GRESHAM COLLEGE
EST. 1597

Lottery-winning Mathematics

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31st January 2023

This lecture looks at games of chance like lotteries, dice, and tossing coins. Can mathematics help us win? We'll see how a Frenchman's gambling problem instigated the field of probability, why you should never buy British Lotto tickets on a Monday, and how Voltaire got rich on an 18th century lottery using a bit of simple mathematics.

The die is cast

Wherever there are people, there are games. We can find ways to turn anything – pebbles, sticks, coins – into a game to play, and often an opportunity to gamble. Among the oldest gaming pieces are dice (one die, two dice). They come in many shapes and sizes, but the basic principle is the same – the dice have a number on each face; you roll or throw your die and look at the number on the uppermost face. Dice are inexpensive to make, and very durable – you can carry them in your pocket and play with them anywhere. Because of that, gambling with dice was a very popular pursuit in the past and people would spend a lot of time and money betting on them. Most dice are cubes with the numbers 1 to 6 on them (and opposite numbers tend to sum to 7, though that doesn't affect the maths – we always assume that the dice are fair, in that every number is equally likely to come up). We'll just look at these for today, but of course you should feel free to play with other possibilities at your leisure! Assuming the die is fair, or "unbiased", we can say that there is a 1/6 probability that we will throw a 1, a 1/6 probability of a 2, and so on for each number up to 6. There are all sorts of games you can play, but here's one that was popular in France a few centuries ago. Your mate Antoine comes up to you and says I bet you 5 francs that I can throw a six; I'm allowed four tries[1]. Should you take that bet?

While you're thinking about that, let me tell you a bit about Antoine, or to give him his full name Antoine Gombaud, Chevalier de Méré (1607-1684). He gave himself the "Chevalier" title, so, while he was a gentleman, he wasn't strictly speaking a nobleman – but everyone called him that anyway after a while. He was a writer, very interested in the idea of how to be a perfect "*honnête homme*" – an urbane, refined, intellectual gentleman. But he was also a gambler who was very interested in the mathematics of calculating odds. That's why he's in this story. He asked the advice of his friend, and one of the best mathematicians in France, Blaise Pascal, to help him resolve some tricky gambling problems, one of them related to this "throw a six" game – we'll hear more about that in a moment. Pascal was born in 1623, by the way, so it's his 400th birthday this year – on June 19th, if you want to throw him a party.

So, back to the game. Remember, to win you have to throw a six, and you have four attempts. Here are the possible outcomes:

- Throw a 6 on the first throw; game ends, you win.
- Throw a non-6, then a 6; game ends, you win.
- Throw non-6, non-6, then 6; game ends, you win.
- Throw non-6, non-6, non-6, then 6; game ends, you win.
- Throw non-6 four times in a row; game ends, you lose.

[1] In fact, the bet may have been to throw an "ace" or 1, but our modern convention is that the "best" throw is a six, so I've gone for that. It doesn't affect the mathematics.

Of course we would certainly not say that because four of the five outcomes result in a win for us, that we will win four-fifths of the time! Each of these outcomes has a different probability of occurring. There's a one in six chance of throwing a 6 first time. Then a $\frac{5}{6} \times \frac{1}{6} = \frac{5}{36}$ chance of throwing "non-6 then 6". Why is this? Well when we want to find out the probability of several events all happening, and these events are independent from each other, then we can multiply the probabilities of each individual event.

If the events are not independent, this is very much not true. For instance, if we know that say 1/10th of the adult population has a beard, and approximately half the adult population are women, that does not mean that one woman in 20 has a beard. But throws of a die *are* independent. So we are OK to conclude that the chance of a non-6 followed by a 6 is $\frac{5}{6} \times \frac{1}{6} = \frac{5}{36}$.

