How to prove $1 = 0$, and other mathematical illusions

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In this lecture I will show you some mathematical illusions: “proofs” that $1 = 0$, that fractions don’t exist, and more.

These “proofs” reveal some very common logical slips that can go unnoticed when we are trying to prove more plausible statements. Some of the more subtle errors involve reasoning that seemed totally correct in the past, until mathematicians realized it could result in seemingly paradoxical outcomes. Understanding what’s gone wrong has led to important developments in mathematics. And the stakes are high. As I’ll show you, once you have “proved” one false claim, you can prove absolutely any statement at all.

I’m going to begin with three truly startling proofs. Since they purport to prove something false, we know that something goes wrong somewhere, but where? I won’t reveal the illusion straightaway, because it’s fun to try and spot what’s going on, but during the course of the lecture we’ll find out, for each one, what has happened.

Don’t worry if some of these seem impossible to believe. Remember the Red Queen’s advice to Alice. “I daresay you haven't had much practice,” said the Queen. “When I was your age, I always did it for half-an-hour a day. Why, sometimes I've believed as many as six impossible things before breakfast.”

Impossible fact #1 $1 = 0$

Proof Note that $1 - 1 + 1 - 1 + 1 - 1 + \cdots = (1 - 1) + (1 - 1) + (1 - 1) + \cdots = 0 + 0 + 0 + \cdots = 0$.

On the other hand, $1 - 1 + 1 - 1 + 1 - 1 + \cdots = 1 \underbrace{-1+1} \underbrace{-1+1} \underbrace{-1+1} \cdots = 1 + 0 + 0 + \cdots = 1$.

So $1 = 0$. ■

What went wrong here? We all probably feel that we are “not allowed to do this”, and that the problem is around infinite sums somehow, but it can be quite tricky to articulate exactly what is wrong. We’ll come back to this later. We’ll also see more “proofs” that $1 = 0$, just in case you weren’t convinced by this one.

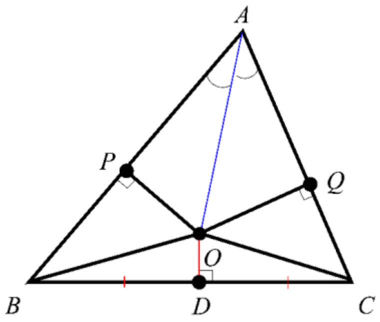
Impossible fact #2 The smallest positive number is 1.

Proof Let x be the smallest positive number. Clearly $x \leq 1$. Now x^2 is also positive, so by minimality of x, we have $x \leq x^2$. Divide both sides by (the positive number) x to get $1 \leq x$. Thus $1 \leq x \leq 1$, which forces us to conclude that $x = 1$. ■

Perhaps this is the moment to give you a couple of quotations. Goethe wrote that “Mathematicians are like Frenchmen: whatever you say to them, they translate it into their own language and forthwith it means something entirely different.” And Saint Augustine advised “Beware of mathematicians and all those who make empty prophesies.” So, don’t say I didn’t warn you.

Impossible fact #3

Let's do a bit of geometry. It's going to involve congruent triangles. Take an arbitrary triangle, ABC. Let O be the point where the angle bisector at A meets the perpendicular bisector of BC (so in the diagram D is the midpoint of BC). From O, drop perpendiculars OP and OQ to sides AB and AC respectively. Finally, draw in the lines OB and OC.



Now we're going to make some deductions about similar triangles. Something is going to go wrong somewhere in this proof so watch out for it.

Firstly, compare ΔODB and ΔODC. Side OD is common to both, and because OD is the perpendicular bisector of BC, we have $\underline{BD} = \underline{CD}$ and $O\hat{D}C = O\hat{D}B$. So ΔODB and ΔODC are congruent (by SAS).

