Chapter 6: The Deceived Brain:

Coding and Illusion

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Visual illusions have bewildered humans and animals for millenia. They have been a constant source of amusement but also a unique doorway for scientists to understand the way the brain processes visual information. In the simplest classic visual illusions, a main object, such as a line or a circle, is deformed or displaced through the interactions of other nearby objects such as line segments or other circles. These interactions can be modelled in terms of effective interactions acting on the object, leading to small deformations. I will show that the introduction of simple phenomenological laws is sufficient to explain many visual illusions. Further, these laws can also be obtained from simplified models of the visual systems, linking visual formation processing to illusions.

6.1 Introduction

Geometrical-optical illusions are relatively simple visual illusions where the geometry of a two-dimensional pattern leads to a misleading perception of the physical reality of its spatial layout. They typically involve easily recognisable geometric shapes such as lines, squares, circles or triangles. A line, for example, may appear longer, shorter, or bent depending on the lines or patterns surrounding it. In other cases, our visual system perceives contours and shapes that are not actually present which is due to the arrangement of surrounding elements.

As an example, consider the illusion in Figure 6.1. It is called the *Shepard table* after the psychologist Roger Shepard [21] from Stanford University. It shows two tables, the one on the left appears to me long and skinny, whereas the one on the right is short and wide. Surprisingly, both table tops are the same geometric shape as shown on the right.

Illusions are not just curiosities, they have been subject to extensive research in multiple scientific fields such as psychology, neuroscience, and computer science, as they provide insights into the workings of the human visual system as already advocated by Helmholtz in 1881 [2]: *“The study of what are called illusions of the senses is however a very prominent and important part of the physiology of the senses; for just those cases ... which are not in accordance with reality are particularly instructive for discovering the laws of those means and processes by which normal perceptions originate."*

The end of the nineteenth century was a particularly fertile ground to natural philosophy as they were no strict disciplinary barriers and scientists from mathematics, physics, biology, and

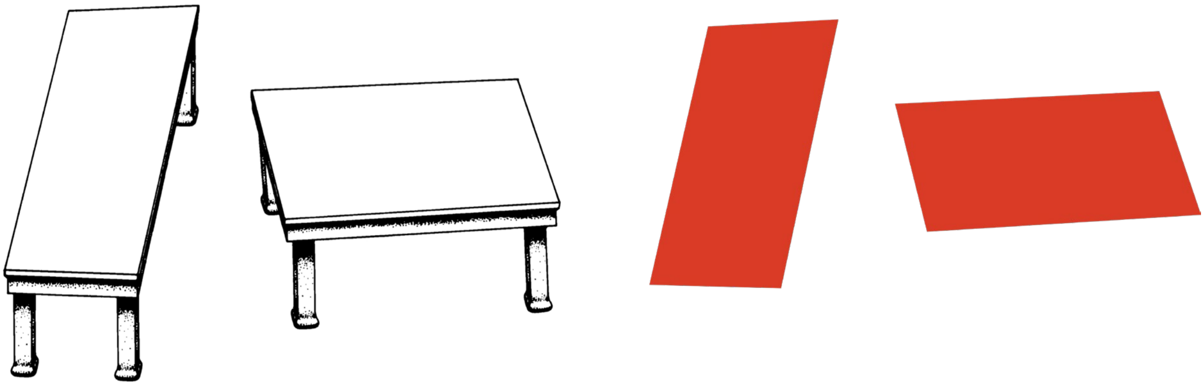

Figure 6.1: Shepard's table tops is for me a striking illusion where the two table tops appear to have different shapes whereas they are actually the same two-dimensional shape as can be appreciated by rotating the figure on the right.

physiology became interested in aspect of human perception. It became the golden age for visual illusions. Rapidly, a few key visual illusions were selected as particularly prominent. Among these, there are essentially three classes of interest for our discussion: illusions that involve changes in orientation and shape such as bending or misalignment; illusions that involve misjudgments of size or length due to the surrounding geometric context; and illusions that suggest a certain shape or contour despite the fact that they are not present. These basic illusions are often referred to as *geometric-optical illusions*, a term coined by Oppel in 1855 [16]. I prefer the term *visual illusions* as distortions or misjudgements are due to the visual system and not to any optical effect. Before we review them, please do not disappointed if you do not see the illusions a I see them. Everyone sees illusions differently and not all appear as illusions. It just means that you are not as easily fooled as I am. Since I do not know what you see, I will describe these illusions as I see them but since I will mostly deal with simple illusions, they are typically observed by most people. In most cases, I will show in red the perceived ***object*** that is misjudged by the effects of extra features, the ***modifiers***, shown in black.