The next way to win is "non-6, non-6, then 6", There's a $\frac{5}{6} \times \frac{5}{6} \times \frac{1}{6} = \frac{25}{216}$ chance of this happening. Finally, there's a $\frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{1}{6} = \frac{125}{1296}$ chance of throwing three non-sixes followed by a 6. So, the total chance of winning is $\frac{1}{6} + \frac{5}{36} + \frac{25}{216} + \frac{125}{1296} = \frac{671}{1296} \approx 0.52$, or about 52%. This is a rather inefficient way of doing the calculation though: a better way is to notice that of all possible dice throws, the only way to lose is the last scenario: throwing four non-6's in a row, which has a probability of $\frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} = \frac{625}{1296} \approx 0.48$. A 48% chance of losing means a 52% chance of winning.

The origins of probability theory

Antoine Gombaud had been playing the "throw a six" game and he knew he had a 52% chance of winning each time. But after a while (perhaps when people tired of losing money to him) he began playing the following variant: we instead throw two dice each time, and win as soon as we throw a double six. There's a 1 in 36 chance of throwing a double six, so it's one-sixth as likely as throwing a six with a single die. Therefore, reasoned Gombaud, if we allow not four, but six times four, or 24, attempts, then we should have the same 52% likelihood of winning, right? So, playing many times over a long period, he should on balance come out slightly ahead. Only that didn't seem to be the outcome. This was one of the problems he asked Pascal about. Pascal wrote to another mathematician, Pierre de Fermat, for his opinion, and they discussed these questions in a series of letters.

Let's deal with Gombaud's question. Do we have a 52% chance of throwing a double six in 24 attempts? When you throw a pair of dice twenty-four times aiming to get a double six, the only way to lose is to fail, 24 consecutive times, to throw a double six. The probability of failing once is $\frac{35}{36}$. The probability of failing 24 consecutive times is therefore $\left(\frac{35}{36}\right)^{24}$, which is approximately 0.51. So, we only have a 49% chance of winning this game. Over a long period, we will come out worse. It's easy to do this with the aid of a calculator, but a formidable challenge without one. Fortunately, there's a really useful shortcut we can use, based on a result called the Binomial Theorem. In the general form it tells you how to multiply out expressions like $(a + b)^n$. The "bi" in "binomial" is because there are two numbers in the bracket. In particular, it tells us that $(1 - x)^n = 1 - ux + vx^2 - wx^3 +$ [terms involving higher powers of x], and there's an easy rule to tell you what the coefficients u, v, w are (and all the subsequent terms). When x is small, in this case $\frac{1}{36}$, we can get a very good approximation to $\left(\frac{35}{36}\right)^{24} = \left(1 - \frac{1}{36}\right)^{24}$ by working out the first few terms, because x^2, x^3 and higher powers of x are tiny. In this case, you can get 3 decimal places of accuracy from the first four terms. Pascal and Fermat knew the Binomial Theorem and how to calculate these "binomial coefficients".

So far, so good. The more difficult challenge that Pascal and Fermat discussed was something that still comes up today in some sporting contexts. It's known as the "problem of points", and it's this: if you have made a wager based on the outcome of a game, and everyone has put money into a pot, or kitty, to pay out to the winner, how should you share the kitty if the game has to be abandoned early? In their series of letters, Pascal and Fermat work through this problem and in doing so, basically invent the mathematical field of probability. (By the way, Gombaud didn't invent this problem; it goes back at least as far as the Italian mathematician Luca Pacioli, who discussed it in a 1494 book, but it does seem to be Gombaud who drew Pascal's attention to it.) A very simple example of the question might be a coin tossing game. Let's say I choose heads, you choose tails, and the winner is whichever is first to three. That is, I win if heads comes up 3 times before tails does. Now, imagine we're at a point where we've tossed the coin 3 times and it's come up H, H, T. At this point we have to abandon the game for some reason; how should we share the kitty? (Or, how should people bet on the outcome if they are betting at this stage?) The possible outcomes at this point are (HHT)H – I win; (HHT)TH – I win; (HHT)TT – you win. Because of this it's tempting to say that I will win two out of three times, so I should get two-thirds of the kitty. The insight that Pascal and Fermat arrived at, after a bit of back and forth, is that you have to include the sort of "ghost" outcomes that would never happen because the game would finish before playing to the conclusion. For instance, HHTHH will never occur. We would stop at HHTH, since three heads have already been thrown. But in order to use the assumption of equal probabilities, we have to include these theoretical outcomes. If we do that, it's easy to see that the theoretical outcomes (though we stop before they are completed) are

(HHT)HH (HHT)HT (HHT)TH (HHT)TT

These are all equally likely, and I win in three of the four cases. So I should get three-quarters of the kitty.

