The Mathematics of Proportion in Art, Design and Nature

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For millennia, proportion has been central to our ideas of form and beauty. This lecture looks at the mathematics of proportion, including the famous golden ratio, discussing where it appears and why. It will also look at why A4 paper is the shape it is and what cookbooks have to do with the Rhind Papyrus.

Introduction

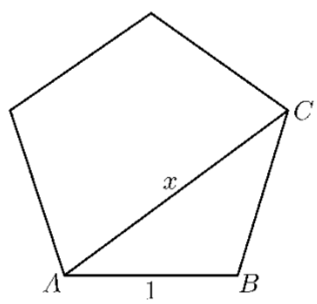
The ancient Greeks and Romans, in their thinking about beauty, talked about ideas of proportion and harmony. Aristotle (384 – 322BC) wrote in his Metaphysica: "The main species of beauty are orderly arrangement, proportion, and limitation, which are revealed in particular by mathematics." If we look at ancient temples such as the Parthenon, we do indeed see a harmony and regularity in their design, which exhibit both symmetry and proportion. Vitruvius, in his *De Architectura*, said that in the human body each part corresponds to the whole in certain fixed proportions, and that by analogy the perfect building could also be obtained by observing particular rules of proportion. Vitruvian ideas about proportion have been highly influential in Western art and architecture, and translations of his work into vernacular languages (such as, in the 17th century, English and French) made the influence more widespread.

But what is meant by "proportion"? The architect Claude Perrault (1613-1688) translated the works of Vitruvius into French, along with explanatory notes, and he explained that this idea of proportion (what Vitruvius called *symmetria* in Latin) "signifies in Latin and Greek relation only. As for example, as the relation that windows of eight foot high, have with other windows of six foot, when the one are four foot broad, and the other three."

It's this idea of looking at *ratios* and *proportions* rather than absolute sizes, that we are going to look at today, beginning with the most famous ratio of them all: the golden ratio.

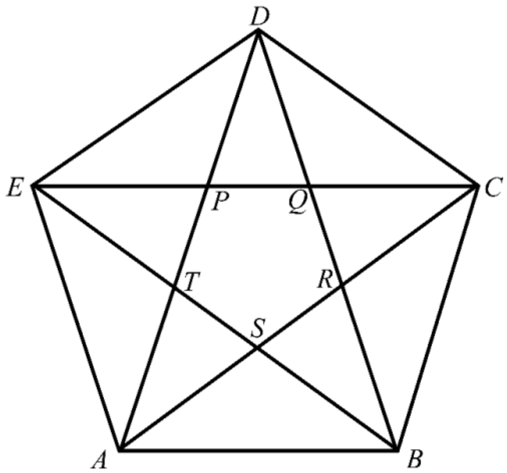
The Golden Ratio

The golden ratio, though not under that name, appears in Ancient Greek texts in the context of geometry. It crops up in several places in the works of Euclid, and much of the reason for that is its relationship with regular shapes like the regular pentagon and the platonic solids. If you want to construct a regular hexagon using straightedge and compass, it's very easy. You just draw a circle and use compasses (leaving the radius the same) to mark off points around the circumference. An equilateral triangle and a square are almost as easy. But the regular pentagon is a bit harder. The faces of the platonic solids are all either equilateral triangles, squares, or regular pentagons, so it's very important to be able to construct these if possible. The classical tools for drawing geometrical figures are the straightedge and compass. There's no guarantee that construction of a particular figure, using just these tools, is possible. It can be proved, for instance, though it takes complicated mathematics to do so, that there is no straightedge and compass construction of a regular heptagon. Imagine we want to draw a regular pentagon of side 1 (as shown on the right). Let's write x for the lengths of the diagonals of this pentagon. How can we construct the figure? We start with a straight line of unit length: this will be the side AB. Now the vertex C is going to lie on the circle with centre B and radius 1, and also on the circle with centre A and radius x. So if we know x we can find C, and from that, we can follow the same procedure to find the other vertices. It all boils down to being able to construct a line of length x.



Looking at the regular pentagon $ABCDE$, we can draw in all the diagonals and this will create a smaller regular pentagon $PQRST$. Each diagonal is divided into a longer part u and a shorter part v, marked by the point where it meets another diagonal. For example $AP = u$ and $PD = v$.