6.1.1 Illusions of orientation

In what I call *illusions of orientation*, we misjudge angular information. In particular, parallel lines may not appear parallel and straight lines may not appear straight. The most famous of such illusions is the *Hering illusion*, first described in 1861 by the German physiologist Ewald Hering [3]. He was exploring how our visual system interprets space and motion when he

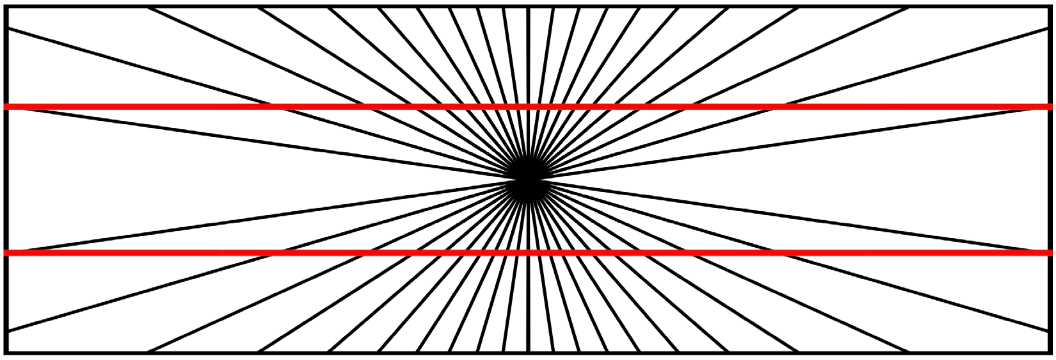

Figure 6.2: In the Hering illusion, two parallel lines (in red) appear bowed outward.

noticed a strange effect: two straight, parallel lines placed over a background of radiating lines looked as if they were bending outward as shown in Figure 6.2 . This simple but powerful illusion quickly became one of the most studied in visual perception.

You too can play the Hering game. All you need is a ruler, a sheet of paper, and a pen. Start by placing a ruler horizontally on a sheet of paper and draw two thick parallel horizontal lines—one on each side of the ruler. Then place a dot in the centre, half way between the two thick lines. Then start drawing light rays emanating from the centre and crossing the two lines, just like in the Hering drawing. How many small lines do you need before the line starts bending? Does the illusion changes with the orientation of the parallel lines or the small lines?

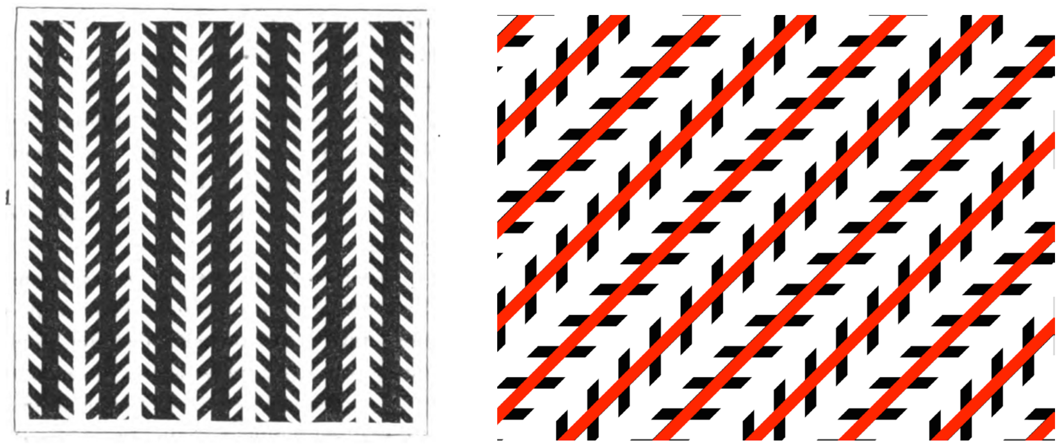

Figure 6.3: In the Zöllner illusion, parallel lines (in red on the right) appear to be converging or diverging.

Another classic is the *Zöllner illusion*, discovered in 1860 by Johann Karl Friedrich Zöllner, a German astrophysicist [25]. With a strong interest in what was called at the time *psychophysics*, he noticed that parallel lines appeared to diverge or converge when crossed by short diagonal lines.[1] Though the main lines are perfectly parallel, they look skewed or tilted

[1]There is a myth that Zöllner discovered these illusions while experimenting with patterns for textiles in his

because of the added intersecting strokes as shown in Figure 6.3.

Now that you have played the Hering game, you should play the Zöllner game. With your ruler make a series of parallel lines on your sheet of paper. Then add series of crossings at about 45° on all the lines, roughly equidistant. Do the lines appear to diverge now? How does it change as you rotate the sheet? Is the effect stronger?

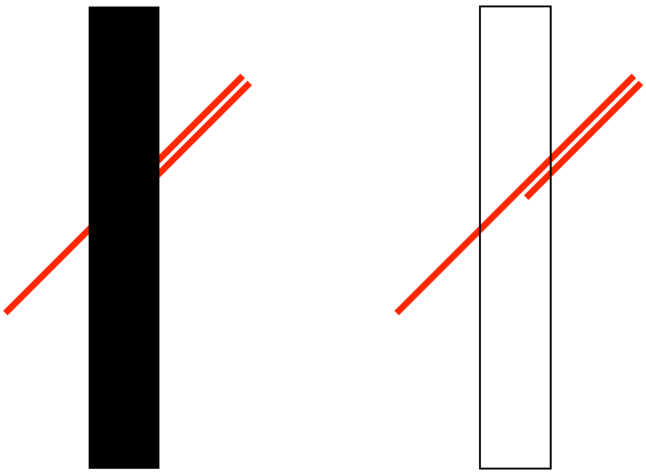

Figure 6.4: In the Poggendorff illusion, a line appear shifted as it passes behind a shape. The lower red line on the right of the rectangle is the corrected version (for the author) where it looks like the continuation of the line on the right.

