The Invention of Mathematical Proof in the Renaissance

Dr Richard Oosterhoff

19th October 2022

Abstract

In practice, mathematicians have been 'proving' their results in many ways, in many places, for thousands of years. In principle, however, what is a proof? Usually, we look to geometry, specifically the geometry of Euclid. But what are the fundamental building blocks of a Euclidean proof? Until quite recently, the Renaissance, this question remained open—due to uncertainties about who Euclid was, the structure of his arguments, and even the layout of his pages. This lecture looks at how the language and practices that we now associate with Euclid hardened into our dominant idea of proof in the 1570s.

Introduction: What's in a Proof?

European intellectuals have, since antiquity, seen Euclid's geometry as the archetype of reasoning, even for logic. Until relatively recently, Euclid was often interpreted in relation to philosophical texts, such as Aristotle's logical treatises (e.g. the *Posterior Analytics*). It's easy to feel like this is depends on Euclid's idea of proof, the step-wise deployment of close reasoning that accompanies each statement, throughout the *Elements*, demonstrating that the new piece of knowledge in fact flows from unshakeable truths already learned earlier. The solidity of Euclidean reasoning—its generality—depends upon this. Descartes even complained that this led some to trust geometry too much: "In geometry everyone has been taught to accept that as a rule nothing is written without there being a conclusive demonstration available; so that inexperienced students make the mistake of accepting what is false, in their desire to appear to understand it, more often than they make the mistake of rejecting what is true."[1] As Peter Dear once noted, Descartes saw his own "geometrical" argument as a matter of organising a text: Definitions, postulates, axioms or common notions, propositions, demonstrations, and corollaries.[2]

Yet this anatomy of geometry is relatively new. Some version of this had been used, in practice, but the idea that a geometrical text necessarily included *all* of these labels was actually a specific historical product—the result of a contingent group of peoples and technologies. The formulation belonged to the late sixteenth century, I shall argue. I want to draw attention to a key episode in the inventing of this sense of proof, in Europe, in the 1570s...

But I'm getting ahead of myself. First I want to set out the long view, with the medieval reception of Euclid. Second, I'll turn to the early sixteenth century, to set out the shifts that set in place our current language of proof against a pedagogical style (or aesthetic) of presenting copious proofs. Finally, third, I'll focus on how Proclus is presented in the Urbino courtly mathematician Federico Commandino, who produced an important translation of Euclid in 1572.

1. The Long View: *Mise-en-page* and Authorship

To make sense of why that moment matters, let's take a very long view on the way Euclid was actually encountered by readers. In particular, consider the *mise-en-page*, which is what historians of the book mean by how a page is organised, the 'white space' between words, the structure of paragraphs, the size and position of margins, and the nature of headers, titles, or different sizes and depth of script.

---

[1] AT 7.5, following Cottingham's translation (modified Dear 1995, 45)

[2] 'Replies to the Second Objections to the *Meditations*, AT 7.160-170.

From this vantage point, the first manuscripts we have of Euclid are quite different from the printed copies of Euclid that we might be familiar with. Let us consider two: MS D'Orville 301.[3] MS Vat. gr.190.[4] In both cases, the *mise-en-page* is slightly disorientating to a modern eye. Both are difficult to break up into smaller sense units; punctuation is missing or unusual, sentences are indistinct, and diagrams seem added into wherever the scribe could find space. The second example is particularly challenging here, though readers familiar with medieval scribal practices may sense a common technique of the time: the main text is at the centre, with commentary squeezed into the margins. There is plenty more to say here, but what we need to note is that enunciations (propositions, theorems, etc.) are not labelled or separated out; moreover, what looks like 'commentary' in this context often involves what we might know as 'proof'.

But what is the status of proof in this context? What here is Euclid, and what is simply added explanation? It's not clear from this kind of page, and the ambiguities only multiply when we understand how medieval readers encountered Euclid over the next centuries. For example, until around the twelfth century, 'Euclid' was just the propositions that were believed to originate in a translation of the late antique Christian writer and philosopher Boethius. (e.g. Bodleian Library MS. Douce 125.) These took the shape of a list of propositions, The implications for **authorship** were clear: that only the enunciations were really what Euclid wrote (or at least cared about), and only enuciations mattered for following a line of geometrical narration or argument.