Probabilistic Pitfalls

Our intuition seems to be particularly poor around probability, and because of that it's very easy, even for good mathematicians, to fall into error. For example, the very good mathematician Jean le Rond D'Alembert wrote an article *Croix ou Pile* for the 1754 Encyclopédie (edited by himself and Diderot) in which he discussed the question "if a coin is tossed twice, what is the probability that you will get at least one head?" and said that the possible outcomes are H, TH, TT, so the answer is 2/3. But this is false, as these outcomes are not all equally likely. We can fix this by listing all potential outcomes, even ones that won't be completed, and then we see that they are HH, HT, TH, TT, each with probability ¼. Therefore, there is a ¾ chance of getting at least one head.

In games of pure chance like dice or lotteries, the numbers that come up are (or should be!) independent of one another, and it's the same with tossing coins. But we have to be very careful when looking at extrapolating to multiple coin tosses, or totals when multiple dice are thrown. Here's the great Leibniz making a mistake of this kind: *with two dice, it is equally likely to throw twelve points, than to throw eleven, because one or the other can be done in only one manner.* But that's not true! It's correct that 12 can only be achieved with two sixes, and eleven can only be achieved with a 5 and a 6, but there are two ways to get a 5 and a 6: the die on the left can be 5 and the one on the right 6, or vice versa. So it's twice as likely to throw 11 as 12. If you throw two dice you can get any total between 2 (two 1's) and 12 (two sixes), but these outcomes are not equally likely. If we temporarily distinguish between the two dice, so we record the possibilities for say the left-hand and right-hand die, then we see that there are 36 possible outcomes, from (1,1), (1,2), (1,3), all the way to (6,4), (6,5), (6,6). Many of the 36 outcomes give the same total. We can obtain 7 in six ways: 1+6, 2+5, 3+4, 4+3, 5+2, 6+1. You are therefore six times as likely to score 7 as you are to score 12.

Another huge pitfall is the gambler's fallacy. We know that red and black are equally likely to come up in roulette, like heads and tails in a coin toss. Therefore, over time, we would expect an equal number of reds and blacks (it's not quite 50-50 odds because only 18 of the 37 slots are red, 18 black and one green, or sometimes two of 38 slots are green). But this does not mean that the probability that the next coin will come up tails is higher if the last five have been heads. Coins, and roulette tables, do not have a memory. The likelihood of any specific sequence of n coins is $\frac{1}{2^n}$. The odds of HHHHHH are $\frac{1}{64}$, but so are the odds of HHHHHT. So don't be fooled.

You can guarantee to win and make back all your losses eventually, by simply doubling your bet every time you lose. If you bet a dollar on black and it comes up red, just bet two dollars on black next time. If it's still red, go again but this time bet four dollars. Then 8, 16, 32, 64, and so on. If black finally comes up on the seventh attempt, let's say, then you'll have bet $64, so you win your $64 back plus another $64, a total of $128. Unfortunately, total outlay was $(1+2+4+8+16+32+64) = $127. However long the game lasts, you'll only ever make a profit of $1. Of course you could start with an initial bet of $1 million, but the risk there is that you run out of ready cash before black finally comes up, or you reach the maximum bet of the casino. A strategy like this wouldn't have worked for Rosencrantz and Guildenstern, in Tom Stoppard's *Rosencrantz and Guildenstern are Dead*. The play begins with them betting on the outcome of coin tosses. But the coin keeps coming up heads, and this happens times in a row. Mathematically speaking, this outcome is just as likely as any other combination, such as 46 heads followed by 46 tails, or heads, tails, heads, tails alternating, or any other specified set of outcomes. Nevertheless, if this happened in real life we would find it extremely unsettling, as indeed do Rosencrantz and Guildenstern. They finally conclude that they must be caught up in some unnatural or even supernatural situation, as indeed they are, being trapped as minor characters in another play without real agency of their own.