When Zöllner sent his newly found illusion for publication to Johann Christian Poggendorff, the editor of the journal *Annalen der Physik*, Poggendorff noticed something strange, the small lines on top of the long lines appeared broken. Doing so, he discovered a new illusion [18], shown in Figure 6.4. In the *Poggendorff illusion* a diagonal line appears to be misaligned as it passes behind a rectangular shape, even though it continues in a straight path. The two ends of the line seem offset, creating a puzzling break in continuity, that I call the *alignment problem*.

Even though the illusion is attributed to Poggendorff, it can be found in many classical and modern paintings, where the effect has been corrected. A striking example of this "correction" can be seen in Peter Paul Rubens' masterpiece, The Descent from the Cross from 1612–1614 that is still on display at his original location in the Cathedral of Our Lady in Antwerp, Belgium. On the right side of the painting, a ladder leans diagonally behind a figure. Although the top part of the ladder is actually shifted to the right, it appears to align perfectly with the lower part. This creates a corrected version of the Poggendorff illusion, where our eyes mistakenly perceive the two misaligned sections as forming a continuous straight line [22].

Since we have no record about the creative process of Rubens, we do not know if this correction was done on purpose. I have now studied this correction in some detail and have observed that it appears in all versions of the same painting and in many other paintings from Rubens. It is clear that Rubens knew how to use a ruler and did not find such a misalignment troubling. Indeed, despite extensive comments and studies of his paintings over the centuries, these misalignments have remained unnoticed, apart from the original paper by Topper in 1984 [22]. Even then, Topper failed to realise that the versions owned by the Courtauld Museum in London also have the same correction, albeit not as strongly. Finally, as long as you are admiring this beautiful painting. Have a look at the left side of the same ladder. You will soon reach

father's factory. It is true that there is a well-known textile brand called Zöllner, but I found no direct evidence that little Johann was influenced by fabric patterns.

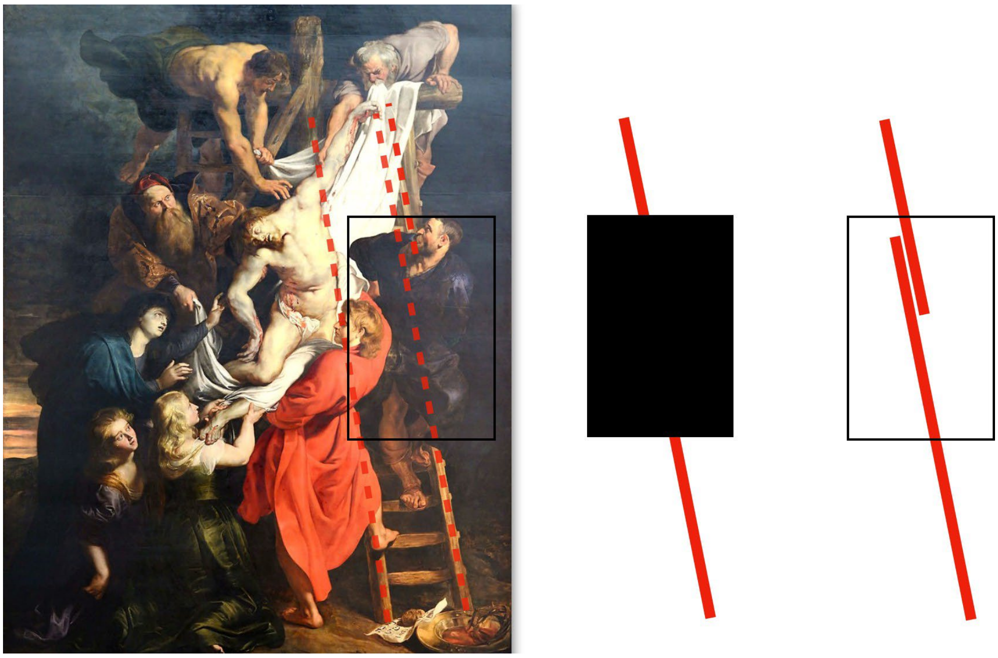

Figure 6.5: In the Descent from the cross, the top part of the rightmost ladder is shifted further right. Despite this misalignment, the ladder seems perfectly straight. To test if the same effect could be related to the Poggendorff illusion, I extracted the shape (black rectangle) and the bottom part of the ladder (in red), I then moved a parallel segment at the top until it looked aligned. The resulting correction of the illusion is exactly the same as the one found in the original painting.

the conclusion that the top part is missing. Where did it go? Why hasn't anybody reported it missing in the last 400 years?

As I was preparing my Gresham lecture, I found many more such corrections in various paintings. The most spectacular one is *The Scream* by Edvar Munch (1893), arguably the most iconic painting in modern art. Look at the set of diagonal railings that frame the walkway in Figure 6.6. They recede into the distance and are partially blocked by the central figure. Although they appear continuous, their alignment is visually adjusted. Munch "corrected" their positions and the net result is that it avoids triggering the Poggendorff illusion, where diagonals interrupted by a shape seem misaligned. The same misalignment appears in all eight versions of the same painting, including the two versions of *Despair* featuring a man standing quietly at the railing at the same location. We can conclude that this misalignment was probably deliberate, the purpose of which is best left for the many art lovers who spent their lives arguing about such questions.

