything from camelids to salt, fish, and maize beer called chicha (Murra 1975; see also Chirinos Rivera 2010). Even finer subdivisions appear to have been noted in the Huanca khipus: for many goods in the account, the cord keepers distinguished between those that were delivered to the Spanish visitors and those that were pillaged (*rancheado*) or stolen by them. How such a distinction would, or could, have been noted in the original knotted accounts is unclear, though we might be tempted to speculate that this was indicated by some binary aspect of the khipus in question, like spin and ply or attachment knot, though of course other elements are possible.

We are fortunate, thanks to the efforts of previous scholars to transcribe myriad archival documents, to have access to published versions of many khipu transcriptions, the largest of which being the *Textos Andinos* corpus compiled by Martti Pärssinen and Jukka Kiviharju (2004, 2010). My recent research on this corpus has involved digitizing about 10,000 lines of khipu transcriptions as originally penned and encoding the result in a machine-readable markup language called XML. In addition to enabling detailed linguistic studies of khipu transcriptions (Medrano 2021a), these digitized texts, like the digitized khipus, allow us to visualize them in novel ways. For example, the map on the slide is the corollary to the maps of khipu locations in museums and private collections that I have already shown. In this case, we see the locations described in 72 khipu transcriptions, grouped by the number of transcriptions per place. Immediately, it is apparent that the corpus of transcribed khipus represents a great variety of locations and climatic zones. This is in some contrast to the overwhelmingly coastal concentration of khipus with known archaeological provenance.

Though I will not go into further depth in this presentation regarding the linguistic aspects of these documents, suffice it to say that they can serve as valuable proxies for gauging transformations that occurred in khipu recording practices following the Spanish conquest, especially in assessing the transition from Inka-era tribute labour to early colonial market exchange.

In addition, the numbers registered in khipu transcriptions represent, as of today, the clearest point of intersection in the search for potential khipu-document “matches,” however broadly defined. In this regard, I would highlight a fantastic analysis by Carrie Brezine, who has used computational methods to compare the distributions of numerical values in extant khipu samples to those of household-level census figures recorded by Spanish observers in the sixteenth century. Brezine’s analysis (published in Urton 2006), some examples of which you can see on the screen, takes as its input several known census documents—each of which listing the number of members of each household surveyed in each location—and treats them as “exemplars” or “templates” for the identification of khipus with similar collections of numbers. Subsequent statistical testing revealed a handful of khipus whose numerical distributions are strikingly similar to those contained in the censuses, enabling their provisional classification as “demographic” khipus which subdivided an area’s or community’s population into a set of distinct social units. The analysis has even identified certain decontextualized khipus in museums and private collections as plausible registers of demographic decline in the decades following the Spanish conquest. Increasingly, it seems that to speak of a “time series khipu” may someday emerge as a realistic possibility.

Looking forward

With that, I hope that this introductory survey has placed our current state of knowledge about khipus within its historical context, especially regarding the mathematical practices of users of Inka-style khipus. We have taken account of various khipu traditions, dispelling the myth of the khipu as an Inka invention and considering its 1,000-year-plus lifespan of active use. A series of case studies have zoomed in on specific mathematical practices, from summation/partitioning and arithmetic at Puruchuco and Inkawasi, in the case of khipus with clear archaeological provenance, to the more abstract multiplicative and geometric possibilities identified by the Aschers and Hugo Pereyra among decontextualized khipus in museums and private collections. Finally, we zoomed out to examine searches for potential “Rosetta” khipus among almost 1,400 extant samples, considering Carrie Brezine’s and my own work with digitized khipu transcriptions.

Where does this leave us, then, in our ongoing efforts to understand the mathematical aspects of Inka-style khipu use? By way of conclusion, I will offer three questions that remain little-treated in the field, but that might nonetheless serve khipu decipherment efforts going forward.

1. **Did khipus record fractions? If so, how?** The Aschers (1981: 135) claim not to have found

evidence for fractional values represented on khipus. However, early colonial transcriptions of khipus sometimes note half-units of various items reported by Andean informants (e.g., 7½ loads of quinoa; Assadourian 1998: 66). Assuming that the canonical, base-10 knot system was in use in the corresponding khipus, how might such quantities have been registered originally, if at all? Were halves or other non-integer values signed by specific knot arrangements, or were other elements of the khipu—e.g., subsidiaries or bright threads interlaced with the individual pendant—used to distinguish them? A similar question arises in terms of khipus that recorded price information. Currency, which was only introduced to the Andes following the Spanish conquest, encompassed various denominations that may have required local conversion. Whether this necessitated the recording of fractional price values, or at least their manipulation during tribute-related calculations, remains unclear. A clearer understanding of fractional values may help to reveal aspects of the decimal numerical knots deciphered by Leland Locke that enabled more complexity than otherwise thought.

2. **What were the largest numbers recorded on khipus?** We cannot state with certainty the maximum order of magnitude that was knotted at the apex of khipu use. Among extant samples, numbers as large as the hundreds of thousands have before been identified—quantities that fall in line with some of the largest groupings of people subject to the administrative structure of the Inkas. But we have to remember that khipus recorded information about more than just people, including accounts of staple goods, for example, that were amassed in highly variable quantities. Frank Salomon has identified a quantity of 1,001,630 fish which members of a central Andean village reported having given to their curate around the turn of the 17th century, for example. "If this is a khipu datum," Salomon (2004: 119) notes, "it corresponds to a very high-capacity khipu." Important work in this realm has been done by Sabine Hyland (2016)—she has proposed a decipherment of two types of colour patterning on khipus, with one pattern corresponding to individual-level data and the other to aggregate figures. Jon Clindaniel (2019: chap. 5) has since found Hyland's decipherment of colour patterning to hold relatively well across hundreds of digitized examples. Moving forward will require assessing these patterns across different types of registered information. Hyland's decipherment emerged from a modern-era communal labour setting. Were similar patterns used to register currency or camelids in the early colony? Did quinoa or chicha command the same khipu design decisions as laborers? Only further investigation will approach an answer.
3. **What is the role of redundancy in khipu recording?** This question is the least well-formed of the three, but it is a profoundly important one for understanding numerical khipus going forward. In short, certain aspects of Inka-style khipus have been shown to provide seemingly duplicative information. For example, I have before observed khipus in museums with colours arranged in bands—that is, six light brown pendant cords, followed by six dark brown pendant cords, followed by six white. However, these same bands of cords are already set off from each other by spacing along the khipu's primary cord. In these cases, are the colours serving a purpose beyond visual embellishment for the user? If not, should they be considered redundant design decisions? Are they recording entirely different categories of information? A firmer footing with regard to redundancy might allow us to assess the interdependency of various khipu signs—to move beyond interpreting a "decontextualized" khipu as a collection of elements to instead "reading" it as a script with its own rules and standards—with a "grammar," as Clindaniel (2019) has referred to it.

The anthropologist Denise Arnold (2006: 237) has before raised the possibility of Inka-style khipus as a "mathematics incarnate." Arnold primarily uses the word in its strict etymological sense, with reference to correlations between a khipu's elements and human anatomy. I think it provides an impressively succinct way, however, for describing an even larger sphere of meaning-making in the Andes. For khipus were as much registers of numerical information as they were dynamic accounts of the administrative and social environments in which numbers operated. As far as I see it, we would be justified in considering them—to repeat the subtitle of this talk—instruments of "mathematics and more."

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