Euclid

Professor Robin Wilson

16 January 2002

In this sequence of lectures, I want to look at three great mathematicians that may or may not be familiar to you. We all know the story of Isaac Newton and the apple, but why was it so important? What was his major work (the Principia Mathematica) about, what difficulties did it raise, and how were they resolved? Was Newton really the first to discover the calculus, and what else did he do? That is next time a fortnight today and in the third lecture I will be talking about Leonhard Euler, the most prolific mathematician of all time. Well-known to mathematicians, yet largely unknown to anyone else even though he is the mathematical equivalent of the Enlightenment composer Haydn.

Today, in the immortal words of Casablanca, here's looking at Euclid author of the Elements, the best-selling mathematics book of all time, having been used for more than 2000 years indeed, it is quite possibly the most printed book ever, apart than the Bible. Who was Euclid, and why did his writings have such influence? What does the Elements contain, and why did one feature of it cause so much difficulty over the years?

Much has been written about the Elements. As the philosopher Immanuel Kant observed:

There is no book in metaphysics such as we have in mathematics. If you want to know what mathematics is, just look at Euclid's Elements.

The Victorian mathematician Augustus De Morgan, of whom I told you last October, said:

The thirteen books of Euclid must have been a tremendous advance, probably even greater than that contained in the Principia of Newton. The sacred writings excepted, no Greek has been so much read and so variously translated as Euclid.

And in his 1843 Lectures on the Principles of Demonstrative Mathematics, P. Kelland wrote at length:

It is certain that from its completeness, uniformity and faultlessness, from its arrangement and progressive character, and from the universal adoption of the completest and best line of argument, Euclid's Elements stand preëminently at the head of all human productions. In no science, in no department of knowledge, has anything appeared like this work: for upward of 2000 years it has commanded the admiration of mankind, and that period has suggested little toward its improvement.

So, what is this work that stands pre-eminently at the head of all human productions? Let's start by placing it in historical perspective.

The period of Greek mathematics lasted for about 1000 years, starting around 600 BC, and falls into three sections. The first of these concerns the semi-legendary figures of Thales of Miletus and Pythagoras. We don't know much about Thales. According to legend he brought geometry to Greece from Egypt, predicted a solar eclipse in 585 BC and showed how rubbing with a stone produced electricity in feathers. Further, as the commentator Proclus said, writing about Euclid's Elements:

The famous Thales is said to have been the first to demonstrate that the circle is bisected by the diameter. If you wish to demonstrate this mathematically, imagine the diameter drawn and one part of the circle fitted upon the other. If it is not equal to the other, it will fall either inside or outside it, and in either case it will follow that a shorter line is equal to a longer. For all the lines from the centre to the circumference are equal, and hence the line that extends beyond will be equal to the line that falls short, which is impossible.

This brings up a couple of interesting points. First, there are no extant primary sources for Greek mathematics, so we have to rely entirely on commentaries and on later versions. Proclus was a commentator from the fourth century AD who derived much of his material from earlier commentaries, now lost, by Eudemus of Rhodes, a student of Aristotle in the fourth century BC. But Proclus lived some 800 years after Euclid. It is rather like us trying to comment on contemporary accounts of Robin Hood, so we have to treat Proclus' commentaries with some scepticism, while recognizing that they are all we have. We also have translations and commentaries by Islamic scholars, which are most useful, but the fact remains that the earliest copy of Euclid's Elements that survives is an Arabic translation in the Bodleian Library, dating from the year 888.

The second point that arises from the extract about Thales is that it was concerned with mathematical proof. Of all the mathematical contributions by the Greeks and they include the introduction and study of conics (the ellipse, parabola and hyperbola) and the study of the three classical problems (on the impossibility of trisecting an angle, doubling a cube and squaring the circle) the idea of deductive reasoning and mathematical proof is the most fundamental. Starting with some initial assumptions or axioms we derive some simple results, and then more complicated ones, and so on, eventually creating a great hierarchy of results, all depending on previous ones. We'll see an example of that later, when we look at Book I of the Elements. There are several allowable methods of proof for the Thales extract we used a proof by contradiction (or reductio ad absurdum), where we assume that the result we want to prove is incorrect and then deduce a result that contradicts our assumptions. We'll see some more examples of this later on.

