The Mathematical Life of Sir Christopher Wren

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We remember Christopher Wren as a great architect. But he was so much more. Today I'm going to tell you about Christopher Wren the mathematician. We'll look at his work on curves including spirals and ellipses, and we'll see some of the mathematics behind his most impressive architectural achievement – the dome of St Paul's Cathedral.

Who was Sir Christopher Wren?

Christopher Wren, who died 300 years ago this year, is famed as the architect of St Paul's Cathedral. But he was also Gresham Professor of Astronomy, and one of the founders of a society "for the promotion of Physico-Mathematicall Experimental Learning" which became the Royal Society. He did research in everything from meteorology to anatomy – he was a mathematician and scientist first and architect later. However this isn't so much of a leap as it might seem now. Architecture was viewed as one of the "mathematical arts", along with navigation, astronomy, surveying and other practical pursuits. Mathematics, and specifically geometry, was for Wren at the heart of beauty. He said "*There are two Causes of Beauty, natural and customary. Natural is from Geometry, consisting in Uniformity (that is Equality) and Proportion. Customary Beauty is begotten by the Use of our Senses to those Objects which are usually pleasing to us for other Causes, as Familiarity or particular Inclination breeds a Love to Things not in themselves lovely. Here lies the great Occasion of Errors; here is tried the Architect's judgment: but always the true Test is natural or geometrical Beauty.*" The portrait I show, by Godfrey Kneller, alludes to this mathematical outlook: Wren is shown not just with the plans of St Paul's, but with dividers and a copy of Euclid's *Elements*. Wren placed mathematics at the top of a hierarchy of truth: "*Mathematical Demonstrations being built upon the impregnable Foundations of Geometry and Arithmetick, are the only Truths, that can sink into the Mind of Man, void of all Uncertainty; and all other Discourses participate more or less of Truth, according as their Subjects are more or less capable of Mathematical Demonstration".* And I can't resist, passing on this compliment that Wren paid London in 1657: *I must congratulate this City, that I find in it so general a relish of Mathematicks*. Thank you to Gresham for continuing in that tradition!

Wren was educated at Oxford and later held the Savilian chair in astronomy there, as well as his Gresham professorship in London. These roles and others place him right at the heart of an exceptionally active and exciting community of scientific thinkers. The group around Gresham College included not just Wren as Gresham Professor of Astronomy but also Robert Hooke, who was Gresham Professor of Geometry at a similar time. Wren was not just a founder member of the Royal Society (which arose out of weekly meetings at Gresham beginning in November 1660) but served as its president. And he was an active contributor in meetings – if perhaps not in subscription fees, which he had to be chased to pay up. In short, he was a key contributor to the scientific and mathematical thought of the time. We can see this, not just from his own work, but by the amount he is mentioned in the writing of others, giving credit to him for certain ideas. For example, when Isaac Newton introduces the idea of a force governed by an inverse square law in his *Principia Mathematica*, he says that one example is the force governing the motion of the planets "as Sir Christopher Wren, Dr. Hooke, and Dr. Halley have severally observed". Wren's name appears seven times in the Principia. In fact, the leading architectural historian John Summerson (1904-1992) wrote that if Wren had died at thirty, he would still have been a "figure of some importance in English scientific thought, but without the word "architecture" occurring once in his biographies". Wren's contributions to astronomy are the subject of a lecture by the current Gresham Professor of Astronomy, Katherine Blundell, which you can watch online: today I want to explore his mathematical contributions.