You too can play the Poggendorff game. Follow the instructions given in Figure 6.7: Start by placing a ruler vertically on a sheet of paper and draw two, parallel vertical lines, one on each side of the ruler. These lines represent the sides of a "barrier." Next, on the left side, draw a diagonal half-line at a 45° angle, starting from the middle of the left vertical line and extending to the edge of the paper on the left. The challenge is to guess where that diagonal line would continue on the right side. To test yourself, place a dot on the right vertical line where you believe the diagonal would emerge if it passed straight underneath the hidden space. It is very important to keep the paper aligned so that the vertical lines appear vertical at all times (no cheating by rotating the paper or your head, using a ruler, or one eye). When you have a dot on the right line, check with your ruler if it is aligned with the diagonal half-line. What do you conclude? What would Rubens do?

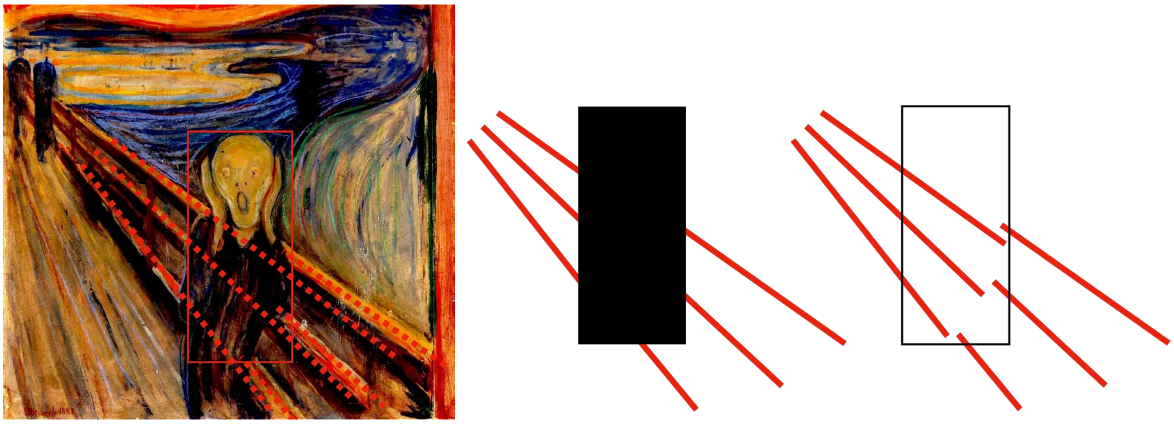

Figure 6.6: The bridge railings in *The Scream* are misaligned. Following the same (non-scientific) self-experiment, I moved three lines parallel to the railings until they seemed to align with their counterpart, matching remarkably the ones drawn by Munch.