We do sometimes hear true stories of mathematically-minded people beating casinos and making money gambling. This is always by playing games that are not totally random, and almost always it's Blackjack. In blackjack, you and the dealer, and any other players, are aiming to get as close as possible to 21, without going over (bust). You get two cards initially, and then you choose at each stage to get another card (hit) or stick with what you have, and so does the dealer. To keep tension in the game, one of their initial cards is kept face down so you don't know what it is, but every other card dealt is face up. The dealer always takes another card if their total is less than 17, so that means your odds of winning are higher the fewer small cards are left in the deck. This means if you keep track you can bet bigger when the odds are in your favour. One simple system is that cards 2-6 are worth +1, cards 7,8,9 are worth 0, and 10 and up are -1. The higher the running total at any point, the better it is for you. Card counting using this or other mental arithmetic systems is legal, but you have to be extremely quick and accurate at it, and even if you do it perfectly it only gives you something like a 1% advantage over the house, which of course is within its rights to end the game or ban you at any stage. Casinos have got better over the years at thwarting card counting, for example by using not just one deck of cards but up to eight, which means it takes a lot longer for odds to improve significantly. Since these methods rely entirely on non-randomness – a slowly diminishing deck or decks of cards, we won't say any more about them here.

What about the lottery?

Let's take a look at the lottery then. There are many lotteries around the world. The first state lottery in England was in 1567, when Queen Elizabeth I decreed there would be a lottery with 400,000 tickets each steeply priced at ten shillings – meaning only the fairly well-off could enter. The top prize was huge for the time though: a "*value of Five thousande Poundes sterling, that is to say, Three thousande Pounds in ready money, Seven hundreth Poundes in Plate gilte and white, and the rest in good Tapissarie meete for hangings and other covertures, and certaine sortes of good Linnen cloth*". You also got immunity from arrest for any crimes other than murder, felony, piracy and treason. In those days, winning tickets, were simply drawn in turn, more like what we'd now call a raffle. Modern lotteries are different (and sadly have much more boring prizes). They all have the same basic structure nowadays. You select a small set of numbers from a bigger set (in the UK currently you choose six numbers between 1 and 59). Then, to determine the winners, numbers are drawn, usually by machine, for example balls out of a big container. The more numbers you match with the winning numbers, the bigger prize you win.

Let's do a toy example: suppose you only use the numbers 1 to 10, and you have to match two numbers. Then there's one winning combination. How many possible pairs of numbers can be drawn? Well here we have to be slightly more careful. There are ten different possibilities for the first number drawn, which leaves nine left for the second number. This looks like $10 \times 9 = 90$. But drawing a 1 then a 5, for instance, gives the same pair as drawing a 5 then a 1. Each pair $\{a, b\}$ can arise in exactly two ways: a then b, or b then a. So we have to halve that 90 to get rid of these duplicates. The probability of winning this lottery is 1/45, representing picking that one winning pair out of the 45 possible pairs.

For three numbers we need to work out the number of ways of picking a set of three numbers out of ten. There are $10 \times 9 \times 8 = 720$ ways of picking a sequence of three numbers in order. But we don't care which number comes out first; we just care about the numbers, not the order. This time each set of three could arise in six different ways: $abc, abc, bac, bca, cab, cba$. To end up with the desired set, you can choose any of the three numbers first, then either of the two remaining numbers, and the final number must be the one that's left. So there are $3 \times 2 \times 1 = 6$ ways. Mathematicians write the product of the first n numbers as $n!$, and call it "n factorial". The total number of ways of choosing a set of three numbers between 1 and 10, is $\frac{10\times9\times8}{3\times2\times1} = \frac{10\times9\times8}{3!} = \frac{720}{6} = 120.$

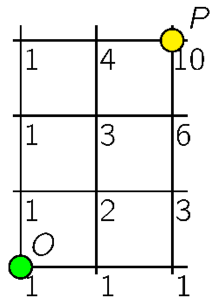
Therefore, the odds of winning the jackpot in this lottery are 1 in 120.