Wicker baskets and lens-grinding

Wren saw the mathematical arts as encompassing direct practical knowledge, like navigation, astronomy, surveying, and of course architecture. He didn't just do theoretical work, he built models, designed machines (like a machine for drawing in perspective), made telescopes, and so on. He was actively engaged with the practical challenges of designing scientific instruments and experiments. One instance of this was his interest in lenses. The basic principle of a lens is that it is a piece of curved glass that focuses light by refraction – Snell's Law details exactly how the direction of a light ray is changed as it passes through and out of glass back into air. Spherical lenses (where the curved part is a section of a sphere) were the only ones that could easily be made in Wren's time, but it was known that other curves were better for the job, producing less aberration. In particular, the question of how to make hyperbolic lenses was key. Just as a reminder, a hyperbola is one of the "conic sections" – curves that are made by slicing through cones (along with the ellipse and parabola). If you want to know lots more about them, you can watch my 2022 Gresham Lecture *The Surprising Uses of Conic Sections* (link at the end of the transcript). One day, Wren was out shopping and he saw a round wicker basket for sale, "of that shape which with us is usually given to salt-cellars", that he realised was made entirely from straight pieces of "osier" (ie willow). Specifically, a series of straight canes all set at the same angle around a central circle, would describe this curved surface that we call a hyperboloid. In other words, the hyperboloid is what we now call a ruled surface, that is, a curved surface that can be made up of straight lines. This would mean that you could make a hyperboloid on a rotating lathe by using a straight-edged tool positioned at a fixed angle. Wren demonstrated this at the Royal Society and also designed a machine for grinding hyperbolic lenses, though it's unclear if the machine was built and used. There's a picture of the design in the subsequent issue of the *Philosophical Transactions of the Royal Society* (November 1669, vol 4, no.53). The historian of science Jim Bennett has described the machine like this: "two revolving cylinders, with their axes inclined, work against each other, to form two hyperboloids in contact along their common generator. At the same time, one of the cylinders also works against a revolving piece of glass, whose axis is set at right-angles to the two revolving cylinders".

A challenge from France

In February 1658, mathematicians in England received a challenge from France. It read "Jean de Montfort [possibly a pseudonym for Pascal] greatly desires that those distinguished gentlemen, the Professors of Mathematics, and others in England renowned for mathematical skill, may condescend to resolve this problem". The problem was, given an ellipse of known dimensions, and a chord of the ellipse crossing the major axis at a known point and angle, to find the lengths of the segments of that chord. Wren solved the problem, and then in return challenged the mathematicians of France to solve another problem about ellipses, which I'll tell you about now.

Kepler's laws about planetary motion say, among other things, that planets move in elliptical orbits with the sun at one focus, and that the line joining a planet to the sun sweeps out equal areas in equal times. A crucial aspect in the mathematics of this is the related question of how to cut a semi-ellipse in a given ratio by a line through one focus. Kepler realised that it's actually enough to solve the problem of cutting a semicircle in a given ratio by a line through any given point on its diameter. This became known as Kepler's problem, and it's this problem that Wren posed back to the French.

Wren had in fact already solved Kepler's problem himself, and his solution involves a fascinating curve known as a cycloid. It's the curve you get by following the path of a point on the circumference of a circle as it rolls along a straight line (or the rim of a wheel rolling along a road). Cycloids were all the rage in the 17th century. They were studied by Galileo, Mersenne, Fermat, Descartes, Pascal, and later Newton and Leibniz, among others. For those interested in appearances of mathematical ideas in literature, cycloids get cameo roles in *Gulliver's Travels*, *Tristram Shandy* and much later *Moby-Dick*. If you want to know more about that – read my book *Once Upon a Prime*!

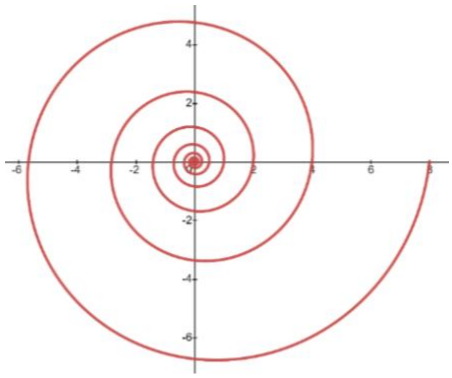
There were two key questions people always had about curves, known as "quadrature" and "rectification". Quadrature is finding the area under a curve. Galileo approximated the quadrature by making a cycloid out of metal and weighing it, but he didn't know the exact formula. We don't know for sure when he did this, but he wrote in 1640 that he'd been studying cycloids for 50 years. At any rate, it took until the 1630s for the correct solution to be found (probably first by Gilles de Roberval): if the rolling circle has area πr^2, then the area under each cycloid arch is $3\pi r^2$. Very nice. But the cycloid had still not been "rectified": this means finding its length. The first person to do this, of all the illustrious mathematicians who had studied it, was

Christopher Wren. He showed that the length is another beautifully simple formula. If the rolling circle has diameter d, its circumference is πd, and each cycloid arch has length precisely $4d$. (Actually, Roberval claimed to have done this first too, but he did that a lot. He only started making this claim after Wren told Pascal the result, and Wren's proof was the first to be published, as far as I know. The general consensus at the time and since seems to be that Wren was indeed the first to rectify the cycloid.)