The triangles EPT and RPT are both isosceles triangles, and the angle between their equal sides is the same: it's the angle between diagonals in a regular pentagon. So ETP and RPT are similar. But they have a common side PT, so they are congruent. In particular, $RT = ET = v$.



The triangles DTR and DAB are both isosceles and have the same angle (at D) between their two pairs of equal sides. Therefore, they are similar triangles, which means their sides are in the same ratios. That is, $\frac{AD}{AB} = \frac{TD}{TR}$.

Finally, triangles CPA and CBA are both isosceles triangles with two equal sides, and the angle between these sides is the internal angle of the pentagon. So they are similar triangles. But they also share the same side AC. This means that they are congruent. From this we can see that $PC = AP = AB = BC = 1$. That is, $u = 1$, which means $v = x - 1$. We can now fill in the values in $\frac{AD}{AB} = \frac{TD}{TR}$, namely $AD = x$, $AB = 1$, $TD = 1$ and $TR = x - 1$, giving $\frac{x}{1} = \frac{1}{x-1}$. That is, $x^2 - x - 1 = 0$, or, if you prefer, $x^2 = x + 1$.

We can solve using the quadratic formula (and remembering that $x > 1$) to get

$$x = \frac{1}{2}\left(\sqrt{5} + 1\right) \approx 1.6180339887 \dots .$$

This ratio between the side and diagonals of a regular pentagon is called the golden ratio or golden number, and denoted by the Greek letter ϕ (phi). Strictly speaking, ϕ is the number, and the ratio is $1{:}\,\phi$. We also saw that $\frac{\phi}{1} = \frac{1}{\phi-1}$. This means that if any line segment is divided into a long part u and short part v such that the ratio of the long part to the whole, is $1{:}\,\phi$, then the short and long parts are in the ratio $\phi - 1{:}\,1$, which is the exact same ratio. We can also turn the equation upside down to get $\frac{1}{\phi} = \phi - 1$, which tells us that the reciprocal of ϕ is just $\phi - 1$.

If we now think of a "golden" rectangle with side lengths in the ratio 1:ϕ, and remove a square from it, then the rectangle remaining has side lengths in the ratio $\phi - 1{:}\,1$, which equals $1{:}\,\phi$. Thus we obtain a smaller version of the same rectangle. We could continue removing smaller and smaller squares indefinitely to obtain successively smaller golden rectangles.

The golden number is obviously important if you want to construct a regular pentagon. It's easy to do this with a straightedge and compass. First make a square and find the midpoint of the base. Extend the compasses from the midpoint of the base to one of the top vertices, and draw a circle with that radius, centred on the midpoint of the base. Mark the place where this circle meets the extended base line. If the square has side a, then the line created has length $a\phi$.

What we now call the Fibonacci sequence (because it was introduced to the Western world by Fibonacci in the 13th century, though it had been known to Indian, Arab, and Persian mathematicians for centuries before that) has a link to the golden ratio. The Fibonacci sequence begins

1, 1, 2, 3, 5, 8, 13, 21, 34, …

Each term, after the initial 1's, is the sum of the previous two. For example, $13 = 8 + 5$ and $21 = 13 + 8$. The next term after 34 is $21 + 34 = 55$. If F_n is the nth term of the sequence, then $F_n = F_{n-1} + F_{n-2}$.

Look what happens when we work out the ratio of consecutive terms. (For non-terminating decimals, I've given the answer to three decimal places.)

1/1	2/1	3/2	5/3	8/5	13/8	21/13	34/21	55/34	89/55	144/89	233/144	$\frac{F_n}{F_{n-1}}$
1	2	1.5	1.667	1.6	1.625	1.6154	1.619	1.618	1.618	1.618	1.618	…?

It looks very much as if the ratio $\frac{F_n}{F_{n-1}}$ is tending to a limit as n grows larger. If so, then for large enough n, the numbers $\frac{F_n}{F_{n-1}}$ and $\frac{F_{n+1}}{F_n}$ will be (approximately) equal. Now, $F_{n+1} = F_n + F_{n-1}$. Dividing through by F_n gives

$$\frac{F_{n+1}}{F_n} = 1 + \frac{F_{n-1}}{F_n}.$$

Putting $x = \frac{F_n}{F_{n-1}} \approx \frac{F_{n+1}}{F_n}$, this converts into

$$x = 1 + \frac{1}{x}.$$

Multiply through by x and we get $x^2 = x + 1$, which is precisely the same equation we solved to obtain the golden number ϕ. This explains the link between Fibonacci numbers and the golden ratio.