It was only a brief step from this to the new editions of Euclid that circulated from the twelfth century, by Adelard of Bath, Robert of Chester, and finally Campanus. In these editions, the division of authorial responsibility was clear: 'Euclid' was the propositions or enunciations, but what we think of as 'proofs' were not only optional, but mutable. That is, one could draw on different commentary traditions to construct a proof, or even come up with one's own. For example, Campanus of Novara (13th c) supplied the most widely used and extensive version of Euclid, but many of the proofs were of his own invention.

2. Proof as Invention: *Copia*

This question of authorship set a series of pretty puzzles for Renaissance editors of Euclid. After all, Euclid was one of the great works of antiquity that intrigued ambitious printers, from the mathematician-printer Erhard Ratdolt in Venice, to Henri Estienne the Elder in Paris, and the Herwagen workshop in Basel.[5] It was challenge enough to set all the complicated diagrams; it was even more difficult to establish a reliable text. The historian of science Robert Goulding has recently showed that questions of authorship underlay many of the textual concerns that animated the generations of humanists, pedagogues, and printers who struggled to come to terms with editions of Euclid. One group of questions circled around authorship itself: Was the original Euclid actually only the enunciations? If so, who wrote the rest? What did that mean for commenting on the text? A second set of questions concerned the implications of texts that were garbled. Should one rely only on the linguistic considerations to correct the text? Or, if you thought your maths were better (or that the commentary tradition was wrong) could you intervene in the text to make the 'proofs' better? What, ultimately, is the relationship between mathematical insight and linguistic considerations?

My own addition to these question is the desirability of *copia*. Let's take the earliest editions, to show what I mean. Ratdolt 1482 was really the medieval Campanus (also reedited by Pacioli). That by Bartolomeo Zamberti (1505) was a brand new translation from Greek, claims author was Euclid of Megara (possibly on the basis of a lost Byzantine tradition)—eloquent, but often mathematically wrong. In the context of these examples, the very identity of Euclid was often uncertain; editors regularly confused ancient sources, crediting another ancient Euclid—but Euclid of Megara—as the author of the *Elements*. This was a mistake with some pedigree, and it directly related to the editorial tradition: following this point of view, the fourth-century philosopher Theon of Alexandria was thought to have written the proofs.

Given confusion over the text itself, and its author, it seemed desirable to have as many options open as possible. This resulted in a particular stream of editions of Euclid, in which the medieval version of Campanus was printed alongside the new humanist translation of Zamberti (where most, if not all, the demonstrations were thought to be Theon's): Lefèvre's 1516 ed. of Euclid.

---

[3] Copied by Stephen the Clerk for Arethas of Patras, in Constantinople in 888 AD.

[4] N. M. Swerdlow, "The Recovery of the Exact Sciences of Antiquity: Mathematics, Astronomy, Geography," Grafton (ed.) *Rome Reborn. The Vatican Library and Renaissance Culture* (1993) 128-29 & plates 101-102)

[5] The very rich publication history of Euclid in the Renaissance has been the target of several key studies. See e.g. https://readingeuclid.org/ ; http://www.sphere.univ-paris-diderot.fr/spip.php?article1065&lang=fr

- This resulted in the main stream of sixteenth-century pedagogical texts, during a season in which Euclid was becoming part of every student's education: (Oronce Fine, Jacques Peletier…all used Zamberti's text, but *modified and added their own demonstrations*, treating 'Euclid' as the narrative of enunciations)

The overriding message here is that mathematical narrative happens in the enunciations—the proof is explanatory, additional (even optional). This assumption can be traced through many other books that proclaimed themselves followered of Euclidean method or structure:

- E.g. Lefèvre, *Elementa arithmetica (Jordanus de Nemore); Elementa musicalia* (1496)
    - Blurs the line with commentary: Often shows multiple proofs, or includes other historical/philosophical information.
- E.g. Bovelles, *Geometria* in the *Compendiosa introductio* (1503).
    - Shows no proofs at all (cf. medieval 'Boethius')
- See themselves as rehabilitating the work of Boethius .
- their own intros to geometry (republished by Oronce Fine in *Margarita philosophica*)

Let's remind ourselves that, to a significant degree, what counted as 'Euclid' was a typographical or (as we said above) decision about *mise-en-page*, which reflected and fostered assumptions about who Euclid was—and what was up for debate, correction, and improvement. Assumptions about authorship then resulted in a very flexible attitude towards demonstrations/proofs. As long as a demonstration was simply a provisional 'working out' of an idea, an explanation or a gloss, it could be rephrased, eliminated, or replaced. Uncertainty about author implied uncertainty about proofs; demonstrations were beneficial, but certainly neither universal or certain!

