Galileo's Journey to the Underworld: The Case for Interdisciplinary Thinking

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Introduction

In 1588, the young Galileo delivered two invited lectures to the Florentine Academy. They were impressive enough to secure him a mathematics professorship at the University of Pisa. His subject in these lectures? The geometry of Dante's Inferno. In my Provost's Lecture, I'll look at some of Galileo's deductions, and how the questions raised may have influenced his later mathematical research. Using this and other examples of creative work in mathematics that crosses our modern ideas of subject boundaries, I will argue that thinking across disciplines is not only intellectually exciting but academically vital.

Poetry

We'll look at the actual content of Galileo's lectures in due course, but first, the obvious question is why, for perhaps the most important lecture of his life so far, the 24-year old mathematician chose to speak about Dante's Inferno. Exploring a fictional world seems more the purview of literature than mathematics. But in fact it is exactly what mathematicians do. In the real world there's no such thing as a mathematically perfect circle, or the mathematical straight line that extends to infinity in both directions. These are abstractions, focusing on the underlying ideas, that help us to uncover the deeper truths. Fiction does the same thing – it creates artificial situations to put characters in that help us explore human nature and emotions. Once we are in that created world, whether it's the axioms of Euclidean geometry or the text of Dante's poem, we can explore it and tease out the logical consequences of the set-up.

And if we're surprised that a mathematician is interested in poetry, it's because we've fallen into that old false binary of being a "numbers person" or a "words person". For most of history, mathematics was a natural part of a cultural education – poets did mathematics, and mathematicians wrote poetry. For example, in the Sanskrit tradition in India, mathematics was passed down in verse. There's a beautiful book by mathematician and poet Bhaskara, the Lilavati, in which he poses problems that we'd now write in terms of algebra, in the form of poems. The rhythms and patterns of poetry gave rise to some of the earliest mathematics. In Pingala's Chanda-sutra, for example, he looks at how many feet there are with a given number of syllables. Any sound can be stressed (heavy/hard, or *guru*) or unstressed (light/soft, or laghu. In English poetry, the words for the possible two-syllable feet are iamb (soft-hard), trochee (hard-soft), spondee (hard-hard), and pyrrhic (soft-soft). There are four possibilities. For three syllables, each of these 2-syllable options can be followed by either a stressed or unstressed syllable, giving a total of eight 3-syllable feet. It keeps doubling, so that we end up with powers of 2, and 2^n possible n-syllable patterns.

But in Sanskrit poetry there's another way to categorise metre – not by the number of syllables, but by the duration. This is because traditionally the light laghu syllables last one beat, and the heavy guru syllables last two beats. This results in the Fibonacci series (1, 2, 3, 5, 8, …) and it was known in the context of poetry hundreds of years before Fibonacci encountered it.

If the mathematicians and poets have often been the same people, it's no surprise that poets have used mathematical metaphors – in *Paradise Lost*, Milton speaks of God creating the universe with golden compasses, for instance, drawing a vast circle in "the vast profundity obscure" and decreeing "this be thy just circumference, O World." Milton was also fascinated by astronomy, and there are three interesting references to telescopes in Paradise Lost: one to "the moon, whose orb/through optic glass the Tuscan artist views", another to the devil flying towards the sun: "There lands the fiend, a spot like which perhaps/ Astronomer in the sun's lucent orb/Through his glazed optic tube yet never saw" (Galileo saw sunspots in 1610), and finally a reference to "the glass of Galileo". The "Tuscan artist" and the "Astronomer" are of course Galileo too (he recorded seeing sunspots with his telescope in 1610), and Milton is even believed to have met Galileo in person.

Dante's Inferno

Let's get back to the Inferno lectures. Galileo was speaking before the Florentine Academy, who at that time had the Mathematics Professorship at Pisa in their gift, so it's a canny political move to talk about Dante, the most celebrated son of Florence. In Inferno, Dante is guided by Virgil through the nine circles of hell, descending through the earth until they encounter Lucifer himself, right at the centre of the Earth. In trying to understand the size, shape and geometry of this realm, Galileo is not claiming this is really "real". What he wants to establish is that Dante has shown us a world that makes sense geometrically. Can we get into the mind of the great poet and explore what he has seen in his mind's eye? Galileo frames his investigation as a comparison between two earlier commentaries on Inferno. The first is by Antonio Manetti (1423-1497), the second by Alessandro Vellutello (1473 – c1550). But Galileo adds a good deal of his own analysis and uses much more advanced mathematics.