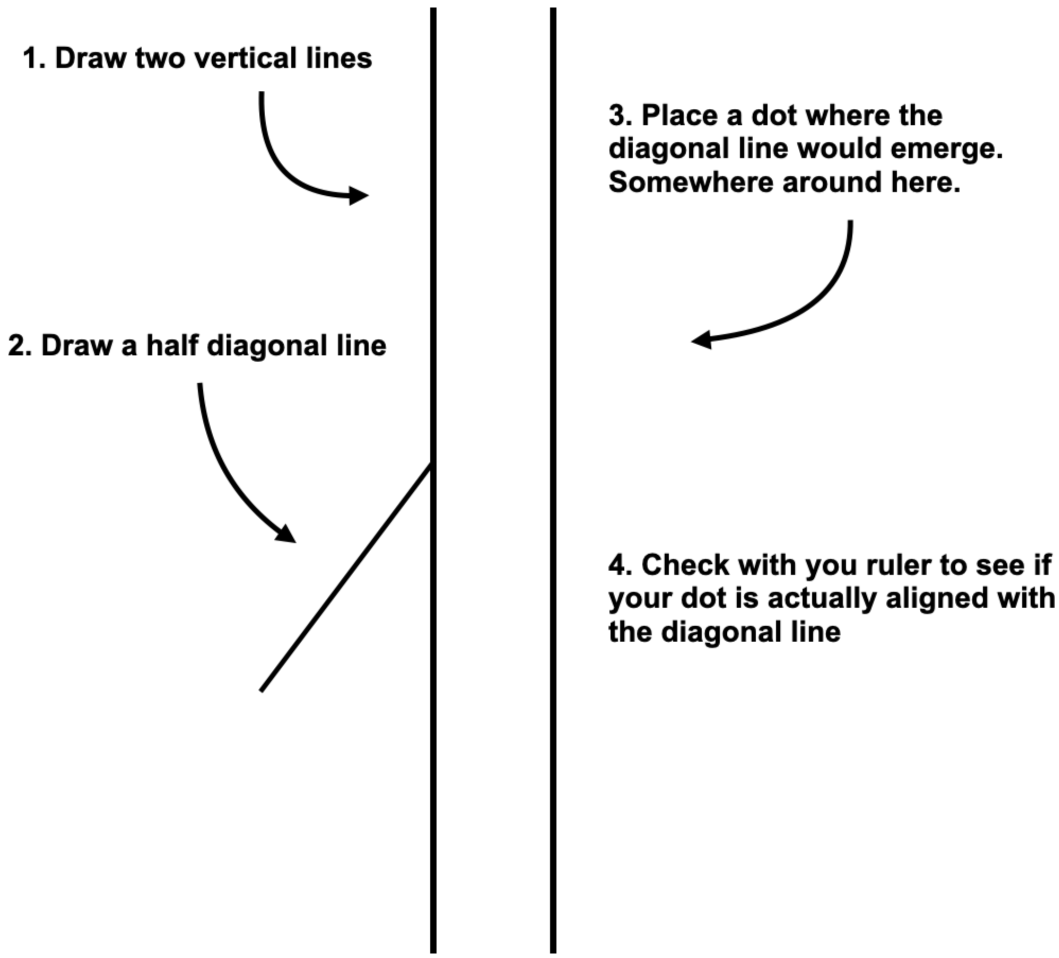

Figure 6.7: Play the Poggendorff game by following these instructions (do use a real sheet of paper, its more fun and do not cheat!).

6.1.2 Illusions of size

Another category of illusions is related to our judgment of relative sizes. The best known illusion in this category is the *Müller-Lyer illusion* introduced in 1889 by Franz Carl Müller-Lyer, a German sociologist [13]. At the time, due to the influence of Wilhelm Wundt who established the first psychology laboratory in Leipzig in 1879, psychology was beginning to shift from philosophical speculation to experimental science. One of the key aspects of this new scientific discipline was the study of visual illusions. In the Müller-Lyer illusion in Figure 6.8, both horizontal lines are identical in length, yet they appear dramatically different (by around 30% for most people). The line with outward-pointing fins seems longer than the one with inward-pointing fins, even though they are the same. Many variations of the original Müller-Lyer illusion exist as I show in Figs. 6.8–6.9.

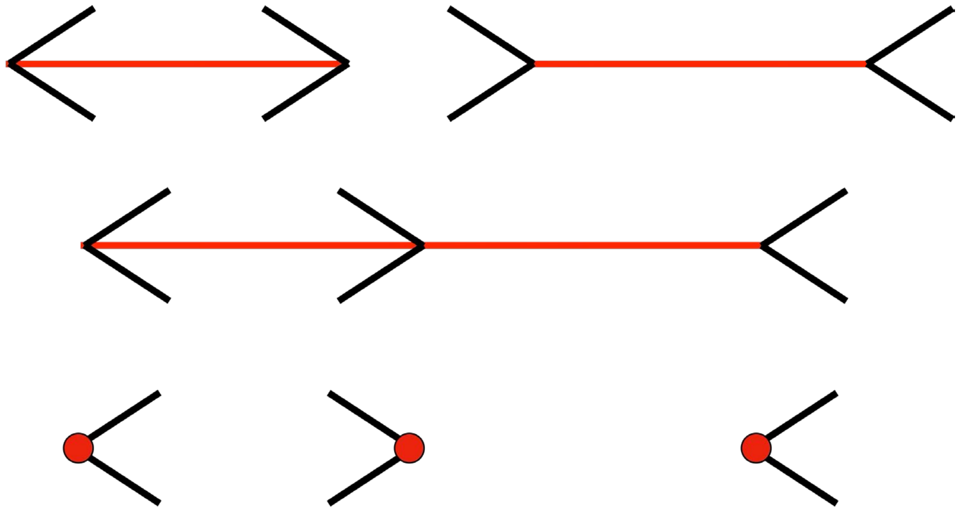

Figure 6.8: In the Müller-Lyer illusion, both horizontal lines have the same length, despite the fact that the one with outward-pointing fins seems longer than the other one. The original version is at the top, the middle one is a version due to Franz Bentrano, and the bottom one is further simplified by removing the lines. In this case, the distance between the two red dots on the left is the same as the distance between the two red dots on the right.

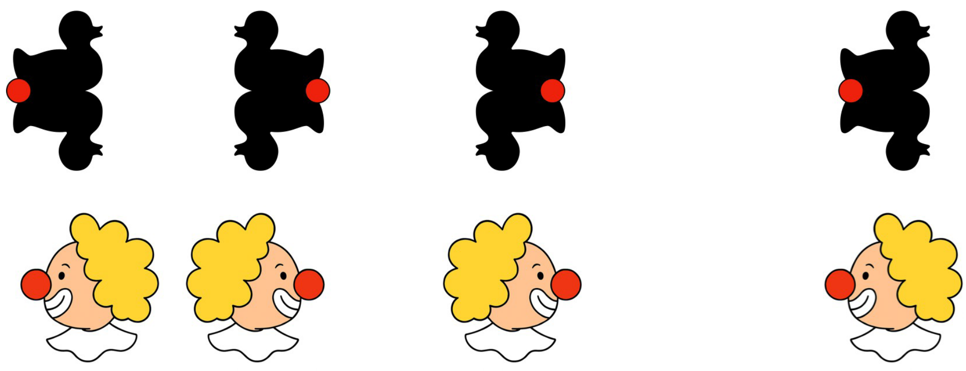

Figure 6.9: The Müller-Lyer illusion is so strong that the fins can be replaced by more complicated patterns and the red lines can be removed as I have done here. I called these the *Müller-Lyer-duck illusion* and the *Müller-Lyer-clown illusion* for obvious reasons. In both cases the distance between the two red dots on the left is the same as the distance between the two red dots on the right.