The total number of ways of choosing a set of four numbers between 1 and 10 is

$$\frac{10\times9\times8\times7}{4\times3\times2\times1} = \frac{10\times9\times8\times7}{4!} = \frac{10\times9\times8\times7\times6\times5\times4\times3\times2\times1}{4!\times6\times5\times4\times3\times2\times1} = \frac{10!}{4!6!} = 210.$$

We've seen here the general formula for choosing a set of r numbers from n. It is $\frac{n!}{r!(n-r)!}$. It's referred to as "n choose r", and can be written in various ways, nCr, or $\binom{n}{r}$ being two of the more popular. You may have seen these symbols on your calculator.

I want to mention another place you see these numbers, both because it's interesting anyway, and because it gives us another way to construct them. Suppose you are moving through a grid city, and you want to walk from a starting point O to a point P. There are many possible shortest routes, but at each stage you will move one block up or one block to the right. You don't want to go further than necessary, so at each stage you will go either up or right, and you won't go further up or right than necessary. So if we are going, for example, to the point (2,3), we will go a total of 5 blocks, comprising 2 moves right and 3 moves up, in some order. We can specify a path completely by choosing which 2 of the 5 moves are "right". That is, there should be "5 choose 2" different possible shortest routes. And we know this is $\frac{5!}{2!3!} = \frac{5\times 4}{2!} = 10$. In general, the number of paths to the point (r, u), writing $n = r + u$, will be $\binom{n}{r}$. But there's another way to think about it that gives us a neat way to calculate the numbers $\binom{n}{r}$. In order to get to P, any shortest path must have passed through either the point X immediately to the left of P, or the point Y immediately below P. A shortest path from O to P that passes through X consists of a shortest path from O to X, followed by the final step to P, and a shortest path from O to P that passes through Y consists of a shortest path from O to Y followed by a final step up to P. Therefore the number of shortest paths from O to P is the number of shortest paths from O to X plus the number of shortest paths from O to Y. We can apply exactly the same reasoning to X and Y, working all the way down to the x-axis and all the way back to the y-axis. For points on the axes, there is always just one shortest path from the origin. Everywhere else, the number of shortest paths from O to a point Z is the sum of the number of shortest paths from O to the point immediately to the left of Z, and the number of shortest paths from O to the point immediately below Z. We can therefore build up to any point by just working out the sums step by step. If we do this for the point $(2,3)$ we get the following, re-confirming that the answer is 10.



If you turn this grid round, you get a triangle where every number is constructed by adding together the two numbers above it. The numbers in this triangle are precisely the numbers $\binom{n}{r}$. So now we have a way to construct them using additions rather than multiplications. Now, remember that calculation we had to do earlier, of $(1-x)^n$. Let's actually think about $(1+x)^n$, just so we aren't bothered by minus signs. When we multiply out, say, $(1+x)^5 = (1+x)(1+x)(1+x)(1+x)(1+x)$, the number of x^2 that we get will be the number of ways to pick two x's from 5 brackets. It's just $\binom{5}{2}$ again, and in general the coefficient of x^r will be $\binom{n}{r}$. So these are exactly the binomial coefficients that we mentioned earlier. In fact, the treatise Pascal wrote about the triangle of "binomial coefficients" is the reason this triangle is known as Pascal's triangle now (though I should mention that binomial coefficients, "Pascal's" Triangle, and later the Binomial Theorem where n is a positive whole number had been known to Indian, Arabic, Persian and Chinese mathematicians for centuries before this.)