When Luca Pacioli, in 1509, wrote *Divina Proportione*, this (the divine proportion) was the name he gave to the golden ratio. Among its marvellous properties, he noted that a regular decagon of side 1 fits exactly inside a circle of radius ϕ (you might like to prove this), any diagonals of a regular pentagon divide each other into segments with a $1{:}\phi$ ratio (we already noticed this), and that the golden ratio is intimately connected to the icosahedron. In fact, the twelve vertices of the icosahedron form the vertices of three mutually perpendicular golden rectangles!

Luca Pacioli said that the “divine proportion” is divine because it’s unique (like God). Its irrationality – that is, the fact that it can’t be represented exactly by any fraction – is like the ultimate ineffability of God. It is defined using three numbers (the whole, the small part and the large part), resembling the Holy Trinity, and like God each of the parts resembles the whole, in that the ratios between the small and large equal the ratios between large and whole. And he gives 13 properties, not because that’s a Fibonacci number, but because there were 13 at the Last Supper. This book is also the source of a grand misconception, because it has sections on both architecture and painting, making people think that Pacioli discusses the divine proportion in these contexts too. But those sections of the book contain not a single reference to the golden ratio! The term “golden ratio” wasn’t even invented until the late 19th century. You can find it, given enough data points, and enough approximating, in lots of famous pictures, in art and architecture, and even in the human body. But mostly that’s just cherry-picking the data. The properties of ϕ are very interesting mathematically, but there’s no clear evidence that we find the golden rectangle more beautiful than other rectangles, or that it’s more prevalent than other numbers in the make-up of our bodies. The term “golden ratio” or “golden section” was first used in an 1888 book by Adolph Zeising who made all sorts of claims about its presence in art, architecture, nature, life, the universe and everything. The letter ϕ was chosen for Phidias, the architect of the Parthenon. There’s no evidence for him having incorporated the golden ratio into that temple. The “golden rule” here is that all that glisters is not gold! In the next two sections, however, I’ll show you a handful of instances where the golden ratio does make a legitimate appearance.

The Golden Ratio in Architecture

Vitruvius decrees exact proportions for the design of buildings. None of them is the golden ratio. The same goes for Pacioli. If the golden ratio is not, after all, something that was laid down by the ancients as the perfect proportion for all building, does it have a role to play in architecture? Well, it has sometimes been used consciously in design. I’ll give one example, the Modulor system of Le Corbusier, the Swiss-born architect and pioneer of mid-20th century design. With new materials like reinforced concrete he was able to free the interiors of houses from structural elements, so that they could be open-plan, with walls, and rooms, only where they would most suit the purpose of the house. A house, as he famously said, should be a “machine for living in”. He despised fancy decoration and ornament. “Decor is not necessary. Art is necessary,” he wrote. This purist style is quite thrilling but there is a lack of empathy sometimes in his attitude to existing buildings and environments. Notoriously, he wanted to bulldoze half of Paris and replace what was there with sixty-storey tower blocks.

Le Corbusier was attracted to the idea of standardization – if Henry Ford can have a production line of Model T’s, why can’t there be a modular house with parts fitting together in a modular way, with standard sizes to make this process more streamlined? He noted that the metric system is in some sense alien to human scales. The metre was defined as one ten millionth of the distance from the equator to the North Pole. This is nothing to do with human beings – a stark contrast to traditional units like the cubit, foot, hand and so on, which derive directly from our bodies. If you are designing a machine for living in, then it makes sense for your scale of measurements to relate to the inhabitants. Le Corbusier’s Modulor system is an attempt to address this. He described it as a “range of harmonious measurements to suit the human scale, universally applicable to architecture and to mechanics”. It is based on the height of a man with his arm raised, and not just any man, but one precisely 1.83m in height because “in English detective novels, the good-looking men, such as policemen, are always six feet tall”. With raised arm, the overall height measurement is 2.26m.