You can play the 3D version of the Müller-Lyer game. Cut two sheet of papers halfway vertically and keep three of them, Now fold each of them in half to make fins as shown in Figure. 6.10. Draw a big dot at the top of the fold. Now place two of them in the same orientation at each side of the table (say a metre apart). The game is now to place the third one as in the picture, so that the middle dot is exactly in between the two other dots (no cheating, just looking straight at them). Then, use a piece of string to compare the distance between left-middle dots and middle-right dots.

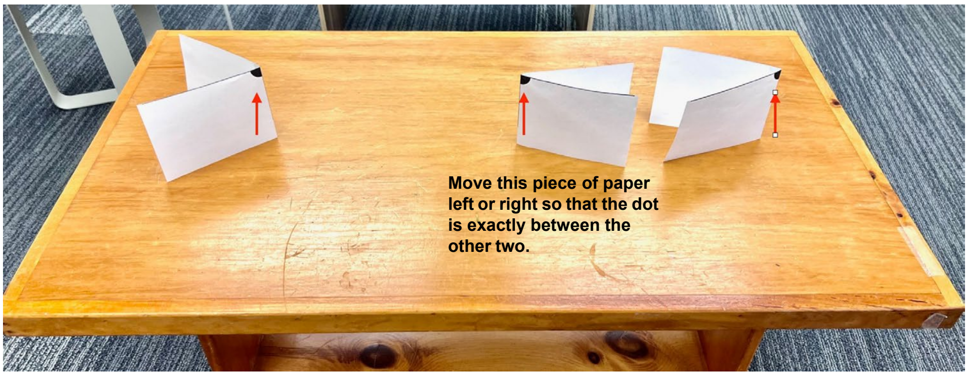

Figure 6.10: The Müller-Lyer game. With three pieces of paper as shown, try to place the middle one so that the middle dot is exactly in between the other two dots (indicated by red arrows). Then check the distance.

6.1.3 Illusory contours

The Kanizsa illusion is a powerful example of how our brains fill in missing information to create shapes that aren't actually drawn. In the classic version shown in Figure 6.11, black "pac-man" shapes and angled lines are arranged so that we perceive a bright white triangle floating on top despite the fact that no such shape is outlined. This illusion was introduced in 1955 by Italian psychologist Gaetano Kanizsa, who used it to show how the mind actively constructs reality by organizing visual information into familiar patterns [7, 8]. The original illusion of this type is credited to Friedrich Schumann, a German psychologist who described it in 1900 [19]. In his version, visual elements were arranged in such a way that the viewer perceived shapes or contours that were not explicitly drawn, an early demonstration of what are now called *illusory contours*.

Before we start with actual mathematics, let's play a last and delicious game, the Kanizsa game. For this game, you will need a kiwi fruit. Slice it to obtain three rings more or less the same size. Now, make a pac-man of each slice by taking a sector of about 60° and place each piece on a white plate so that their centres are more or less at the vertices of a equilateral triangle. Rotate each pac-man until you see your own Kanizsa triangle appear. You can move them around until the triangle disappear, or change the angles to bend the sides of the triangles. Make sure to eat your kiwi when you are done. No wasting food and it's full of vitamin C.

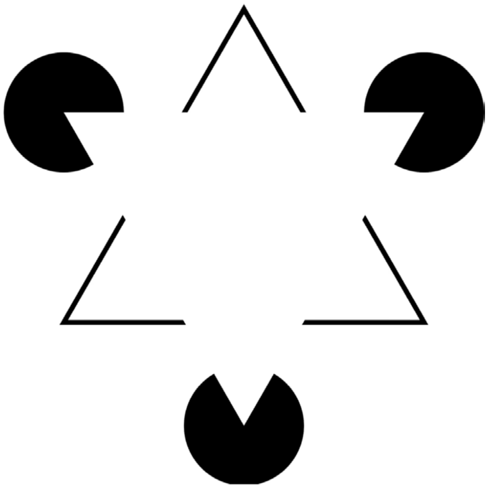

Figure 6.11: See the triangle? It’s not really there. The Kanizsa illusion shows that your brain fills in the gaps, creating edges and shapes that do not exist.

Figure 6.12: The Kanizsa game is self-explanatory. This time you are allowed to play with your food by slicing kiwi fruits and placing the slices on your plate until a triangle appears, even if it is not really there. The best experiments are the ones that you can eat!

6.2 A general framework

Our goal now is to build a mathematical theory for such illusions. What do I mean by that? Take for instance, the Hering illusion, the original ***object*** is a straight line that is bent when we add ***modifiers***, smaller lines on top of it. Hence, the line is transformed into a curve. Can I predict the shape of the curve that I see? I want to do this in a manner that is sufficiently general that it can be applied not only to Hering's illusion, but any variations on the theme where a well recognisable curve is modified by other features. For instance, if in the Hering illusion, I change the orientation of the modifiers, I obtain the *Wundt illusion* of Figure 6.13[23]. If I can predict the shape I see with the Hering illusion, can I predict the one I see in the Wundt illusion, or any such variations for that matter?