Second, compare ΔAPO and ΔAQO. By construction, $P\hat{A}O = Q\hat{A}O$ and $A\hat{P}O = A\hat{Q}O$. Hence (angles in a triangle add up to 180) $P\hat{O}A = Q\hat{O}A$. Since AO is common to both triangles, we have that So ΔAPO and ΔAQO are congruent (by AAAS).

Thirdly, $\underline{OB} = \underline{OC}$ because ΔODB and ΔODB are congruent, and $\underline{OP} = \underline{OQ}$ because ΔAPO and ΔAQO are congruent. The angles at P and Q are right angles, and therefore, by (RHS) we see that ΔOPB and ΔOQC are congruent.

Now, $\underline{AB} = \underline{AP} + \underline{PB}$. From the congruence of ΔAPO and ΔAQO, we have $\underline{AP} = \underline{AQ}$. From the congruence of ΔOPB and ΔOQC we have $\underline{PB} = \underline{QC}$. Therefore $\underline{AB} = \underline{AP} + \underline{PB} = \underline{AQ} + \underline{QC} = \underline{AC}$.

That is, $\underline{AB} = \underline{AC}$. We have now proved the incredible fact that ***all triangles are isosceles***. ■

What is a proof?

At this point we may be questioning what a proof actually is. There are two answers to this, which it's important not to mix up.

- Answer (a) A sequence of logical deductions that establishes the truth of a statement.
- Answer (b) Half a percent of alcohol

It's very important to remember: don't drink and derive!

Assuming we are trying to construct a genuine proof, every step must follow logically from what has gone before. Sometimes we get the direction wrong. Suppose you are asked to prove that, for instance, that $(x+y)^2 - 4xy = (x-y)^2$. It's tempting to write something like this:

$$(x+y)^2 - 4xy = (x-y)^2$$

$$x^2 + 2xy + y^2 - 4xy = x^2 - 2xy + y^2$$

$$-2xy = -2xy$$

$$0 = 0$$

and then say "$0 = 0$ is true, therefore $(x+y)^2 - 4xy = (x-y)^2$ is true". But that's very dangerous. If this is allowed, we could equally say:

$$1 = -1$$

$$1^2 = (-1)^2$$

$$1 = 1$$

And now, by the same reasoning as above, "$1 = 1$ is true, therefore $1 = -1$ is true". The vital point is that every step in the proof has to follow from what has gone *before*, not what is going to come after!

Let's have another impossible fact.

Impossible fact #4: $1p = £1$.

Proof: $1p = £0.01 = (£0.1)^2 = (10p)^2 = 100p = £1$. ■

This is a unit-mixing nightmare, and converting units can definitely cause problems in the real world. The most infamous example is probably the Mars Climate Orbiter, which got too close to Mars in 1999 and was lost, wasting $125 million, because a team on the ground sent thruster data in pounds of thrust, while NASA was expecting Newtons (one pound of thrust is the force required to prevent a mass of one pound from falling under earth's gravity, so it's $0.454 \times 9.81 \approx 4.45N$). The oldest example I know is the Swedish warship *Vasa*, which sank shortly after being launched in 1628. The ship had several problems – its centre of gravity was too high, among other things – but after it was raised in 1983, archaeologists found that a critical part of the problem was the uneven weight distribution. They found four rulers that the workers who built the ship had used. Everyone was working in feet. But unfortunately, on one side of the ship they seem to have been working in Amsterdam feet (28.31cm), and on the other they were using Swedish feet (29.69cm). That difference was enough to make the ship significantly heavier on the port side, which contributed to the instability of the ship. I think my favourite calculation error in a construction project is perhaps the High Rhine Bridge connecting the Swiss city of Laufenberg to the German city of the same name. The problem originated with the fact that altitude measurements depend on where you are counting as zero altitude. In Switzerland, the official reference base point for height is the Repère Pierre du Niton, the "pierre" in question being a specific rock sticking just above the surface in Lake Geneva. In Germany (and much of the EU) the reference point is based on mean water level in Amsterdam. The discrepancy of 27cm was known about and accounted for, but unfortunately, a rogue minus sign meant that instead of raising the Swiss side by 27cm, they lowered it. When the two halves of the bridge approached each other, it became clear that they were 54cm apart, and corrections had to be hurriedly made.