Le Corbusier was already making conscious use of the golden ratio in the 1920s, for example in his 1927 Villa Stein in Garches, (along with other ratios like 1:2) but it is really with Modulor that it comes to the fore in his work. The system has a series of golden rectangles of various sizes based on this standing figure – some based on half the total height – which is the position of the navel, and some based on the full height. For example, the distance from the navel to the top of the head is ϕ times the distance from the top of the head to the end of the raised arm. That 2.26m, by the way, is 89 inches, which, not by coincidence, is a Fibonacci number. It means that the height can easily be split in a golden ratio with lengths of around 55 inches and 34 inches. In fact, the very word “Modulor” hints at the explicit use of the golden ratio, because it blends the word module with “or”, the French for gold.

The idea was that this scale could be used for all measurements, from furniture to cities, and that this would lead to an overall harmony in sympathy with the human body. Le Corbusier used the system in his housing blocks, known as Unités D’Habitation. The first of these was in Marseille, and known as La Cité Radieuse, and was followed by buildings in Nantes, Berlin, and elsewhere. Photos of the interior of La Cité Radieuse show squares and golden rectangles in abundance, all following the specified Modulor measurements.

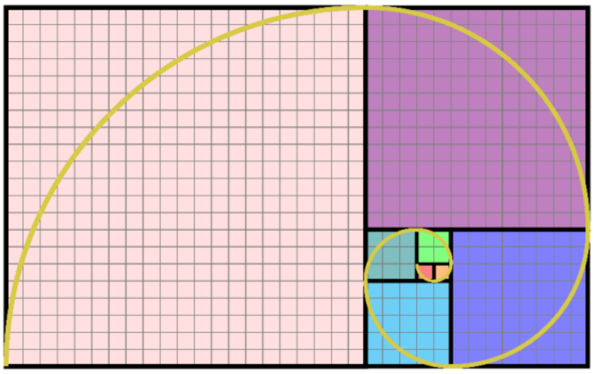
My own view is that, although the linking of ϕ to the human body is spurious, the pleasing property that removal of a square from a golden rectangle results in another smaller golden rectangle means that its use in design can indeed create appealing harmonies across different scales.

The Golden Ratio in Nature

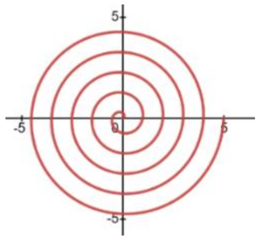
A lot of claims are made about the golden ratio and the Fibonacci numbers in nature. Most are false, but there are one or two places where they really do appear.

Spirals

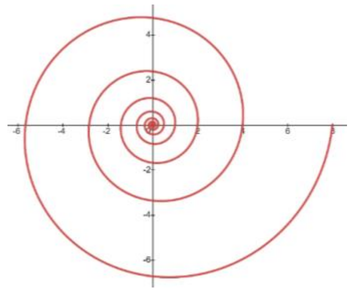
If we take a series of squares whose side lengths follow the Fibonacci sequence, then we can fit them around the initial unit square, so that they appear to make a spiraling design of squares. This works because at each stage the next square’s side length is the sum of the previous two squares’ side lengths, so its side matches up exactly with the two adjacent squares that precede it. We can then make a spiral-like curve by drawing a quarter-circle arc in each square.



This approximates a spiral, and even looks rather like the shape of some shells – in particular Nautilus shells. However, the resemblance is mostly an illusion. Firstly, there’s no reason to have this disjointed curvature that we see in the construction with squares and quarter-circles. The Fibonacci “spiral” is really an approximation to a special type of spiral, called a logarithmic spiral. These spirals were first mentioned by the German artist and engraver Albrecht Durer, and studied in great detail by the mathematician Jacob Bernoulli – he gave them the name “spira mirabilis”, or “miraculous spiral”. Unlike so-called Archimedean spirals, where the rings of the spiral are a constant distance apart, with logarithmic spirals the distance between consecutive rings forms a geometric progression – it is multiplied by a fixed amount each time.