Let's work out the odds of winning the jackpot in the UK National Lottery. In this lottery, you buy a ticket for £2 and choose six numbers between 1 and 59 (you also choose a seventh "bonus ball" number, but that's not relevant for the jackpot. With our brilliant binomials it's now an easy calculation. The number of ways to choose six numbers from 59 is $\binom{59}{6}$, which is 45,057,474. So our odds of winning the jackpot are about 1 in 45 million.

(Originally, when the National Lottery was introduced, it was £1 to play and you chose 6 numbers between 1 and 49. The odds of winning the jackpot were about 1 in 14 million. But in 2015 they added ten more numbers, and the odds worsened to 1 in 45 million. AND, it now costs £2, so with the same budget you can only play half as often. The prizes have gone up a bit though, to compensate.)

By the way, there are other prizes for matching fewer than six numbers. If you match five balls plus the bonus ball you win £1,000,000. Matching five balls without the bonus ball gets you £1,750, while four balls has a prize of £140. Matching three balls wins £30, and matching 2 gets you a free go at the next draw.

We can work out the chances of matching exactly three numbers. If you are going to match exactly three numbers, three of your numbers are from the six winning numbers, and there are $\binom{6}{3} = 20$ such sets. The other three numbers are from the 53 remaining non-winning numbers, and there are $\binom{53}{3} = 23{,}426$ ways to choose them. This means there are $20 \times 23{,}426 = 468{,}520$ ways to match exactly 3 numbers, out of the 45,057,474 possible lottery tickets. If you work that out, you get odds of approximately 1 in 96 of matching exactly 3 numbers. Combining all these prizes together, the organisers of the National Lottery say that about 53% of the ticket money is paid back in prizes, over time (the actual proportion for each draw varies as the jackpot isn't always claimed).

So, are there any strategies to actually win? Well it's easy, simply invent a time machine and go back in time to buy your lottery ticket. I remember an old TV series where some scientists found a way to go back in time by a few days, but through some convenient plot device the universe somehow prevented them from changing anything dramatic. They needed funding (being scientists) and decided it would be morally justifiable to win the lottery. So they printed out the winning numbers, as they were announced, something like this: 12 5 21 25 6 11, and then jumped in the time machine to buy the ticket. But somehow, they found that they had only matched 4 numbers. The cute plot twist was that they had looked at the paper upside down so it read 11 9 52 12 5 21. So that's time travel out as a way to win the jackpot.

But I've toyed with you long enough. Here are six ideas to maximise your winnings, none of which involve breaking the laws of physics!