Put another way, context here goes directly to understanding the 'style' of proof, what it is and what it can be. In the context of the early sixteenth century, following on from the major studies of Terence Cave and others, we know that intellectuals particularly valued the notion of *copia*, or abundance and variety.[6] In particular, the ideal of copia was part of the literary practices of invention, explicitly understood as the collection of ideas from various authors in order to recombine them for one's own use. 'Invention', in this sense, was not merely an effort to 'come up with something new', as we might think; it was a more subtle practice of recombining, in order to diversify; of imitating, in order to produce an abundance of ideas. The ideal of copious invention, therefore, was partly about responding to other texts, other authors, and imitating them. But the goal was something we might think of as 'maximalist', focused on amplification, abundance, *copia*, *varietas*.[7]

For mathematics, I would suggest two implications, on the basis of the examples just discussed and those they influenced. First, that the idea of *copia* opened up a big space for intuition and practice within the narrative structure of mathematical argument. (E.g. Bovelles 1511, Oronce Fine 1530s: Heavy emphasis on *construction* as a form of proof. Petrus Ramus in 1550-1560s similarly made grand statements about *practice* as proof enough.)

A second implication is that proof itself was conceived as an 'invention', in the sense humanist applied to their mode of authorship. If we look at the Paris edition of 1516, which had a tremendous influence and readership due to its ability to offer the range of possibility in Latin: it implies that there *ought* to be an original set of proofs, but that these aren't those, yet, and that the reader's task is to harmonise, make connections, and ultimately inscribe themselves within this tradition of commenting on the larger narrative structure, just like Campanus and Theon. If we turn to other examples from that editor's oeuvre, such as his Euclidisation of musical harmonics (*Elementa musicalia* 1496), we find multiple versions of proofs, multiple kinds of explanations, and the author acting as a commentator submitting them all to the reader.

By conceiving of a proof as an 'invention' of this kind, we witness a kind of attitude towards mathematics that is quite different than the ones we are accustomed to, being acclimated to the spare, algebraic modes of understanding proof that Descartes left us. This pulls in two directions: towards such spare treatment of geometrical narrative that it may drop entirely away, becoming optional—or towards copia that is, to our eyes, overwhelming, as when Lefèvre tends not to take the shortest way to proof, but presents several

---

[6] Terence Cave, *The Cornucopian Text* (1979), who takes as starting point that Erasmus' *De copia* was the preeminent textbook of the day.

[7] (This is a bit counter-intuitive in light of the 'back to the sources' narrative we normally are told of the Renaissance; instead, the commentary habits of the Renaissance, you might say, were about layering the text with more, not less.)

demonstrations, some embodying the network of 'necessity' from axioms to deductive conclusions, but others more or less linguistic in focus, or amplifying in aim.

3. Inventing Proof: Commandino and Clavius

Not until 1572 do we find a fairly standard language for describing Euclid's mode of argument, in Federico Commandino's entirely new translation of the *Elements* from the Greek. Commandino is a key figure in the humanist reception of ancient mathematical texts, as a scholar in the household of the Duke of Urbino. Whereas Lefèvre and his collaborators worked within the university context, Commandino tutored the young Duke—but his prestige came from his translations and editions of classical mathematical and engineering texts, which became standard reading among philosophers and mathematicians over the next century. (Isaac Barrow and Isaac Newton encountered the tradition of ancient mechanics, for example, in Commandino's editions.)

Commandino's 1572 edition of Euclid was particularly influential. This is already because of his preface, which offered a particularly wide-ranging defence of the power of geometry. But one part of his preface is his 'anatomy of geometry', where for the first time we meet the range of terminology used to describe the different narrative elements in Euclidean geometry: he distinguishes axioms from suppositions, postulates, 'problems', and theorems. As Commandino says, this terminology (or what I'm calling an 'anatomy of geometry') comes from Proclus, the fifth-century teacher of the Platonist school at Alexandria, who left an extensive Neoplatonic commentary on Book 1 of the *Elements*. The sixteenth-century reception of Proclus is one of the big questions in the history of mathematics, because in his commentary Proclus makes explicit the Platonic philosophy of mathematics that contains the seeds of seventeenth-century mathematical philosophy—Kepler, Galileo, and Descartes, we might say as a shorthand. There are several aspects of Proclus that were attractive and allusive to Renaissance educational reformers: the idea that all philosophy should begin on a platform of mathematical knowledge; that the mental faculty of imagination is fundamentally geometrical; or, most powerfully, that behind the appearances of all reality there is a *mathesis universalis*.