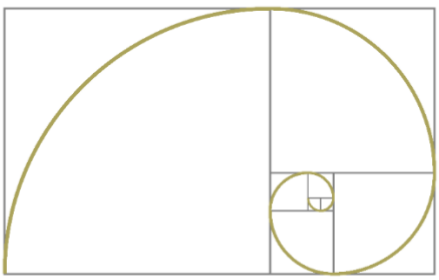
Archimedean spiral



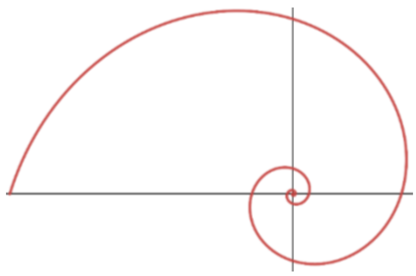
Logarithmic spiral

That is, with Archimedean spirals the distance from the origin is a *multiple* of the angle, whereas with logarithmic spirals the distance is a *power* of the angle (or conversely the angle is a logarithm of the distance, hence the name). A simple example of an Archimedean spiral is the spiral $r = 2\theta$, where r is the distance from the origin and θ is the angle travelled from the x-axis. Usually it's more convenient instead to divide by the total angle in a full revolution, so we might prefer to work with $r = 2\left(\frac{\theta}{360}\right)$ if we use degrees, or $r = 2\left(\frac{\theta}{2\pi}\right)$ if we use radians. Then the spiral crosses the x-axis at 0, 1, 2, 3, 4 and so on. Meanwhile, an example of a logarithmic spiral is $r = 2^{\theta/360}$ (if using degrees). We might call this a 2-spiral. It has the property that with every complete turn, the distance of the spiral from the origin doubles. It crosses the x-axis at 1, 2, 4, 8, 16 and so on. We can also allow the angle to be negative, and this means that the spiral can curve further inwards to every tinier distances, because, for example, $2^{-1} = \frac{1}{2}$, $2^{-2} = \frac{1}{4}$, and so on.

Our Fibonacci "spiral" is an approximation to the logarithmic ϕ^4-spiral. Every quarter-turn, the distance of the spiral from the origin increases by a factor of ϕ, so that every full revolution it has increased by ϕ^4. The spiral $r = \phi^{4\theta/360}$ (or $r = \phi^{4\theta/2\pi}$ if working in radians) intersects the positive part of the x-axis precisely at the powers of ϕ^4 (or about 6.854). If you try and work out what the Fibonacci spiral's distances from the origin are after each quarter turn, it's a bit of a fiddly calculation. It turns out that the square of the distance from the origin after n steps is the sum of the squares of the first n Fibonacci numbers. But it can be shown that $1^2 + 1^2 + 2^2 + 3^2 + \cdots + F_n^2 = F_n F_{n+1}$. This means that the distance after n steps is $\sqrt{F_n F_{n+1}}$. Compared to the distance after $n-1$ steps, we get a ratio of $\frac{\sqrt{F_n F_{n+1}}}{\sqrt{F_{n-1} F_n}} = \sqrt{\frac{F_n}{F_{n-1}} \times \frac{F_{n+1}}{F_n}} \approx \sqrt{\phi \times \phi} \approx \phi$. And that's why the Fibonacci spiral looks very like the ϕ^4 –spiral.



Fibonacci "spiral"



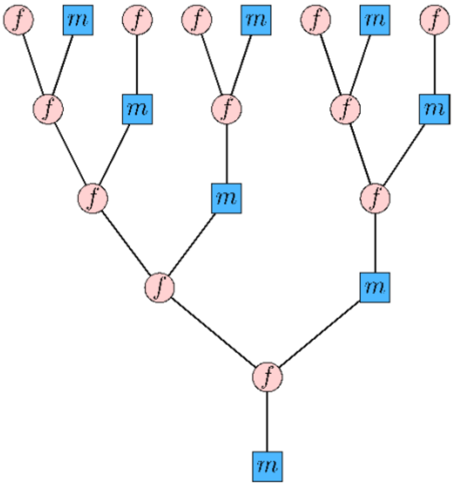
ϕ^4 –spiral

All logarithmic spirals have a lovely self-similarity property, that you can zoom in or out as far as you like and see precisely the same shape (if we allow the full range of angle values from as large and negative as we like to as large and positive as we like). In nature, if we think of how plants and animals grow, if they are growing out from a central point at a fixed rate, as happens with a Nautilus shell, then the outer parts continue to grow while they expand out from the centre. Logarithmic spirals allow for this to happen while keeping precisely the same shape. The spiraling makes room for new growth. With the nautilus shell, that's where it ends. There's no reason for a particular choice of power in the equation. Indeed, people have tested various shells and found k-spirals for a range of values of k. However, if we are looking at plant growth, additional factors come into play. For example with sunflowers, it's important to get the greatest number of seeds into the flower head, and to spread them efficiently to maximise the amount of sunlight each one gets. If the angle of growth were something like $90°$, then every fourth seed would be trying to fill a position with a seed already in it. Here, a ϕ-spiral works well, because it turns out ϕ is in some sense the hardest number to approximate with a fraction, which means that whole numbers of rotations, and hence overlaps, are minimised. This was discussed by Chris Budd in his lecture on mathematical myths, so I won't say any more on that here.