We don't usually try to square units of currency, but if we tried the same trick with centimetres and metres, we would see the problem instantly: $0.01m$ does not equal $(0.1m)^2$ because $0.01m$ is a measurement of length and $(0.1m)^2 = (0.1)^2m^2$ is a measurement of area.

More proofs that $1 = 0$

Proof 2 Observe that $x^0 = 1$ for all x. Also, $0^y = 0$ for all y. Set $x = y = 0$. Then $1 = 0^0 = 0$. ■

Proof 3 Just as $1 = 1^2 = 1$, $2 + 2 = 2^2 = 4$, and $3 + 3 + 3 = 3^2 = 9$, more generally, for any x, we have that $x + x + \cdots + x = x^2$ (where on the left there are x copies of x). Differentiating gives $1 + 1 + \cdots + 1 = 2x$. Hence, for any x, we have $x = 2x$. Setting $x = 1$ gives $1 = 2$, from which $0 = 1$ follows. ■

Everything is true

It's a disconcerting fact that if you allow even a single false statement into the temple of mathematics, then the entire edifice crumbles to the ground. A consequence of proving $1 = 0$ is the following.

Corollary Let *S* be any statement. Then *S* is true.

Proof Let *F* be any false statement. If we can prove "at least one of *S* and *F* is true", then *S* is true. Let *F* be the false statement "$1 = 0$". We just proved $1 = 0$. So *F* is also true. Hence, we have proved "at least one of *S* and *F* is true". Therefore *S* is true. ■

Impossible fact #5: $0 > 0$.

Proof Let $x = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots$ and $y = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \cdots$ Now, $2y = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = x + y$. This gives $x + y = 2y$, hence $x - y = 0$. But $x - y = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) + \cdots$. So $x - y > 0$. Hence $0 > 0$. ■

What is going on with infinite series?

Infinite series are dangerous. For example, if $x = 1 + 2 + 4 + 8 + \cdots$, then $1 + 2x = 1 + 2(1 + 2 + 4 + \cdots) = 1 + 2 + 4 + 8 + \cdots = x$. Hence, $1 + 2x = x$, so $x = -1$. But this series clearly sums to infinity. We are dangerously close to proving that $\infty = -1$ (of course, the computer scientists have known this for some time). In this example we could argue that $x = \infty$ is actually an alternative solution to $1 + 2x = x$, or we could disallow trying to calculate with infinity. But that wasn't the issue with our first proof of the lecture. We claimed that $1 - 1 + 1 - 1 + 1 - 1 + \cdots = (1 - 1) + (1 - 1) + (1 - 1) + \cdots = 0 + 0 + 0 + \cdots = 0$. But, on the other hand, $1 - 1 + 1 - 1 + 1 - 1 + \cdots = 1 \overbrace{-1 + 1} \overbrace{-1 + 1} \overbrace{-1 + 1} \cdots = 1 + 0 + 0 + \cdots = 1$. Something has gone wrong, but what? To understand this, we need to think about what we really mean when we try and add infinitely many terms together. This is something we want and need to do on occasion. For example, we are all happy with expressions like $\frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10000} + \cdots$, otherwise known as $0.3333\cdots$ or $0.\dot{3}$. Even if we wrote these numbers as fractions, we still need to be able to cope with decimals like π that genuinely do go on for ever without a recurring pattern, and cannot be expressed as fractions.

What we have to work out is whether the series *converges*. That is, if we make a sequence whose n^{th} term is the sum of the first n numbers in the series, does that sequence approach some limit? We can make this mathematically rigorous, but in essence we say that a sequence converges to a limit L if its terms get as close as we like to that limit. That is, you can pick as tiny a number as you like, and we can be sure that there is a point beyond which all the terms of the sequence are closer to L than this tiny number. A series converges precisely when the sequence of partial sums converges.