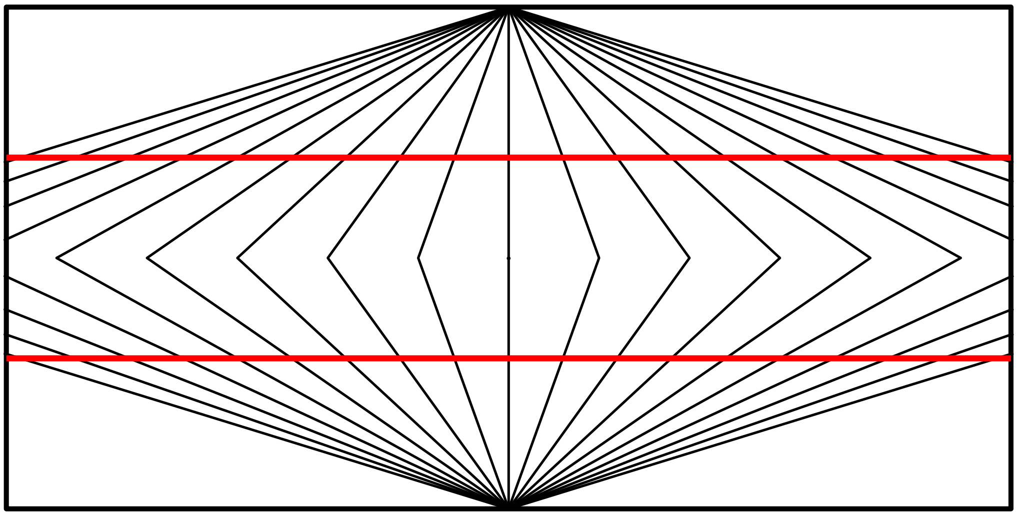

Figure 6.13: The Wundt illusion is a variation of the Hering illusion where the modifiers are inverted creating the perception of inward bowing red lines.

To build such a theory, we will follow the footsteps of Wassily Kandinsky [6] in his attempt to build a 'science of art' in *Point and Line to Plane*, in which he states *"The ideal of all research is: (i) the precise investigation of each individual phenomenon—in isolation, (ii) the reciprocal effect of phenomena upon each other—in combinations."*

We will start with the basic Gestalt assumption that the brain perceives some objects rather than pixels. We will start with points, lines, and curves, and see how they are modified by the presence of extra features. In trying to understand such perceptual effect Kandinsky follow the general Gestalt philosophy that there is some kind of "forces" acting on these objects. As Kandinsky expresses: "*The original source or every line remains the same: the force*" [6]. The same view is echoed by his friend Paul Klee at the Bauhaus school: *"The universal cause is reciprocal tension, a pull in two directions at once"* [10]. Or, in the words of Kurt Kofka, one of the founding figures of Gestalt psychology: "*"Force" has a definite meaning in the physical world, but what can it mean in a behavioural environment?*" [11]. Therefore, for modelling purpose, we can use our understanding of mechanical forces and build models that integrate particular local effects that alter the shape of objects. It is important to stress that this is an *analogy*. There is no actual physical forces acting in your brain or in the image that you see. But for mathematical purpose, we can use mechanical theories to built a theory of deformation triggered by the interaction between an object and the modifiers, just like a real object would be modified by some forces. But, as we will soon discover, forces are not quite enough to explain illusions.

6.3 Illusory contours

We start with illusory contours. If I look at the drawing at the top of Figure 6.14, what I see is the continuous curve at the bottom that is interrupted by white spaces, In fact, the top drawing is just a set of 11 curves. In Gestalt-speak, this is the law of good continuation, where we naturally perceive visual elements as connected or flowing together in a smooth path, even if they are not physically linked. How is it actually accomplished? According to David Mumford, a simple model for that response is that the brain connect smoothly such pieces in such a way as to minimise the total integrated curvature of the curve [14]. If the curvature of a curve connecting two pieces is too high, the brain does not register it as a connection. This model can be justified by the special type of neural networks that is used by our visual system. Indeed, simple cells in the primary visual cortex (V1) respond best to edges or bars of light at specific orientations within a particular part of the visual field. These cells act like tiny line detectors, becoming activated when a stimulus matches their preferred angle and position. This discovery was made by David Hubel and Torsten Wiesel in the 1960s, earning them a Nobel Prize for revealing how the brain begins to process visual information at the cortical level [5]. In the brain, neurons that encode proximity and close by directions tend to fire together, reinforcing the signal and giving us the perception of a connection.

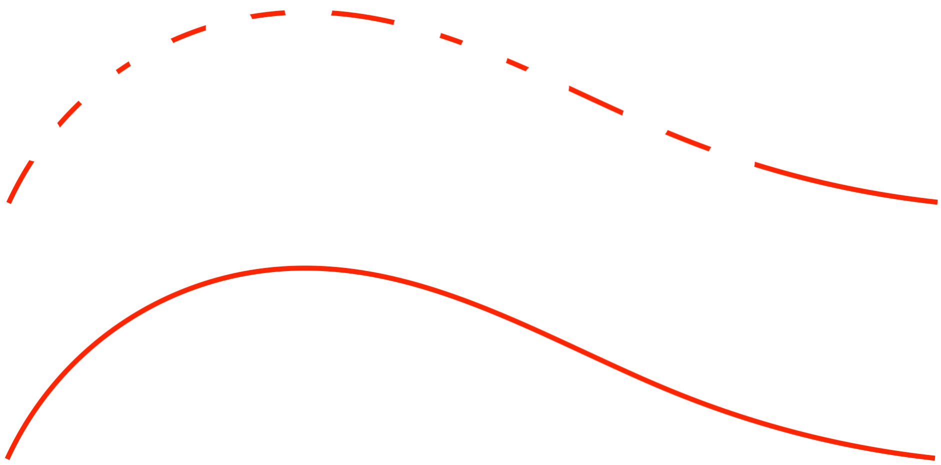

Figure 6.14: In the top drawing, 11 individual curves are shown but registered by the brain as the continuous curve below interrupted by white space.