He begins by looking at the shape of hell. It is a spherical cone with its apex at the centre of the Earth directly below (as Dante tells us) Jerusalem. From other contextual clues, Virgil and Dante enter hell at one edge of the cone in Italy, which makes the domed roof of hell encompass one sixth of the earth's surface. This agrees with Manetti's conclusion, but he wasn't able to work out the volume of such a shape. But Galileo was familiar with the work of Archimedes, and he deduced from it a formula for the volume of a spherical cone, from which he could calculate that the volume of hell is about $1/14^{th}$ the volume of earth. He also throws in that Vellutello's version of hell would be laughably puny, a mere one thousandth part of Manetti's. (It may not be a coincidence that Manetti was from Florence but Vellutello was from Lucca.)

Once we know the basic shape, we have to fit in nine circles, plus a few other landmarks, getting as close as possible to an accurate representation of Dante's vision. Dante helps quite a bit here, because as he descends with Virgil through the nine circles, he mentions some of the distances. As they move along a corkscrew-like path, around the rim of the cone of hell, for example, there's a point at the bottom of the 8^{th} circle where there are ten layers of "malebolgia", sometimes translated as pits, or "rottenpockets". Virgil says in the ninth of these, "*che miglia ventidue la valle volge*", that is, "*this hollow circles twenty-two miles round*". Then the tenth bolgia is "*eleven miles around*". The obvious reason for these numbers is that $\pi \approx \frac{22}{7}$, making the diameters 7 miles and 3.5 miles respectively. We can deduce from this that the ten pits form a series of concentric circles going up 3.5 miles in diameter at each stage, and then their height above the centre of the Earth is fixed, because all we have to do is work out at what height above the apex the cone has that diameter.

The most entertaining part of Galileo's analysis is when he is dealing with the final chamber at the deepest part of hell, where Lucifer is entombed in ice. His naval is at the exact centre of the earth. To determine the height of this space, we have to know Satan's height. There are two key passages that help with this. Firstly, Dante says "*The emperor of the woeful kingdom/rose from the ice below his breast/And better with a giant I compare/Than do the giants with those arms of his:/Judge, then, what the whole must be/that is proportional to such a part"* (Canto XXXIV). So we have a comparison between Dante, a giant, and Satan's arm. Knowing the rough proportions of the human body, and Dante's height, we could arrive at an estimate if we knew the height of a giant. Fortunately, in Canto XXXI, we meet a giant called Nimrod, and his face "*was as long and broad/as is, in Rome, the pinecone at St. Peters, /His other parts as large in like degree*".

The pinecone mentioned is a large statue dating back to Roman times, and 5.5 braccia high, according to Galileo. (A braccio is, as the name suggests, about the length of an arm. It varied widely across Italy over time and location, but at this time it was approximately 58cm or 23 inches.) Using this, and a height of 3 braccia for Dante, he is able to estimate that Lucifer is 2000 braccia high (about ¾ of a mile, or approximately 365 pinecones.)

There's one riddle wrapped up in these lectures, and it has to do with domes. Hell is a vast cone, covered by a portion of the Earth's crust. Wouldn't this vast dome collapse in on itself? Along with Dante, one of the other things Florence was most proud of at the time was the huge dome of Florence Cathedral, an architectural marvel designed by Brunelleschi. One can imagine Galileo gesturing down the street to the Duomo as he tells his audience that “a dome 30 braccia wide can be 4 braccia thick”, and so, he says, we can just scale this up and all will be well. But the problem is that scaling doesn’t work this way. It’s why we can make scale models out of cardboard, but not the buildings themselves. At some point the structure becomes too weak to support its weight, because when you scale a structure, the cross-sectional area increases with the square of the scaling factor, and the volume, and hence mass, increases with the cube of the scaling factor. This means the pressure on the supporting structure increases until the structure cannot support itself.

We can be absolutely sure that Galileo became aware of this at some stage, because the law governing scaling, called the “Square-Cube Law”, was discovered by Galileo himself. He didn’t publish it until many years later, when among other things he explained how the law helps to determine what animals look like. So the question is: did Galileo know this point about the dome was wrong – was it a tongue in cheek reference to the fact that domes are notoriously difficult to build and at that time often fell down – a knowing nod to the brilliance of the Florentine Brunelleschi? If Galileo hadn’t worked out scaling laws yet, how soon did he realise? And did he have this realisation because of thinking more about these structures? In short, did Galileo make one of his most important mathematical discoveries directly as a result of reading poetry? I like to think he did.