Wren's solution of Kepler's problem manages to relate the areas into which the semicircle must be divided to lengths of specific circle arcs. These are then equated to carefully positioned "stretched" or "prolate" cycloids – which of course Wren already knew how to find the length of, from his own earlier work. And so he was able to solve Kepler's problem. His solution was published by John Wallis in a 1659 treatise on the cycloid (which also included Wren's rectification of the cycloid). If your Latin is tip-top, you can give it a read: John Wallis: *Tractatus duo, prior de cycloide et corporibus inde genetis: posterior, epistolaris in qua agitur de cissoide*. In a 1668 letter, the English mathematician John Wallis said that although the challenge of Kepler's problem had been issued to the French mathematicians almost a decade previously, "there is none of them have yet (that I hear of) returned any solution". Take that, Jean de Montfort!

Seashells and antlers

Spiral-like shapes crop up regularly in nature. There's a particular kind of spiral, called a logarithmic spiral that was familiar to Wren. Logarithmic spirals were first mentioned by the German artist and engraver Albrecht Durer, and studied in great detail by the mathematician Jacob Bernoulli – he gave them the name "spira mirabilis", or "miraculous spiral". In a logarithmic spiral, the distance r from the centre is a power of the angle we've moved through (or conversely the angle is a logarithm of the distance, hence the name). This means that the gap between consecutive rings of the spiral is increasing each time. One example of a logarithmic spiral, shown below, is $r = 2^{\theta/360}$ (where we are measuring our angles in degrees). With every complete revolution, the distance of the spiral from the origin doubles. It crosses the x-axis at 1, 2, 4, 8, 16 and so on.



John Wallis had shown in the 1650s how to "rectify" a logarithmic spiral, in other words how to find its length (or more properly the length of any part of it), by transforming, or "convoluting", it into a straight line without changing the length. Wren managed to show that a version of this idea could work a dimension higher, and could be used in reverse to convolute or twist a cone into a kind of three-dimensional or solid logarithmic spiral. He suggested these spirals could be behind the growth of snail shells and seashells. And it's since been found that this is absolutely right.

All logarithmic spirals are self-similar, in that they retain precisely the same shape as they grow. In nature, if we think of how plants and animals grow, if they are growing out from a central point at a fixed rate, as happens with something like a Nautilus shell, then the outer parts continue to grow while they expand out from the centre. Logarithmic spirals allow for this to happen while keeping the same shape. The spiraling makes room for new growth. The three-dimensional version of a logarithmic spiral that Wren studied is just the right solution for shells, and is achieved in nature by one side of the structure growing at a faster rate than another. By varying the parameters in the general equation for a solid logarithmic spiral, many different shell-like shapes can be created. Wren's ideas continue to inspire. In 2021, a team at Monash University came up with a "power cone" construction generalizing the cone-to-spiral idea (and Wren is referenced extensively in their article) that gives a mathematical basis for the formation of animal teeth, horns, claws, beaks and other sharp structures.

Building St Paul's

There is a lot of mathematics in architecture – balancing the different forces acting on load-bearing structures, designing a building whose proportions are aesthetically pleasing while also satisfying the practical requirements of its intended use, and so on. We can't talk about all of that in a single lecture or even a series of lectures, so I'm just going to speak about St Paul's Cathedral, and in fact just one aspect of it. Wren went through several designs for St Paul's trying to make everyone happy. One design was felt not to be magnificent enough, another cost too much, another was disliked by the church authorities. There were site-specific challenges too, like the fact that the cathedral had to be built on soft London clay. Unusually, the crypt extends under the entire cathedral – this isn't so much to give extra space as to allow for very thick columns and piers (much thicker than what's visible in the main cathedral) to support the structure. I'm going to restrict myself here to discussion of one key mathematical challenge: the dome. For both aesthetic and symbolic reasons, Wren wanted the dome that you see from all across London, the defining shape of the cathedral, to be a hemisphere. It's the perfect symmetrical shape, symbolizing the universe (that phrase, the celestial sphere), and its geometrical construction as an infinitude of circles, which themselves symbolize eternity. This is the shape of the dome that we see from the outside, crowned with its large lantern. But there's a problem. A hemisphere is not actually a very strong structure. A masonry hemisphere of the scale required would struggle to support its own weight, and would have no chance of supporting the additional (850 ton!) weight of the lantern. So Wren needed to find some other way of supporting that weight without ruining the aesthetics of either the outside or the inside of the cathedral.