One example is $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}\cdots$.The sequence of partial sums goes $\frac{1}{2},\frac{3}{4},\frac{7}{8},\frac{15}{16},\cdots$ and the n^{th} partial sum is $1-\frac{1}{2^n}$. This gets arbitrarily close to 1 as n increases, and so we can be confident that the series converges with limit 1. Now look at $1-1+1-1+\cdots$. This time, the sequence of partial sums goes $1,0,1,0,1,0,\cdots$. It never settles to anything, just alternates between two values. So, it definitely doesn't converge – we say it's *divergent*. (By the way, if you define $x=1-1+1-1+\cdots$, then $1-x=x$, from which $x=\frac{1}{2}$. There is actually a way in which this can be made mathematically rigorous, called the Cesàro summation, but it's beyond what we have the time to discuss here.)

If these partial sums have to stick closer and closer to a finite limit, then at minimum the terms we are adding have to get smaller and smaller. But that's not enough. Think about the harmonic series:

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots=1+\left(\frac{1}{2}\right)+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\cdots>1+\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right)+\cdots$$
$$=1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots$$

This will eventually exceed every finite number. The series has an infinite sum: it diverges. On the other hand, the so-called alternating harmonic series $S=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots$ behaves differently. If you plot a graph of the sum to n terms you can see that it appears to be converging to a number around 0.69. It can be proved rigorously that this is indeed what's happening, and the exact value is $S=ln\,ln\,2\ \approx 0.6931$ ..., so the alternating harmonic series does converge. But there's a problem. When we add a finite sequence of numbers, it doesn't matter what order we add them in, we'll get the same outcome. We know that $1+2+3=3+1+2$, for instance. But if we order the terms of the alternating harmonic sequence differently, we run into trouble. In the following rearrangement, we would eventually see every $\frac{1}{n}$, with the correct sign, exactly once. Terms 1, 4, 7 and so on are $\frac{1}{n}$ for odd n. Terms 2, 5, 8 and so on are $-\frac{1}{n}$ for even n where n is not divisible by 4, and terms 3, 6, 9 and so on are $-\frac{1}{n}$ for n a multiple of 4.

$$S=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\cdots=\left(1-\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{6}-\frac{1}{8}\right)+\left(\frac{1}{5}-\frac{1}{10}-\frac{1}{12}\right)+\cdots$$
$$=\left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{6}-\frac{1}{8}\right)+\left(\frac{1}{10}-\frac{1}{12}\right)+\cdots\frac{1}{2}\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots\right)=\frac{1}{2}S.$$

Which gives $S=\frac{1}{2}S$, and that unfortunately appears to imply that $S=0$. Studying this exact series was what prompted the mathematician Augustin-Louis Cauchy to say, correctly, that we can't assume that infinite series can be rearranged at will leaving the sum the same. Later, Bernhard Riemann proved an amazing result called the Riemann Series Theorem. A series is called *absolutely convergent* if not only does it converge, but if you replace all its terms with their absolute values (in other words, replace each x with $|x|$), the resulting series still converges. The alternating harmonic series is not absolutely convergent because the corresponding series of absolute values is $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$, the standard harmonic series, which is divergent. Anyway, the theorem says that if an infinite series is convergent but not absolutely convergent, then its terms can be rearranged so that it converges to any number you choose, and can also be rearranged so that it diverges! On the other hand, if the series is absolutely convergent, then you can rearrange as much as you like with no issue.

Impossible fact #6 $\frac{1}{2}=\frac{1}{3}$

Proof This proof is due to French mathematician Joseph Bertrand and uses probability theory. We're going to calculate the same thing in two different ways. What's the probability P that a random chord of a circle is longer than the side of its inscribed equilateral triangle?