Mathematics and the other arts

In celebration of the positive consequences for Galileo of thinking about literature, I want to spend the rest of my talk showing you just a handful of examples of the spur to creativity that comes when we work across the disciplines, and the benefits it can bring to both sides. I’m going to give two instances of fruitful connections between mathematics and art, and two between mathematics and music.

Back to Florence for our first example, where the rules of linear perspective were worked out by Filippo Brunelleschi, whom we’ve already encountered today. Understanding these rules, and how to apply them, allowed artists to create incredibly realistic imagined architectural backdrops for their paintings, in the process making some of the most beautiful art in history. It also allowed them, by choosing things like vanishing points, to add extra symbolism to their work, as for example in The Annunciation, by Domenico Veneziano. Here, the vanishing point is not Mary or the Angel Gabriel, but the locked door to a secret garden at the back of the scene, a symbol of Mary’s virginity.

Getting perspective right is all about geometry. The fact that, for instance, the images of the parallel front-to back edges of the floor tiles in the Annunciation painting all converge on a single vanishing point is a mathematical consequence of what happens when you project an image onto a plane. Leon Battista Alberti, who helped to popularize the rules of perspective in his book *Della Pittura* (1436), began by talking about the mathematics, and gave geometric constructions of things like perspective drawings of a tiled floor. Meanwhile mathematics benefitted too, because exploring the consequences of the strange idea parallel lines can sometimes meet, led to new and interesting adventures in what we now call projective geometry.

Our second artistic example involves something mathematicians are very good at: counting. In particular, counting the different ways to fit shapes together. Throughout history, many forms of decorative art have involved repeating patterns, or tilings. This is particularly evident in classical Islamic art, where there is traditionally a prohibition against representing people and animals. The Alhambra Palace in Granada, Spain, with its Moorish influences, is full of beautiful tiled walls, and for the artist it’s an important question as to what possible shapes can be used to create a repeating pattern with no gaps and no overlaps. In one sense the number is infinite because you can make minor tweaks to existing shapes that work, but any repeating, or “periodic”, tiling has one of just three so-called regular tilings underlying it. A regular tiling is one where every tile is the same, and it is a regular polygon (one where all the sides are equal and all the angles are equal). To make this work, you need the internal angle of your polygon to go an exact whole number of times into 360. The only solutions are the equilateral triangle, the square, and the regular hexagon.

So on a flat plane, there are three basic tilings. Can maths help the artist who wants more possibilities? One such artist was M.C. Escher, who was inspired by visits to the Alhambra to create many pictures featuring tilings. He explored all the options on the plane, but Euclidean geometry isn’t the only kind. There are regular tilings of the sphere, for instance: to make one start with a regular polyhedron (one each of whose faces is the same regular polygon, with the same number of faces meeting at each vertex) and then essentially inflate it so that it becomes spherical. There are exactly five regular polyhedral tilings – known as the Platonic solids – giving rise to five regular tilings of the sphere. That's still only eight tilings total. But one day Escher saw a picture in an article by the mathematician Donald Coxeter, and he saw that this was a tiling of a completely different sort. He immediately got to work to produce a picture based on it. The image is a tiling of hyperbolic space. The picture is a projection of that space onto a disk, and just as planar maps of the spherical world necessarily distort some shapes and sizes, so too does this. The boundary of the circle is in fact infinitely far away from the centre – all the triangles in the tiling are in fact congruent, but they appear to get smaller as they approach that boundary. The triangles in the picture aren't equilateral (even in hyperbolic space); their angles are 90, 45, 30 (angles in hyperbolic triangles add up to less than 180), but underlying this periodic tiling is a regular tiling where four hexagons meet at every point.

Escher corresponded with Coxeter asking how these designs are made, and how many possibilities there are, and Coxeter gave him the good news that there are infinitely many regular hyperbolic tilings. For example with squares: putting 3 round a point gives a spherical tiling – the cubic one. Putting 4 round a point gives the square grid planar tiling. If you have five or more, there is a hyperbolic tiling for each possibility. It's an infinite playground to explore. Escher produced several more of these "circle limit" designs, and with modern computers it is possible to make your own – I've used Malin Christersson's excellent (and free!) tiling tool at www.malinc.se to create a Gresham example where four regular pentagons meet at each vertex.

We'll now move onto some examples from music. Two days after this lecture is the Gresham Festival of Musical ideas, where I'll be talking with our Music Professor Milton Mermikides about the mathematics of rhythm and harmony, so I wanted to pick different ideas today.