These ideas were developed around 550 BC by the Pythagoreans the brotherhood that existed around the Greek seaport of Crotona, now in Italy. The Pythagoreans believed that all is number that everything worthy of study can be quantified and divided the mathematical sciences into four parts: arithmetic, which deal with quantities; music, which deals with intervals regarded as ratios between quantities; geometry, which deals with magnitudes at rest; and astronomy, which deals with magnitudes in motion. These four mathematical areas were later combined into the quadrivium which, together with the trivium of grammar, rhetoric and logic, formed the seven liberal arts subjects studied in academies and universities for the next 2000 years or so.

The Pythagoreans were concerned with many areas of mathematics. For them, arithmetic meant not only ordinary calculations with whole numbers, but also the study of particular types of numbers. For example, they knew that the square numbers are obtained by adding consecutive odd numbers (for example, 16 = 1 + 3 + 5 + 7) and that the triangular numbers are obtained by adding consecutive integers (for example, 15 = 1 + 2 + 3 + 4 + 5). They also studied prime numbers, as we'll see later.

The Pythagoreans were also interested in commensurable and incommensurable numbers, which play an important role in Euclid's Elements. We say that the numbers 8 and 12 are commensurable because they can both be measured a whole number of times by a ruler of length 4, and the numbers 3p and 5p are commensurable because they can both be measured a whole number of times by a ruler of length p. Essentially, two numbers are commensurable if their ratio can be written as a fraction (a ratio of whole numbers) so 3p divided by 5p is just 3/5, which is a fraction. However, as they discovered, the diagonal and side of a square are not commensurable. The current proof of this, which I'll now give, is well known and is typical of the Greek approach, except that it uses algebra, where the Greeks would have couched everything in geometrical terms.

The proof is by contradiction: by Pythagoras' theorem the ratio of the diagonal and side of a square is v2, and we must prove that this number cannot be written as a fraction as a/b, where a and b are whole numbers. So, to obtain a contradiction we assume that v2 can be written as a fraction, and we can further assume that this fraction is written in its lowest terms, so a and b have no common factor. By squaring, we can rewrite this as a2 = 2b2, which means that a2 must be an even number. But if a2 is even, then a must also be even (because otherwise, a is odd, so a2 is odd). Since a is even, we can write a = 2k, for some whole number k. So 2b2 = 4k2, which gives b2 = 2k2, so b2 is even, and so b is even. But this gives us a contradiction a and b are both even, so are both divisible by 2, contradicting the fact that a and b had no common factor. This contradiction arises from our original assumption that v2 can be written as a fraction so this assumption is wrong: v2 cannot be written as a fraction, and so the diagonal and side of a square

are incommensurable.

The second great period of Greek mathematics took place in Athens, with the founding of Plato's Academy around 387 BC, in a suburb of Athens called Academy thats where the word Academy comes from. Plato's Academy soon became the focal point for mathematical study and philosophical research, and it is said that over its entrance were the words 'Let no-one ignorant of geometry enter these doors'.

Plato believed that the study of mathematics and philosophy provided the finest training for those who were to hold positions of responsibility in the state, and in his Republic, he discussed the importance of the mathematical arts of arithmetic, geometry, astronomy and music for the so-called philosopher-ruler. His book Timaeus includes a discussion of the five regulars, or Platonic, solids: the tetrahedron, cube, octahedron, dodecahedron and icosahedron in which the faces are all regular polygons of the same type and the arrangement of these polygons at each corner is the same: for example, the cube has six square faces, with three appearing at each corner.

One of the students at the Academy was Aristotle, who remained there for some twenty years. He was fascinated by logical questions and systematised the study of logic and deductive reasoning. Another early student there was the mathematician and astronomer Eudoxus of Cnidus, who advanced the hypothesis that the sun, moon and planets move around the earth on rotating concentric spheres, a hypothesis later adopted in modified form by Aristotle. I mention Eudoxus because he is often credited with developing the theory behind two of the books in Euclid's Elements Book V on proportion and Book XII on the method of exhaustion.

Around 300 BC, with the rise to power of Ptolemy I, mathematical activity moved to the Egyptian part of the Greek empire. In Alexandria Ptolemy founded a university that became the intellectual centre for Greek scholarship for the next 800 years our third period of Greek mathematics. Ptolemy also started its famous library, which held over half-a-million manuscripts, before eventually being destroyed by fire.

There are a number of important mathematicians associated with Alexandria Apollonius (who wrote the standard work on conics), the great astronomer Ptolemy (after whom the Ptolemaic system of planetary motion is named), the neoplatonists such as the geometer Pappus and Hypatia, one of the most important women mathematicians of all time. But arguably the greatest of all, and the earliest important mathematician to be associated with Alexandria, was Euclid, who lived and taught there around 300 BC.