Firstly, we can randomly choose a chord by picking two random points on the circumference. Given the chord AB, we can compare to the inscribed triangle with one vertex at A. The chord will be longer than the side of the triangle if and only B is somewhere on the arc between the other two vertices of the triangle. The length of that arc is exactly one third of the circumference of the circle, and so the probability of this occurring is $\frac{1}{3}$. Thus, $P = \frac{1}{3}$.

Now let's calculate it a different way. The perpendicular bisector of any chord passes through the centre of the circle. So we can randomly choose a chord by choosing a radius, and then any point on the radius to be the midpoint of our chord. Put one vertex of the inscribed triangle at the other end of the diameter corresponding to this radius, so the base of the triangle is parallel to our chord. The chord is longer than the side of the triangle if and only if its midpoint is farther from the centre than the base of the triangle is. The inscribed triangle meets the radius halfway along its length. Therefore, the probability of this occurring is $\frac{1}{2}$. That is, $P = \frac{1}{2}$. Hence, $\frac{1}{2} = \frac{1}{3}$. ■

Bertrand was not, of course, "proving" that $\frac{1}{2} = \frac{1}{3}$. He devised these arguments to show that we have to be incredibly careful about what we mean when we say we are going to choose something randomly. He gave another argument that if you choose the chord randomly in a different way, you can also arrive at $P = \frac{1}{4}$. The three collectively are known as Bertrand's paradox. The upshot of it all is that if there are infinitely many possibilities, we can't just say we are assuming they are all equally likely and that we are picking one at random, unless we define very carefully first how we are doing that selection. It's not that one answer is right and one is wrong, rather that our method of selection needs to be part of the definition of the question.

It's rather like our assertion earlier that $1 = 0$ because $0^0 = 0$ and $0^0 = 1$. In fact, what we should conclude here is that 0^0 is undefined. If you once believe it has a value, then say $x = 0^0$. Then $x = 0^0 = 0^{2-2} = \frac{0^2}{0^2} = \frac{0}{0}$. So $\frac{0}{0}$ suddenly has a value, this x. That means we can calculate with it, and we're forced to conclude, using the standard rules of fractions, that $\frac{0}{0} + \frac{0}{0} = \frac{0}{0}$, which implies $2x = x$ and so $x = 0$. On the other hand, clearly $x = \frac{1}{x}$, as the fraction is the same on the top and bottom, which forces $x = \pm 1$. Nothing works. We could argue then that $\frac{0}{0} = \infty$. As we just saw, that implies $0^0 = \infty$, which is something of a surprise!

Spotting the errors

I promised I'd resolve the first three impossible proofs. We've already dealt with the infinite series proving that $1 = 0$. How about the smallest positive number being 1? The problem here is saying that there is such a number. Really this is a "proof by contradiction" that no such number can exist. A proof by contradiction (also known as a *reductio ad absurdum*) goes like this: assume that the thing you want to prove is false. Then deduce something clearly incorrect. That means your assumption must have been wrong. Hence the original statement must be true. Here, our statement would be "there is no smallest positive number". We prove it by saying "suppose, for a contradiction, that there is a smallest positive number x". Every step of our original argument follows correctly from this, and so we would find that $x = 1$. However, that's clearly false because $\frac{1}{2}$ is a smaller positive number. So, our original assumption was false, and we can conclude that there is in fact no smallest positive number. Proof by contradiction is a very powerful technique. I won't say that the next theorem is false, but you can decide what you think of the proof.

The Optimist's Theorem All positive integers are interesting.