The first is to do with bellringing, specifically change ringing, a tradition which developed and is still mostly done, in English churches. A key reason for this is that in England our churches tend to have sets of bells (or "rings"), tuned to different pitches, and moreover we have full-circle ringing where the bells go all the way around. The first of these means we can explore the possibilities of music made with multiple different pitches, rather than just having a tolling bell, and the second means that we can't just play any tune we like, because the bells are in motion, they are very heavy, and cannot be stopped or started at will. There are rules. And rules mean structure. And structure means mathematics. The basics of change ringing are these: bells sound in a sequence of *rows*, with each bell sounding exactly once in each row, and in the change between rows each bell can move at most one place in the order. With three bells (labelled 1, 2, 3 from highest to lowest), we can ring 123, then 132, but we can't then ring 213 because this would force bell 2 to move two places. An *extent* on n bells is where we play a sequence of rows, following these rules, starting and finishing with 123…n with every other row played exactly once between these endpoints. One extent on 3 bells could be 123, 132, 312, 321, 231, 213, 123. Exploring what patterns are possible is a highly mathematical challenge.

There are two key questions: is an extent on n bells always possible? And is there a musically "interesting" extent? A change between rows where only two bells out of eight, say, swap places, would be quite dull, sonically speaking, because most of the pattern would be the same. So the "best" extents are where as many bells as possible move from one row to the next. For seven bells, as any swap involves a pair of bells changing places, the best we can do is a "triple swap" where three pairs are interchanged and one bell is left in the same position. Over 300 years ago, Fabian Stedman, one of the founders of bellringing as a codified practice, conjectured that there exists an extent on seven bells (5040 changes) featuring only triple swaps. He was right: a "triples-only" extent was finally found by mathematicians in 1994. There remain unsolved problems in bellringing which will no doubt entertain mathematicians for many years to come.

Our final example is a use of a mathematical concept to compose music: that concept is fractals. The most famous fractals are things like the Mandelbrot set, which, in common with other fractals, has the property of self-similarity, it contains versions of itself at infinitely many scales.

In music, composers such as Kaija Saariaho have used fractal sequences in composition. One example of a fractal sequence is the Thue-Morse sequence. Start with the numbers 1, 2, 3, 4 and so on, but expressed in binary. We get 1, 10, 11, 100, 101, 110, 111, 1000, 1001, 1010, 1011, 1100, 1101, 1110, and so on. Now make from this the sequence where each term is the number of 1's appearing in the corresponding number. The result is 1, 1, 2, 1, 2, 2, 3, 1, 2, 2, 3, 2, 3, 3, and so on. If we highlight alternate terms in bold we get 1, **1**, 2, **1**, 2, **2**, 3, **1**, 2, **2**, 3, **2**, 3, **3**, and so on. Picking out these terms results in 1, 1, 2, 1, 2, 2, 3, … We retrieve the same sequence we started with. Our original sequence contains infinitely many copies of itself! A fractal sequence can be used in musical composition to specify pitch of notes, duration of notes, or really anything to which one can assign a number. The self-similarity property gives a richness to the music where echoes of the larger scale sound appear at smaller scales or different speeds throughout the composition. Composers have used everything from probability distributions to magic squares as frameworks for composition. Ideas from across disciplines really can enrich both fields.

Conclusion

There's a passage in a novel by Henry Longfellow (*Kavanagh*, 1849), in which a schoolteacher and his wife are talking about mathematics. He says "There is something divine about the science of numbers. Like God, it holds the sea in the hollow of its hand. It measures the earth; it weighs the stars; it illumines the universe; it is law, it is order, it is beauty. And yet we imagine […] that its highest end and culminating point is book-keeping by double entry. It is our way of teaching it that makes it so prosaic". Then he takes an old book from his shelves. "Now here is a book of mathematics of quite a different stamp." "It looks very old. What is it?" "It is the Lilavati of Bhaskara Acharya, translated from the Sanskrit."

Teachers do amazingly well at keeping their subjects exciting in spite of ever-changing diktats from on high decreeing what must be packed into the syllabus this year. But in a healthy society learning doesn't stop with school, or even a university degree. We must cherish and feed our curiosity. That's where Gresham College comes in – it's a chance to go beyond – to explore ideas without being bound by the strictures of a defined subject curriculum. For over four hundred years, it has been a place where people can come and hear exciting ideas, to be challenged, to think. I'm honoured to be giving this Provost's lecture, and I encourage you all to explore Gresham's online archive, come to our in-person lectures, and spread the word about this precious institution. Thank you.