The question of the best shape for a masonry arch had been recently addressed both by Wren and his friend, Royal Society colleague and fellow Gresham Professor, Robert Hooke. On 19 January 1671, the Royal Society reported that: *Dr Wren delivered to the President his demonstration of what line it is, which an arch, fit to sustain any assigned weight, makes. The President was desired to examine it, and to give an account of it to the Society. Mr Hooke, being called upon for his demonstration of the same subject answered, that he had already declared the substance of it to the President, who yet desired him to give it also in writing, that so it might be with more leisure and conveniency examined.*

Neither of these demonstrations have been preserved, and it's not clear if they were mathematical proofs or the outcomes of physical experiments. However, some years later Hooke did write down in anagram form a phrase which indicates that he had determined the solution to the problem (even if he had not necessarily found a mathematical proof): it's a catenary. A catenary is the curve made by a chain or rope allowed to hang freely between two points. Galileo had talked about this problem; he thought that to a good approximation the solution was a parabola, but it was discovered later to be a subtly different curve. Hooke found that the equations describing the forces acting on a hanging chain are equivalent to those describing the forces acting on an arch (this time not tension and gravity but compression and gravity). That would imply that the most stable, strongest shape for an arch is a catenary, but upside-down. You can make the actual curve of the arch a slightly different shape but the line of thrust is still a catenary curve, so that needs to be part of the structure of the arch. This means the shape that requires the least amount of material, the most efficient shape, is indeed a catenary. So, we now have an outer hemispherical dome with a gigantic lantern, that can't support itself and needs some kind of internal structure. To hide that internal structure, Wren built an inner dome whose cross section is a catenary, fitting in very nicely with other elements of the internal design.

But what about the support for the outer dome and lantern? What Wren did there was to build a third, middle dome – and for this he wanted the strongest possible dome shape. While the catenary is optimal for an arch, that doesn't guarantee it's optimal for a dome. Wren and Hooke believed that the perfect shape would in fact be the positive half of the curve $y = x^3$. Why did they think this? Well, we can do a bit of investigation here. It's similar in flavour to the fact that a parabola ($y = ax^2$) is a good approximation to a catenary. If we think about trying to find the equation of a catenary, we see that in equilibrium, the forces at every position along a hanging chain must balance. If we think about a point (x, y) on the chain, the weight W of the section of the chain between 0 and x will be pulling vertically downwards, the force F exerted by the tension from the entire left-hand half of the chain will be acting horizontally to the left, and the tension T from the remaining upper right-hand part of the rest of the chain will be acting upwards along the chain, at an angle of θ to the horizontal. The vertical forces balance, so we get $W = T \sin\theta$, and $F = T\cos\theta$. That means $\tan\theta = \frac{W}{F}$. We can make an approximation that $\frac{y}{x} = \tan\theta$ as well (this would be true if we had a straight line from the origin to (x, y), but we actually have a curve). The final step is to make another approximation, that W is proportional to x; this would again be true if we had a straight line from the origin to (x, y). So we get the approximation that $\frac{y}{x} = ax$ for some constant a, and hence that $y = ax^2$, a parabola. This is a reasonable approximation and gets better the smaller the curvature. The actual general equation of a catenary curve passing through the origin is $y = \frac{1}{2b}(e^{bx} + e^{-bx} - 2)$, where b is a chosen fixed constant. There's an infinite series we can use to calculate this expression: $y = \frac{bx^2}{2} + \frac{b^3x^4}{24} + \frac{b^5x^6}{720} + \cdots$ (higher powers of x). If x is small, then successive powers of x are even smaller, so the term doing all the hard work here is $\frac{bx^2}{2}$. If we choose $a = \frac{1}{2}b$, we can see that the parabola matches this very closely. Right, that was the warm-up. Now think about a dome. If we try to resolve the forces this time, the weight pulling downwards at a given point will be (approximately) proportional, not to a length, but to a surface area, and so our equivalent of $\frac{y}{x}$ this time is going to be proportional, approximately, to x^2, not x. (This is all extremely rough and ready!) So we can understand why Hooke and Wren arrived at the approximation of a cubic curve, $y = ax^3$, for (a cross-section of) the ideal dome. Again, the true equation has been found since then. It's extremely complicated! There's a series expansion of it that begins $y = a(x^3 + \frac{x^7}{14} + \frac{x^{11}}{440} + \cdots)$ so for small x the cubic equation is a good approximation.