But a *mathesis univeralis* isn't what Commandino takes from Proclus in 1572. Rather he takes, in the first instance, the language of proof that differentiates different aspects of mathematical narrative. The significance of such a distinction is important to highlight; medieval figures such as Jordanus or Campanus did, in their *practice*, distinguish demonstrations of theorems from the working out of problems, but as a matter of *principle* did not have a consistent vocabulary for doing so—not least, as I have argued so far, because *all demonstrations were conceived not as a part of the text of Euclid, but rather as glosses and commentary added by others*.

Commandino's own approach differs from those earlier editions, not least because his own philological expertise—combined with the witness of Proclus—allows him to clear up problems of authorship. He points out that Euclid of Megara could not be the author of the *Elements* (because too early), and that the proofs <u>do</u> belong to Euclid, but as <u>edited</u> by Theon of Alexandria (4th century).

These insights inform the whole *mise-en-page* of Commandino's Euclid:

- Each proposition classed as a 'problema', 'theorema', etc.
- Each proposition followed by an 'authoritative' proof
- The significance of this canonical structure then is emphasised by being set off from Commandino's further commentary (which is emphatically *not* further proof).

The significance of these choices is embedded in another influential edition, that by the mathematical professor of the Jesuit Collegio Romano, Christoph Clavius. His edition of Euclid, first published in 1574 but revised and republished regularly, adopted the same interest of Commandino in distinguishing proofs or demonstrations from commentary. Where he differed is in the range and depth of commentary, which made his version particularly powerful over the following decades. Clavius is pedagogically minded, and he presents plenty of additional proofs, dialogue with the text, reflection on competing or alternative proofs, etc. But the fundamental continuity with Commandino remains: his *mise-en-page* distinguishes these from the authoritative proof of "Euclid". In fact, then, Clavius' layering of commentary does not *compete* with Euclid's proofs, but rather cements the status of "Euclid's" proof as particularly authoritative, itself deserving of further commentary.

The idea of reformulating Euclid's work with greater rigour, or more simply, never goes away: John Wallis and Girolamo Sacchari are only the most famous of many figures who contested the validity of Euclidean proofs.[8] But these are never again confused with the task of clarifying the *text* of Euclid. (The philology has been entirely separated from the mathematical insight…)

Conclusion

The implications I see are for the very idea of proof, of what an axiomatic system is, and of what doing geometry means. My claim is that this shifts in the sixteenth century, and it is because of these shifts that Descartes can take "geometrical method" to mean a particular way of organising a text—that axiomatic reasoning becomes what it does for the seventeenth century.

This is a complex story—it's probably unwise to put too much weight on a single hinge moment. But Commandino's edition of 1572 presented, in Latin, the vocabulary of axiomatic proof, with the authoritative example of Euclid. And the ensuing decades developed Commandino's approach, circling around the question of exactly how Euclidean geometry might inform or improve philology and philosophy more generally:

- The famous 'question of the certitude of mathematics', a controversy fought in Italy, especially Padua
- Followers of the iconoclastic Peter Ramus, in Paris and Germany
- Followers among the Dutch practical mathematicians, such as Adriaan van Roomen
- Henry Savile in England, against Joseph Justus Scaliger
- c. 1628 Descartes argued in his *Regulae ad directionem ingenii* for a universal mathematics

The outcome of these debates varied—but what they held in common was mutual reference to the one, pre-eminent Latin translation of Euclid by Commandino. No matter which side of these debates one held, the language was the same: all referrred to Commandino's Latin translation of Euclid as the authoritative one. The vocabulary of geometrical proof was the one Commandino had, with the help of Proclus, invented.

We could take this to be a triumphal story, in which Commandino is the hero. But I'd prefer a different lesson, one that helps us understand medieval and Renaissance mathematics in Europe better. I've tried to show how, in the medieval period, there was no consensus on these questions (such that "doing geometry" is chiefly about intuiting enunciations). This was what I've called a 'copious' style of mathematics, which allowed a wider range of definitions of geometry, and is the context in which geometry could be applied to the natural world; it also opens up fertile points of comparison, I hope we shall see in the next lectures, to doing mathematics in other cultural contexts.

[8] E.g. Girolamo Saccheri, *Euclid Vindicated from Every Blemish*, ed. Vincenzo De Risi, trans. L. Allegri and George Bruce Halsted (Cham: Birkhäuser, 2014).