We know virtually nothing about Euclid's life. As well as the Elements, to which we turn in a minute, he his credited with writing many other books, including several in geometry (the Data and On Divisions of Figures), the Porisms (a three-book work on problems which has not survived), a four-book work on Conics (which also has not survived), and books on Astronomy and Optics.

We now come to the main topic of this talk which is an outline of Euclid's Elements. The first thing to note is that it was not the earliest such work, a compilation of results known at the time, organised in a systematic way since Hippocrates of Chios and others had written Elements, though these have not survived. However, Euclid's was the most important. As the commentator Proclus observed:

It is a difficult task in any science to select and arrange properly the elements out of which all other matters are produced and into which they can be resolved. Such a treatise ought to be free of everything superfluous, for that is a hindrance to learning; the selections chosen must all be coherent and conducive to the end proposed, in order to be of the greatest usefulness for knowledge; it must devote great attention both to clarity and to conciseness, for what lacks these qualities confuses our understanding. Judged by all these criteria, you will find Euclid's introduction superior to others.

Euclid's Elements consists of thirteen books, traditionally divided into three main parts, on plane geometry, arithmetic and solid geometry. Here's a quick overview. Books I and II deal with the foundations of plane geometry and the geometry of rectangles; Books III and IV then proceed to the geometry of circles. Book V is on proportion, which is then applied to the geometry of similar figures in Book VI. Books VII, VIII and IX are on arithmetic, and include basic properties such as the divisibility of integers, as well as a discussion of prime and perfect numbers. Book X, the longest and most difficult book amounting to a quarter of the whole work, is on incommensurable line segments. The final three books are on solid geometry and conclude with a construction and classification of the five Platonic solids. Since there are several hundred propositions in all, I won't go through all thirteen books systematically. Instead, I've chosen a few propositions that indicate the types of proof that Euclid gives, as well as the range of topics discussed.

Book I, Foundations of Plane Geometry, starts with 23 definitions, such as the following:

1. A point is that which has no part.

2. A line is breadthless length. . .

8. A plane angle is a surface which lies evenly with the straight lines on itself. . .

15. A circle is a plane figure contained by one line such that all the straight lines falling upon it from one point among those lying within the figure are equal to one another...

17. A diameter of the circle is any straight line drawn through the centre and terminated in both directions by the circumference of the circle, and such a straight line also bisects the circle. . .

That one is interesting, because Thales found it necessary to prove the last statement, that a diameter bisects a circle.

23. Parallel straight lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction.

There are then five geometrical Postulates, beginning with three allowable constructions:

1. To draw a straight line from any point to any point.

2. To produce a finite straight line continuously in a straight line.

3. To describe a circle with any centre and distance.

4. That all right angles are equal to one other.

5. That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than two right angles.

This last postulate doesn't seem to fit in with the others and, as we'll see, it caused no end of problems over the next 2000 years.

These are followed by five Common Notions, such as 'Things which are equal to the same thing are equal to each other (that is, in algebraic notation, if $x = a$ and $y = a$, then $x = y$)', and 'The whole is greater than the part'.

Finally, we get down to our first proposition:

Proposition 1. On a given straight line to construct an equilateral triangle.

Notice first that, although this proposition concerns the construction of an equilateral triangle, Euclid then proceeds to prove that the construction actually works that the resulting triangle is equilateral. Notice also that at each stage there is a reference to a definition or a postulate, and in later propositions there are frequent references to earlier propositions. The following diagram shows how some of the propositions in Book I build on one another, with Proposition 47 at the top. We'll see what Proposition 47 is in a moment.

Here are some of the other Propositions in Book I. Proposition 5 is the famous Pons Asinorum, the bridge of asses, that in an isosceles triangle, the angles at the base are equal to each other. This result is credited to Thales, and in medieval universities it was often as far as students of Euclid ever reached: if you could cross the bridge of asses, you could then go on to all the treasures that lay ahead! Propositions 1 to 26 are all basic results and constructions in plane geometry, such as the bisection of an angle and congruence theorems for triangles, and these are followed by nineteen propositions on parallel lines and parallelograms. These include the results that the angles of a triangle add up to two right angles, and that given any triangle, we can construct a rectangle with the same area. Since any polygon, such as a pentagon, can be split into triangles, we can construct a rectangle with the same area as any polygon. In Book II, Euclid shows how to construct a square with the same area as a given rectangle. Combining these results, we can then find a square with the same area as any given polygon that is, we can square any polygon. Unfortunately, we cannot similarly square the circle that is, find a square with the same area as a given circle. This was one of the Greek classical problems, and it was not until the nineteenth century that it was finally proved to be impossible.