Population Proportions

One place we see proportions and ratios in nature is in populations. The ratio of predators to prey, for example, in an ecosystem, is not going to be 1:1. Within an animal population, we may observe a relatively stable proportion of juveniles to adults over time, if there are no environmental changes. On an even more basic level, about half of humans are biologically male and half are female. This makes sense when you think that every baby has one male and one female parent. Half of our ancestors are biologically male, half male. So the most efficient ratio of males to females in the bit of the population that's at reproductive age is 1:1. But there are some species where reproduction works differently, and this leads to different outcomes. In bees, there is a phenomenon called haplodiploidy. Male honeybees (drones) hatch from unfertilized eggs – they have half the chromosomes of their mother, and they have no father. So a male bee just has one parent, a female. Female bees hatch from fertilized eggs, and so they have both a mother and a father.

We can work out the number of bees of each sex there are in each generation of ancestors of a single bee. A drone's family tree looks like this (male bees are marked m, females are marked f).



$n = 5$ generations back

$n = 4$ generations back

$n = 3$ generations back

$n = 2$ generations back (grandparents)

$n = 1$ generations back (parent)

$n = 0$ generations back

You can see there are lots of Fibonacci numbers going on. The number of female bees and the number of male bees in each generation seems to be a Fibonacci number, as is the total number of ancestors in each generation. To see why, write A_n for the number of ancestors n generations back, $male(n)$ for the number of male bees and $female(n)$ for the number of female bees in that generation. Only the female bees in generation $n-1$ have a male parent, so $male(n) = female(n-1)$. But *every* bee in generation $n-1$ has a female parent, so $female(n) = A_{n-1}$. We can combine these two observations:

$$male(n) = female(n-1) = A_{n-2}.$$

And $A_n = male(n) + female(n) = A_{n-2} + A_{n-1}$. That is,

$$A_n = A_{n-2} + A_{n-1}.$$

But this is exactly the same recursive relationship that holds for Fibonacci sequence! So the n^{th} generation of the bee family tree has a Fibonacci number of ancestors in it! Moreover, the proportion of male bees to female bees is $A_{n-2}: A_{n-1}$. We know that the ratio of successive Fibonacci numbers approaches the golden ratio as n gets larger. Therefore, the proportion of male to female bees in the family tree is (roughly) the golden ratio. Just as our reproductive system, with 1:1 male to female family trees, lends itself to a population that's roughly 1:1 male to female, with bees, the $1:\phi$ family tree ratio results in a bee population where the proportion of male to female bees is $1:\phi$. Amazing!

Paper and Other Rectangles

The Golden Ratio or golden number is not the shape of standard paper. A standard piece of paper that we use for printing is A4 paper, and this is part of a range of sizes, A3 is bigger, A5 is smaller, A6 smaller still. These paper sizes are an international standard, in use across the world – though America still hangs onto its Letter size paper. By definition, A4 paper is 210mm wide by 297mm high. This seems rather idiosyncratic, why not 20cm by 30cm or something? Is it a conversion from some old measurement in inches? No – it has always been defined in metric. Given the topic of this lecture though, the first question to ask is why this shape: why are the length and width in this ratio? And what is the ratio? The answer is to do with a very useful property of the "A" series paper. Way back in 1786, the German scientist Georg Christoph Lichtenberg (1 July 1742 – 24 February 1799) proposed a standardized paper system where each size has the property that if you fold or cut it in half, you reproduce paper the same shape but half the area. This property is very useful because it means you can create smaller pieces in the series by folding and cutting, starting with the largest size, with no wastage. Starting with a single large sheet or folio, you can fold once to make two sheets, twice to make four (quarto), again to make eight (octavo) and so on. In modern times this has proved particularly useful now that we all have printers and photocopiers, because we can shrink and expand things, make A5 booklets out of A4 paper, the width of each size equals the length of the next size down, which reduces the number of different paper tray dimensions needed, and so on. We can work out exactly what the proportion should be in order to get this effect. If the short side is 1 and the long side is x, then when we cut the rectangle in half along its long side, we get a smaller rectangle whose short side is $\frac{1}{2}x$ and whose long side is 1. To preserve the same proportions in this smaller figure, it must be the case that $\frac{1}{x} = \frac{\frac{1}{2}x}{1}$. If we rearrange this, we end up with $x^2 = 2$, so that $x = \sqrt{2}$. We can check that indeed this is the proportion of A4 paper, because $210 \times \sqrt{2} \approx 210 \times 1.41421 \approx 296.985 \approx 297$.