So, Wren and Hooke's best guess for the ideal shape of a masonry dome is a cubic curve in cross-section. They took the part of the curve $y = x^3$ for positive x, and rotated it around a vertical axis to create what Hooke called a "cubico-parabolical conoid". And it's this shape that Wren used for the middle dome, which supports the hemispherical outer dome and its central lantern. By the way, if you stand inside the cathedral and look up, you think you can see through the dome to the lantern, but in fact what you are seeing is a painting of the lantern on the base of the middle dome! In summary, the dome of St Paul's is in fact a triple dome: a catenary inside a cubic curve inside a hemisphere. Pretty amazing, and a tour de force of Wren's mathematical and architectural skill.

Wren: a mathematician of note

The first half of my lectures this academic year were about mathematics and money – we had coins and currencies, game theory, and how to win the lottery. The second half are about unexpected mathematical lives, with Christopher Wren as the first – architect but also mathematician. Coming up in May we have the mathematical life of Florence Nightingale – famous for nursing but also a pioneer of statistics, and then in June Alan Turing, this time we do think of him as a mathematician, but in the context of cryptography. I'm going to tell you about his brilliant work in mathematical biology. There is a link between the two parts though, maths and money, and unexpected mathematical lives – I wonder if you have spotted it? I actually wanted to call this part of the year "Mathematicians of Note", because all three of these mathematical heroes have appeared ON money, Christopher Wren on the £50 between 1981 and 1996, Florence Nightingale on the £10 between 1975 and 1992, and Alan Turing on the £50 from 2021.

So, tune in next time to hear about our next mathematician of note: Florence Nightingale.

Find out more

- If you'd like to read more about Wren's life, two very good places to start are Lisa Jardine's 2002 biography *On a Grander Scale*, and Adrian Tinniswood's 2001 biography *His Invention so Fertile.*
- The wicker basket story, along with much else of interest, is recounted in J. A. Bennett's book, *The mathematical science of Christopher Wren* (Cambridge University Press 1982).
- You can play with the effects of different shaped lenses – spherical, parabolic, and hyperbolic – using Lenore Horner's Geogebra simulation at https://www.geogebra.org/m/Ddbpxd5X
- I've given Gresham lectures previously on both conic sections and the cycloid. Both are available, along with transcripts, online: https://www.gresham.ac.uk/watch-now/conic-sections for conic sections and https://www.gresham.ac.uk/watch-now/geometrical-curves for the cycloid. A lovely video showing how to make a hyperboloid from straight sticks is at https://www.youtube.com/watch?v=ECT8SPWzliE or you can make one with string at https://www.youtube.com/watch?v=mDYY9oOa0Js
- There's an excellent article by Tony Philips on the mathematics of shells at http://www.ams.org/publicoutreach/feature-column/fcarc-shell1. I created my designs in Geogebra3D, using a modified version of the general solid logarithmic spiral equation discussed in the article.

- The 2021 article about generalisations of Wren's ideas on shell growth, is Evans, A.R., Pollock, T.I., Cleuren, S.G.C. *et al.* A universal power law for modelling the growth and form of teeth, claws, horns, thorns, beaks, and shells. *BMC Biol* **19**, 58 (2021). https://doi.org/10.1186/s12915-021-00990-w
- The following paper is a helpful summary of Wren's mathematical work which gives detail of the original sources, for example the places in Wallis's Tractatus de Cycloide where he explain's Wren's rectification of the cycloid and solution to Kepler's problem. *Wren the Mathematician*, D.T. Whiteside, Notes & Records of the Royal Society, 15, pp107-111 (1960).
- The story of the $y = x^3$ approximation to the perfect masonry dome, and a derivation of the correct equation, is given in *Hooke's Cubico-Parabolical Conoid*, by Jacques Heyman, in *Notes and Records of the Royal Society of London*, Vol. 52, No. 1 (Jan., 1998), pp. 39-50 https://www.jstor.org/stable/532075.

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