Book I ends with Pythagoras' theorem and its converse.

Proposition 47. In right-angled triangles, the square on the side subtending the right angle is equal to the squares on the sides containing the right angle.

This result may or may not have been known to Pythagoras but was certainly known in some form to the Babylonians a thousand years before Pythagoras, as a famous tablet Plimpton 322 indicates: it has lists of numbers, including two of the sides of right-angled triangles. Notice that, in Euclid, it is a theorem concerning the areas of squares it is not an algebraic result, that a2 + b2 = c2 for the sides a, b and c of a right-angled triangle.

The proof in the Elements is believed to be by Euclid himself. He adds a few lines, as shown, and proves that the triangles ABD and FBC are similar triangles. From earlier results on parallelograms he deduces that the area of the square ABFG equals the area of the rectangle BDLN. In the same way, he proves that the area of the square ACKH equals the area of the rectangle CELN. By adding these two results together, he deduces that their sum, the area of the square BCED is the sum of the squares on the other two sides.

Charles Dodgson, better known as Lewis Carroll, was a great enthusiast for Euclids Elements, and said of this theorem:

It is as dazzlingly beautiful now as it was in the day when Pythagoras first discovered it, and celebrated the event, it is said, by sacrificing a hecatomb of oxen a method of doing honour to Science that has always seemed to me slightly exaggerated and uncalled-for. One can imagine oneself, even in these degenerate days, marking the epoch of some brilliant scientific discovery by inviting a convivial friend or two, to join one in a beefsteak and a bottle of wine. But a hecatomb of oxen! It would produce a quite inconvenient supply of beef.

Book II, on the geometry of rectangles, is very short and quite controversial. On the surface, it seems to be about what it says, the geometry of rectangles, but the interesting thing is that we can recast these in algebraic terms. Opinions differ as to what Euclid had in mind when he presented these results, especially since many of them do not reappear later in the Elements. Let's look at a couple of examples.

Proposition 1 shows that if we split up a rectangle, then the sum of the separate areas is equal to the sum of the areas of the separate rectangles. In algebraic terms, this says that a(b1 + b2 + b3 + b4) = ab1 + ab2 + ab3 + ab4 which is an example of the distributive law of algebra.

Proposition 4 is even more interesting. It states:

If a straight line is cut at random, the square on the whole is equal to the squares on the segments and twice the rectangle contained by the segments.

We can interpret this algebraically as (a + b)2 = a2 + b2 + 2ab. In a similar way, later propositions in Book II can be interpreted in more modern terms as (a + b)(a b) = a2 b2, as completing the square for solving a quadratic equation, and as the cosine law for triangles.

Book III introduces the properties of circles including their chords and tangents. I'll mention just a couple of results here:

Proposition 20. In a circle the angle at the centre is double the angle at the circumference when the angles have the same arc as base.

Two special cases of this are:

Proposition 31. In a circle the angle in a semicircle is right.

Proposition 22. The opposite angles of quadrilaterals in circles are equal to two right angles.

Book IV, Regular polygons in circles, is the most homogeneous and tightly constructed book in the Elements. Given a circle, it is shown how to construct equilateral triangles, squares and regular hexagons inside and circumscribing the circle. Later, Archimedes attempted to estimate p by calculating the perimeter of the hexagons inside and outside a circle and this gave a very poor result, so he repeatedly doubled the number of sides until he reached 96 sides, giving him the result that p lies between 310/71 and 31/7. 500

years later this result was extended by Chinese mathematicians to polygons with 24,576 sides, yielding p to six decimal places.

In Book IV, Euclid's main achievement was to construct a regular pentagon in a circle. By combining this construction with that of an equilateral triangle, he was then able to construct a regular 15-sided polygon. In the 1790s, Gauss extended this idea to determine exactly which regular polygons can be constructed. They are ones based on the numbers 3, 5, 17, 257 and 65537 - the so-called Fermat primes.

Book V, The general theory of magnitudes in proportion, is probably based on the work of Eudoxus, and is the most abstract book in the Elements. It deals with the theory of proportion: results involving ratios, such as

if a : b = c : d, then a : c = b : d.

Using algebraic notation, results of this form are very easy to prove:

if a/b = c/d, then a/c = b/d.