1. The obvious first one: don't play the lottery. Instead, invest the cost of one lottery ticket each week. If this is £2 then after a 52-week year your lottery-playing alter-ego has won, on average, £55, but you have £104. This more or less doubles your expected "winnings". If you are playing, just be honest with yourself that what you are paying that money for is a little frisson of excitement that you might win the jackpot, and/or the knowledge that you are supporting some good causes: 28% of ticket price goes to good causes. Don't spend more than you can cheerfully afford to lose!
2. Play a different lottery. The US Powerball lottery requires you to choose five numbers between 1 and 69 AND, for the jackpot, a "powerball" number, between 1 and 26. The odds of doing this are 1 in 292,201,338. This means the prize regularly rolls over. After 40 draws without a winner in 2022, there was finally a jackpot winner on November 7^{th} 2022; they won \$2.04 billion. But think of the 40 draws without a winner, and all the tickets bought and money lost. For much better odds, you can win the jackpot of the Polish mini lotto by picking 5 numbers from 42 correctly. The odds of doing this are 1 in $\binom{42}{5}$ = 850,668. At the other extreme, Italy's SuperEnalotto requires you to choose six numbers from 90. The odds of winning the jackpot in this game are 1 in $\binom{90}{6} = 622{,}614{,}630$. The odds of winning a prize of some sort are 1 in 16, but we are only interested in the big money here. In the case of a rollover, would it be worth buying one of each possible ticket? Even for the Powerball win, it "only" costs $2 \times 292{,}201{,}338$, or about \$584 million to buy one of each ticket. So you'll guarantee to win the jackpot and make a massive profit if it rolls over to a billion dollars. Well, not quite. First, in the US you have to pay tax on the win. Second, what's everyone else doing? When prizes get that big, the chance of having to share the jackpot increases massively, as hundreds of millions of tickets are bought. Even if just one other person wins with you, you would make an \$84 million loss, before paying any taxes. That doesn't sound very good. There is one success story though. Back in 1992, the Irish Lottery was a "pick 6 from 36" game. This means there are 36C6 = 1,947,792 possible winning combinations. Each ticket cost IR£0.50, so you could buy one of each possible ticket for £973,896. After a couple weeks with nobody winning the jackpot, the rollover amount reached £1.7 million. A man called Stefan Klincewicz had been watching the figure rise, and a group led by him tried to buy up one of each ticket. They didn't quite get there but they managed to get about 80% of the possible tickets, and did have a winning ticket amongst them (as did two other winners). With their share of the jackpot along with lots of subsidiary prizes, the group managed a profit of around £310,000. Shortly after that, the Irish Lotto changed its rules so that this wouldn't be possible again!
3. Spread the risk – syndicates. Statistics show that one in five top prizes on the UK National Lottery are won by syndicates.[2] If a group of people buys tickets, then each individual raises their chance of having a share of a jackpot for the same outlay. Of course, you do have to share the winnings. It's an interesting question as to how many tickets you have to buy to guarantee winning at least something. You'd have to make sure that at least one ticket had at least three of the winning numbers, as matching three numbers is the fewest numbers that guarantees a cash prize. (If you match two you get a free lucky dip but that doesn't guarantee you anything.) OK so how many

[2] https://www.cdn-national-lottery.co.uk/c/files/syndicate-pack.pdf

tickets do you need to buy? Should be easy to work out, right? Amazingly, this is an open problem in mathematics! It's called the lottery problem, or the lotto design problem, and it fits in the area of mathematics known as design theory. For our lottery ticket question, the number we are interested in is called $L(59,6,6,3)$, the minimum number of subsets of size 6 of a set of size 59, so that given any "winning" set of six numbers, at least one of these subsets contains 3 of the winning numbers. There is no known formula to work this out. Just as an indication, a lower general bound for $L(m,n,n,2)$ was found in 1964, this would be the minimum number of tickets required in a "pick n numbers out of m" lottery to guarantee matching at least 2 of the winning numbers. It was shown that $L(m,n,n,2) \geq \frac{m(m-n+1)}{n(n-1)^2}$. In the case of our lottery, the minimum number of tickets needed to match 2 numbers is $L(59,6,6,2)$, which, if you plug the numbers into the bound, must be at least 22 (we don't know the exact minimum number). More tickets than that would be needed to guarantee matching 3 numbers. The cost of those tickets, at £44, is already more than the smallest prize of £30. So, for most lotteries, this is pointless. There used to be a lottery in Massachusetts which capped its jackpots and so if there was no win for long enough, it increased the amounts for smaller prizes. This meant occasionally it became worth following a strategy of this sort. But it's hard work!