I do sort of miss those lovely old Imperial paper size names, like Super Royal, Grand Eagle, and Double Elephant, but the fixed proportion system is clearly a sensible one. The final question is why we have this precise measurement for A4 paper – we know what the length should be, given the width, but why that width? Well, it's all down to the French. They invented the metric system, of course, and there's a record of a "Loi sur le timbre" (a law on taxing paper), dated the 13 Brumaire, Year 7 of the French revolutionary calendar, giving paper sizes with our A series proportions, defined such that the areas of the sheets are nice fractions of a square metre. The "Moyen Papier", for example, is one eighth of a square metre. It is exactly what we now call A3. This has crystallized into our modern system which defines A0 (the largest in the A series) as paper in a 1:$\sqrt{2}$ proportioned rectangle with an area of exactly one square metre. Since the area halves each time we cut the paper in half, by the time we get to A4, the area will be 1/16th of a square metre. It's then a simple matter to find the side lengths: if the shorter side is a metres, then we have to solve $a \times a\sqrt{2} = \frac{1}{16}$, and so $a = \sqrt{\frac{1}{16\sqrt{2}}} \approx 0.2102241m \approx 210mm$, and that's where we get our 210mm width for A4 paper.

I wondered about analogues of this kind of question in higher dimensions. We don't use three-dimensional paper, but there is a kind of cuboid we do produce in standard sizes: bricks. Do they have this halving property? We don't routinely halve and quarter bricks starting from one giant brick, so that's not necessarily what we want to be achieving. For strength, we offset the layers (the proper name is courses) of bricks in a wall, and often have a double-thickness too, so at the corners it's useful to have the ends of bricks being half the length. That is, the width should be half the length. This also gives lots of scope for more interesting brick bonding – that is, the patterns made by the bricks. Bricks can be "stretchers" (laying lengthways), "headers" (widthways) or "soldiers" (upright). For additional flexibility, the height of the brick is such that if you rotate a header 90 degrees, you can fit three of these into the length. Putting all this together, we require a length that's twice the width and three times the depth. Or, almost. We have to allow space for mortar and that means we remove 10mm from each of these dimensions to get the actual dimensions of the brick. The brick "unit", including mortar, is 225mm long by 112.5mm wide by 75mm high. This is a $6{:}3{:}2$ ratio. The bricks themselves are therefore 215 mm long by 102.5 mm wide by 65 mm high.

Ratios and Recipes

If you have ever baked a cake, you will know that there's lots of mathematics going on. You need to know about proportions of ingredients for the recipe, involving volumes for the cake, surface areas for the icing, and even perimeters for the cake band. Some cookbooks have a handy table to help you work out quantities for different shapes of cake tins – which are almost always either round or square, and you'd usually fill them to the same height. This means that the volume of ingredients needed just depends on the base area of the tin. For a round tin, that area is going to depend on π. Unless you are a professional chef, you don't need to have extreme accuracy here. My trusty *Good Housekeeping* cookery book tells me that for the same quantity of cake ingredients you can either make a 9-inch round cake or an 8-inch square cake.