But in the Elements they are proved geometrically, which seems much more complicated. These results on ratios are then applied in Book VI: The plane geometry of similar figures to geometrical objects of the same shape, but not necessarily the same size in particular, to the theory of similar triangles and quadrilaterals.

Book VII, Basic arithmetic, and its successors Book VIII, Numbers in continued proportion; and Book IX, Numbers in continued proportion; the theory of even and odd numbers, perfect numbers take a completely new tack. This is arithmetic and number theory, and these books contain several of Euclid's best-known results.

Book VII starts with a long list of definitions: whole numbers, even and odd numbers, prime numbers, square and cube numbers, and perfect numbers. Proposition 1 is as follows:

Two unequal numbers being set out, and the less being continually subtracted in turn from the greater, if the number which is left never measures the one before it until an unit is left, the original numbers will be prime to one another.

For example, if we take 23 and 17, and continually subtract the smaller from the larger, we get 23 = 17 + 6, 17 = 6 + 11, 11 = 6 + 5, 6 = 5 + 1: the 1 here indicates that the original numbers 23 and 17 are prime to one another that is, they have no factors in common. Using this idea, Euclid goes on to develop his method for finding the greatest common divisor of any two whole numbers the well-known Euclidean algorithm:

Proposition 2: Given two numbers not prime to one another, to find their greatest common measure.

In all of this, Euclid's descriptions are all in terms of lengths of lines, rather than in terms of operations on numbers. A similar remark can be made about Euclid's proof in Book IX, Proposition 20, that there are infinitely many prime numbers which is one of the most famous proofs in the whole of mathematics. Here is the modern way of writing out the proof: recall that a prime number is a number whose only factors are itself and 1 for example, 11, 13, 17 and 19 are all prime numbers.

Suppose, for a contradiction, that the only prime numbers are p1, p2, . . . pn.

Form the number N = p1 p2 . . . pn + 1.

Since the primes p1, p2, . . . pn all divide their product p1 p2 . . . pn, none of them can divide N. It follows that N is itself a prime number, or else is divisible by some prime number that is different from p1, p2, . . . pn. In either case, there is a prime number different from the given ones, giving the required contradiction. Thus, there are infinitely many prime numbers.

Euclid's proof involves the lengths of lines, and starts with only three lines of prime length, representing the general case. Apart from this, the proof is the same.

The final result in this section of the Elements is Euclid's result on perfect numbers. A number is perfect if it is the sum of the numbers that divide it: for example, 6, 28 and 496 are all perfect numbers: 6 = 1 + 2 + 3;

28 = 1 + 2 + 4 + 7 + 14; and

496 = 1 + 2 + 4 + 8 + 16 + 31 + 62 + 124 + 248.

Euclid's result is

If as many numbers as we please beginning from an unit be set out continuously in double proportion, until the sum of all becomes prime, and if the sum multiplied into the last make some number, the product will be perfect.

What this means is that if the sum 1 + 2 + 22 + 23 + … + 2k is a prime number, then the number 2k times this is a perfect number: in other words, summing the series, we get: 2k × (2k+1 1) is perfect if 2k+1 1 is prime for example:

for k = 1, we have 6 = 2 × 3;

for k = 2, we have 28 = 4 × 7;

[for k = 3, we get 8 × 15, but 15 is not prime, so we ignore this one];

for k = 4, we have 496 = 16 × 31;

and so on. The next two are 8128 (= 64 × 127) and 33,550,336 (= 4096 × 8191).

I shall return to the subject of perfect numbers in my lecture on Euler.

Book X, Incommensurable line segments, is the longest book in the Elements, and is too complicated to discuss here. In this book, the Euclidean algorithm of Book VII is applied to general magnitudes in order to obtain criteria for commensurability.

Book XI, Foundations of solid geometry; Book XII, Areas and volumes, and Eudoxus' method of exhaustion; and Book XIII, The Platonic solids, deal with various aspects of three-dimensional geometry. Book XI covers such topics as lines and planes, angles between planes, parallel and perpendicular planes, and lines perpendicular to planes, while Book XII deals with the area of a circle, and the volumes of pyramids, cylinders, cones and spheres.

Book XIII is particularly remarkable. It introduces the five regular solids: the tetrahedron, cube, octahedron, dodecahedron and icosahedron and then describes how they can be constructed. For example, to construct a dodecahedron, we take a cube and add to each face a roof whose proportions are such that the faces become pentagons; these proportions involve the so-called golden ratio, whose geometrical properties are worked out in detail at the beginning of Book XIII. In this book, Euclid also proved such remarkable results as the following:

If an equilateral pentagon is inscribed in a circle, the side of the pentagon is equal in square to that of the hexagon and that of the decagon inscribed in the same circle. In other words, if we calculate the lengths of the sides of a pentagon, a hexagon and a decagon inscribed in a circle, then these turn out to be the lengths of the sides of a right-angled triangle.