Proof (by contradiction) If this were false, then there would be at least one boring positive integer. And that would mean there would be a smallest such. Let n be the smallest boring number. But that's an interesting property for n to have! So n is interesting, which we said was not the case. So our assumption that not all positive integers are interesting led us to a contradiction. Hence, our assumption was wrong and in fact all positive integers are interesting. ■

Let's think about isosceles triangles. Whenever you are reading a proof, impossible or not, it can be a very helpful tool to work it through for yourself, even trying a specific example just to get a feel for it, and also to try for the most extreme cases you can think of. For geometry proofs there are some great online tools nowadays, but you can also just draw your own diagram. That isosceles triangle proof began with a construction. Take a triangle and construct the angle bisector of one side, and the perpendicular bisector of the opposite side. You'll see that if the triangle is isosceles, these are the same line. As soon as it becomes scalene, that intersection point O exists but appears to be outside the triangle. When we drop perpendiculars we now find that we may have to extend the sides to create the points P and Q. Now, all the triangles we showed are congruent are still congruent, because that reasoning is not affected. The problem happens in the very last line, because instead of $\underline{AB} = \underline{AP} + \underline{PB} = \underline{AQ} + \underline{QC} = \underline{AC}$, all we can conclude is $\underline{AB} = \underline{AP} \pm \underline{PB} = \underline{AQ} \pm \underline{QC}$, which may or may not equal $\underline{AC}$. (They will be equal, of course, precisely when the triangle is isosceles.)

Let it be true

We've proved all triangles are isosceles, so we should have no trouble with the following theorem.

Theorem There exists a triangle with three right-angles.

In fact, this can be true, but we have to change our assumptions. In the Euclidean geometry of the plane, triangles cannot have three right-angles. But in the geometry of the surface of a sphere, where "lines" are the geodesics – the shortest path between two points – our triangles are made of arcs of great circles and angles in a triangle add up to more than 180 degrees. There are plenty of triangles with three right angles. Asking "when could this be true" is one of the most powerful questions in mathematics.

To finish, here's a less exalted example. We all know how to simplify fractions – we cancel the common terms. To simplify $\frac{sinsin\, x}{n}$ we just cancel the n's to get $\frac{sinsin\, x}{n} = 6$, right? No, that's silly. What we mean is that if there's the same numerical factor on the top and bottom, we can cancel that. For example, $\frac{16}{64} = \frac{1}{4}$, by cancelling the 6's. That's wrong too, of course, but in this case it happens to give us the correct answer. Are there any other examples like this? Yes, $\frac{19}{95} = \frac{1}{5}$, by "cancelling the 9's". In fact, $\frac{19\cdots9}{9\cdots95} = \frac{1}{5}$ whenever same number of 9's on top and bottom, and we can prove it (it really is true!) in a single line.

$$\overbrace{9\cdots9}^{n}5 = 10^{n+1} - 5 = 5(2 \times 10^{n} - 1) = 5 \times 1\overbrace{9\cdots9}^{n}.$$

If you want to play around with this, can you find other instances where the wrong kind of cancellation actually does give the correct answer? And why stop there – is it ever true that $\frac{a}{b} + \frac{c}{d} = \frac{a+c}{b+d}$? Can you find examples, or if there are none, can you prove there are none? Is there other spurious arithmetic out there?

I hope you've enjoyed these mathematical conjuring tricks. My next and final Gresham lecture is on real proofs; we'll talk about what mathematicians mean by proof, and I'll show you some of my favourites.

Further Reading

If you want to explore geometric constructions as we did to test the isosceles triangle argument, try Geogebra https://www.geogebra.org/

Unfortunately, in terms of spurious proofs, there isn't a great deal of further reading to recommend. I have been collecting them for many years. All the proofs here, and many others, seem to be part of the mathematical folklore and impossible to attribute to any individual. I've certainly never seen any of them associated to any one person. The first such proof I ever saw was when I was 14 years old and attending a mathematics masterclass – it was the $1 - 1 + 1 - 1 \cdots$ "proof" with which this talk began, and it was shown to us students by a mathematician called Heather Cordell. The proof that all triangles are isosceles was told to me by my mother, who was a maths teacher. Since then I've been shown some by mathematician friends, have found more online, and encountered others in the one book I'm aware of having been published in the UK on this topic, which I picked up second-hand some years ago: *Fallacies in Mathematics*, by E. A. Maxwell, Cambridge University Press (1959).