Over 3500 years ago, the Ancient Egyptians also needed an approximation for π. Problem 50 of the Rhind Mathematical Papyrus begins: "Method of reckoning a circular piece of land of diameter 9 khet. What is its area in land?" The method given is to "subtract one-ninth of it, namely 1; remainder 8. You are to multiply 8 eight times; it becomes 64. This is its area in land." This method is applied in other problems too. What is says is that to find the area of a circle of diameter d, you subtract $\frac{1}{9}^{\text{th}}$ of the diameter to get $\frac{8}{9}d$, and then square the result. That is, the area of a circle of diameter d is $\frac{64}{81}d^2$. We know that the real area is $\frac{\pi}{4}d^2$, meaning that the Ancient Egyptians were effectively using the approximation $\frac{256}{81}$ for π. At around 3.16 this is very close, and good enough for most practical purposes. Coming back π-and-a-bit millennia to the present day, and my 9-inch diameter round cake tin has a base area of $\frac{81\pi}{4}$ square inches, while my 8-inch square tin has an area of 64 square inches. The cookbook tells me these require the same amount of cake ingredients, and so its approximation to π is exactly the same $\frac{256}{81}$ as the Ancient Egyptian scribe all those centuries ago!

How Square is Your Peg?

We'll finish, just for fun, with a multi-dimensional proportion conundrum. We all know the saying: you can't fit a square peg in a round hole. But what's a worse fit, a square peg in a round hole, or a round peg in a square hole? Again, the answer is all about proportion. The best fit is the one that fills the greatest proportion of the hole. A circular peg of radius r in a square hole fills πr^2 out of the total hole area of $4r^2$, so a proportion of $\frac{\pi}{4}$, or roughly 79%. A square peg in a circular hole of radius r, by contrast, fills $2r^2$ out of the total hole area of πr^2, and so the proportion filled is $\frac{2}{\pi}$, or roughly 64%. This means it's better to have a round peg in a square hole than a square peg in a round hole. In 1964, the mathematician David Singmaster asked what happens in higher dimensions? What proportion of the volume of a cube is contained in a sphere that just fits inside it? (Or, if you like, a spherical peg in a cubic hole.) This isn't hard to work out if you know that a sphere of radius r has volume $\frac{4}{3}\pi r^3$, and a cube of side $2r$ has volume $8r^3$. Meanwhile if we put the biggest possible cube inside the sphere, it would have diagonal equal to the diameter of the sphere, namely $2r$. If the side of a cube is a, then the diagonal b of each face satisfies, by Pythagoras' Theorem, $b^2 = 2a^2$, and then the diagonal c of the cube satisfies $a^2 + b^2 = c^2$. This means $c^2 = 3a^2$, and so in our case $a^2 = \frac{4}{3}r^2$. The volume of the cube is a^3, which is $\left(\frac{4}{3}\right)^{3/2} r^3$. Putting it all together, a spherical peg fills $\frac{\pi}{6} \approx 52\%$ of a cubic hole, and a cubic peg fills $\left(\frac{4}{3}\right)^{3/2} \div \left(\frac{4}{3}\pi\right) = \frac{2}{\pi\sqrt{3}} \approx 37\%$ of a spherical hole. So both alternatives are worse than the 2-dimensional versions, but still the "round in square" option wins. But a curious thing happens if you go to higher dimensions. The proportions keep declining, but "round in square" is always better than "square in round", up until a certain point. At nine dimensions, it swaps over! From that point, "square in round" is better than "round in square". This is just one of the many times when mathematics can confound our expectations.

I hope that you have enjoyed our tour of the mathematics of proportion. I've suggested a few links and references below if you want to find out more.

References and Further Reading

- My predecessor Professor Chris Budd discussed some of the more overblown theories about the golden ratio in his lecture on mathematical myths, which you can watch on the Gresham YouTube channel (along with all Gresham Geometry lectures dating back many years). Links to all these are available at https://www.gresham.ac.uk/professorships/geometry-professorship/
- There's a good article on the golden ratio which discusses what Pacioli said, which is useful if, like mine, your French is marginally less dreadful than your Latin or mediaeval Italian: *Au carrefour des mathématiques, de la nature, de l'art et de l'ésotérisme: le nombre d'or*, by J Mawhin, in Rev. Questions Sci. 169 (2-3) (1998), 145-178.
- Vitruvius can be read in translation at the Project Gutenberg website. https://www.gutenberg.org/files/20239/20239-h/20239-h.htm
- Dr Ron Knott's encyclopaedic website on Fibonacci numbers is excellent: find it at www.tinyurl.com/fibandphi.
- If you want more links between mathematics and cookery, look no further than *How to Bake Pi*, by Eugenia Cheng, published by Profile Books in 2016 (ISBN 1781252882)
- You can view the Rhind Papyrus online at https://www.britishmuseum.org/collection/object/Y_EA10058

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