Euclid concludes the Elements by proving that the only possible regular solids are the tetrahedron, cube, octahedron, dodecahedron and icosahedron. There can be no more. This is the first ever classification theorem in mathematics and forms a fitting climax to this great work.

Euclid's Elements was warmly received, and quickly replaced all its predecessors and competitors. Commentaries on it were produced by many people first by the neoplatonists Boethius and Nicomachus, and then by Islamic scholars a few centuries later, in what is now Iraq.

Baghdad was on the trade routes between the West and the East. But it wasn't just silk and spices that went along these routes, but also scholarly contributions from Europe and India such as the ancient texts of Euclid, Archimedes and others from the Greek empire, and the Indian number system from the East. The latter was developed and became our familiar Hindu-Arabic numerals, while the former works were translated into Arabic, commentaries were written, and their results extended in many directions. Indeed, most of our knowledge of Greek mathematics comes from this period either from the Arabic commentaries themselves, or from the Latin translations of them that started to appear in Europe from about the twelfth century. Later, after the invention of printing, a number of printed versions appeared, and the Elements became the most frequently reprinted book after the Bible.

I would like to conclude this talk by discussing one difficulty that kept on resurfacing, which I touched on earlier. This was Euclid's fifth postulate that if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than two right angles.

The problem was that this looks much more like a proposition than an assumed postulate, and very many attempts were made, over a period of more than a thousand years, to deduce it from the other postulates and axioms. One approach was to find other results to which it is equivalent, such as the following, which explains the name parallel postulate: given any line l, and any point p not lying on l, there is just one line through p parallel to l. If we can prove this directly, we can then deduce the fifth postulate. Another equivalent statement is that the angles of any triangle add up to two right angles.

The Arabic scholar Alhazen, known for his contributions to optics, attempted to prove the parallel postulate by invoking the idea of motion. Take the perpendicular from p down to l, and then slide this line along, from left to right, always keeping it at right angles to l: then the top end will trace out a line through p parallel to l. This approach, which does not prove the parallel postulate, was later criticized by the Islamic scholar Omar Khayyam, remembered in the west mainly for his collection of poems known as the Rubaiyat.

Another attempt to prove the parallel postulate was given by the Oxford professor John Wallis, in a lecture on 11 July 1663. However, his approach assumed that similar, but non-congruent, copies of given geometric figures can be constructed. A consequence of this is that if there were a geometry for which the parallel postulate does not hold, then any two similar figures would have to be congruent which seems highly unlikely.

The most successful approach was made by the Italian mathematician Gerolamo Saccheri. He considered the sum of the angles around a triangle and proved that there can be no geometry for which this sum is greater than two right angles. He next tried to prove that there can be no geometry for which the sum of the angles is less than two right angles. If he had been able to prove this, then he could have deduced that the sum of the angles is always exactly two right angles, and this would have proved the parallel postulate.

To cut a long story short, the reason that Saccheri was unsuccessful is that there are geometries for which the sum of the angles of any triangle is less than two right angles. These geometries were discovered independently around 1830 by the Russian mathematician Nikolai Lobachevsky and the Hungarian mathematician János Bolyai. In such a geometry, which must necessarily be very strange, the first four postulates are satisfied, but the fifth postulate is not. Also, in this geometry, any two similar triangles are necessarily congruent, and given any line l and any point p not lying on l, there are infinitely many lines through p that are parallel to l.

As you can imagine, such non-Euclidean geometries, as they are now known, are complicated to describe, but I want to show you such a geometry, constructed by the French mathematician Henri Poincaré in 1882. We'll consider a disc whose boundary is the edge of our geometry. The lines of the geometry take two forms: ordinary straight lines, or curved lines that meet the boundary at right angles. Two lines are parallel if they meet on the boundary. It turns out that this is a geometry that satisfies the first four postulates, but not the fifth, and that all the above strange properties hold.

Weve come a long way from the geometry of Euclid and his contemporaries, and I hope that you have considered the journey worthwhile. If not, I conclude by recounting the story that someone who had begun to read geometry with Euclid asked him what advantage he would get by learning these things? Euclid called his slave and said Give him threepence, for he must make profit out of what he learns. I do hope that you don't all want to claim threepence from me.