4. NEVER buy your ticket on a Monday. This is a slightly macabre piece of advice, but the odds of you dying on any given day are considerably higher than one in 45 million. In the UK, lotto draws are on Wednesdays and Saturdays. So you are more likely to die before the draw than to win. If you must gamble, do it at the last minute. In fact, to take this to the extreme, incredibly, based on accident statistics, you have about a 1 in 30 million chance of dying from falling over while trying to put trousers on. So death by trousers is more likely than winning the lottery. Be careful out there.
5. Don't try and prove a point about being rational. Any specific set of six numbers is equally likely to come up, because it's a random draw. If I happen to attach special significance to the numbers 3, 1, 41, 59, 26 and 54 that's fine, but they aren't more or less likely to come up just because they are the first few numbers in the decimal expansion of π. Looking at which numbers have come up before and picking "popular" or "unpopular" ones, or saying that since 12 came up last week it won't come up again this week, are all false reasoning. The lottery balls do not have memories! The only thing we can try to do is minimize our chance of having to share the jackpot. Lots of people pick birthdays, so numbers between 1 and 31, which can be dates of a month, tend to be chosen by more people. But don't be tempted to exhibit your knowledge that 1, 2, 3, 4, 5, 6 has the same odds of coming up as a more "random" looking collection, because an estimated ten thousand people every week do the same. You don't want to win a £10 million jackpot only to have to share it ten thousand ways and end up with just a thousand pounds.
6. Be Voltaire. This might be quite hard to achieve, I admit. But I mentioned at the beginning how Voltaire managed to make money from a lottery, so I'll tell you the story now, to finish off. Voltaire had a friend called Charles Marie de La Condamine (1701-1774), a French mathematician, explorer and scientist. Among other adventures, he spent a decade in South America making detailed measurements of the length of a degree of latitude near the equator, while colleagues did the same in the Arctic, to test Newton's hypothesis that the Earth is not a perfect sphere, but bulges slightly at the middle. La Condamine and Voltaire cooked up a plan based on a new lottery in France. The French government at the time issued government bonds. But because the interest rate being offered was low, they were having trouble selling enough of them. So the government hit on an idea a bit like the premium bonds we have in the UK now. If you had a bond, it entitled you to buy (for a small extra price, equivalent to 1p for each £10 of the bond's value), a lottery ticket. If you won the lottery, you got the price of your bond plus a huge prize: 500,000 livres. I've tried to convert this to modern money and my best estimate is that it's around £6 million. A lot of money, anyway. But there was a huge flaw in the design. The cost of the ticket depended on the cost of the bond, but the main jackpot prize did not. So what Voltaire and La Condamine did was to buy up a huge number of bonds worth very little, so that they could buy a huge number of lottery tickets, giving them a high chance of winning. They did this month after month, and did indeed make a fortune. Eventually the French government cottoned on, and even tried to prosecute Voltaire and La Condamine, but the court ruled that there was nothing illegal about profiting from a government's stupidity. So, the pair kept their winnings (but the government cancelled the lottery).

I hope you've enjoyed this guide to chance, fate and the lottery. Have a look at the further reading for more information about some of these ideas.

Further Reading

- There's more detail about the origins of probability in Keith Devlin's *The Unfinished Game: Pascal, Fermat, and the Seventeenth-Century Letter That Made the World Modern*, Basic Books, 2008.
- You can read translations of some of the letters between Pascal and Fermat at https://www.york.ac.uk/depts/maths/histstat/pascal.pdf
- For more examples of missteps in probabilistic thinking by otherwise brilliant mathematicians, see *Errors of Probability in Historical Context*, by Prakash Gorroochurn, in *The American Statistician*, Nov 2011, Vol. 65, no. 4, p246-254. Professor Gorroochurn has also made the article available online at http://www.columbia.edu/~pg2113/index_files/Gorroochurn-Errors of Probability.pdf
- The 2008 film *21* is based on the true story of a group of MIT students who made money at Blackjack by counting cards. It's based on *Bringing Down the House* (2002), by Ben Mezrich.
- There's a nice article by Brendan Mackie about Voltaire and La Condamine's lottery-winning exploits at https://www.damninteresting.com/the-enlightenment-guide-to-winning-the-lottery/
- You can play around converting 18th century currencies to each other (including Livres to Pounds) at http://www.pierre-marteau.com/currency/converter.html

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- I produced the calculator-style numbers using the free text input box at https://www.fontspace.com/digital-7-font-f7087
- The website www.hiclipart.com is a useful source of free-to-use png clip art and photos which many of the images come from. The portraits (for example of Voltaire and La Condamine) are, unless otherwise mentioned, from Wikimedia Commons, a great source of public domain images.