Mathematically, a smooth curve in the plane can be characterized by the angle $\theta(s)$ between the tangent and a given direction (choose the horizontal) as a function of the arclength s as shown in Figure 6.15. The curvature (literally, a measure of how curved the curve is) is then simply the first derivative $\theta'(s)$. One can show that the curve that minimises the curvature is a solution of the following second-order ordinary differential equation:

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}s^2} + a\sin\theta = 0. \tag{6.1}$$

The parameter a in this equation plays the role of a tension (obtained by pulling both sides of the curve). It is necessary to explain the Zöllner illusion but not needed for the simpler illusions that we will study. So, for the purpose of this discussion, it will be ignored. This leaves us with the much simpler equation

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}s^2} = 0. \tag{6.2}$$

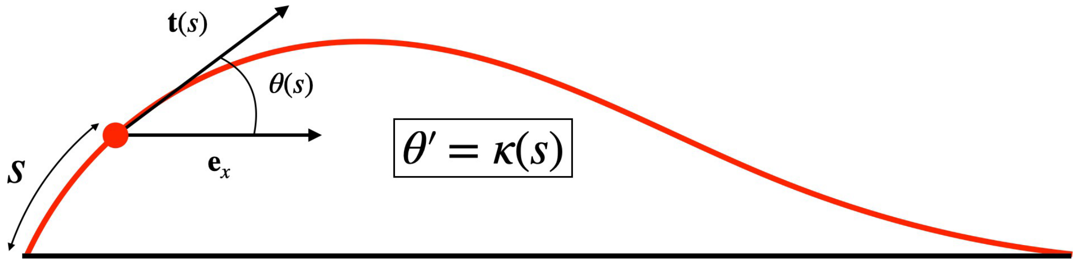

Figure 6.15: A smooth curve in the plane can be described by a function $\theta(s)$ where θ is the angle between the curve's tangent $\mathbf{t}(s)$ and the horizontal direction $\mathbf{e}_x$, and where s is the arclength (the length from the origin to a point following the curve).

In particular, note that if θ is constant, it is a solution which corresponds to a straight segment. This is exactly the case of the regular Kanizsa triangle. Given the two tangents at the end of any two pac-men, an edge of a simple triangle is the curve that minimises (piecewise) the curvature.

What if we close the mouth of the pac-men a bit as shown in Fig 6.16? In that case, θ constant does not match both tangent conditions at the end of the pac-men. However, the general solution is $\theta = C_1 s + C_0$ (with arbitrary constants C_1 and C_0 set by the tangent angles at the boundaries), which creates bent curves (in mathematics these curves are called *Hermite splines*).

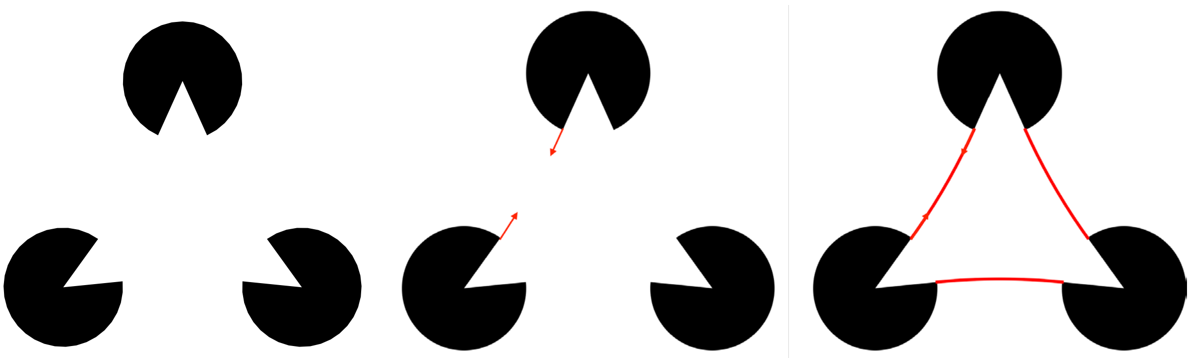

Figure 6.16: For a smaller opening of the pac-men, the triangle appears to have bent edges. In this case a triangle is not compatible with the constant solution of the elastica equation with the two tangents given by the red arrows. However, the general solution can easily accommodate this condition, in which case the angle is linear in the arc length and the predicted illusory contour is a triangle-like shape with bent edges.

6.4 Bending illusions

Now that we understand how smooth or straight curves are perceived as minimisers of the total curvature, we can model the interaction between such simple objects and small modifiers crossing the curves as in the Hering and Wundt illusions. Notice that when we reverse the orientation of these modifiers, the effect is reserved (and nullified when we had both of them as shown in Figure 6.17). Hence, we are led to conclude that it is the angle between modifiers and object that is important.

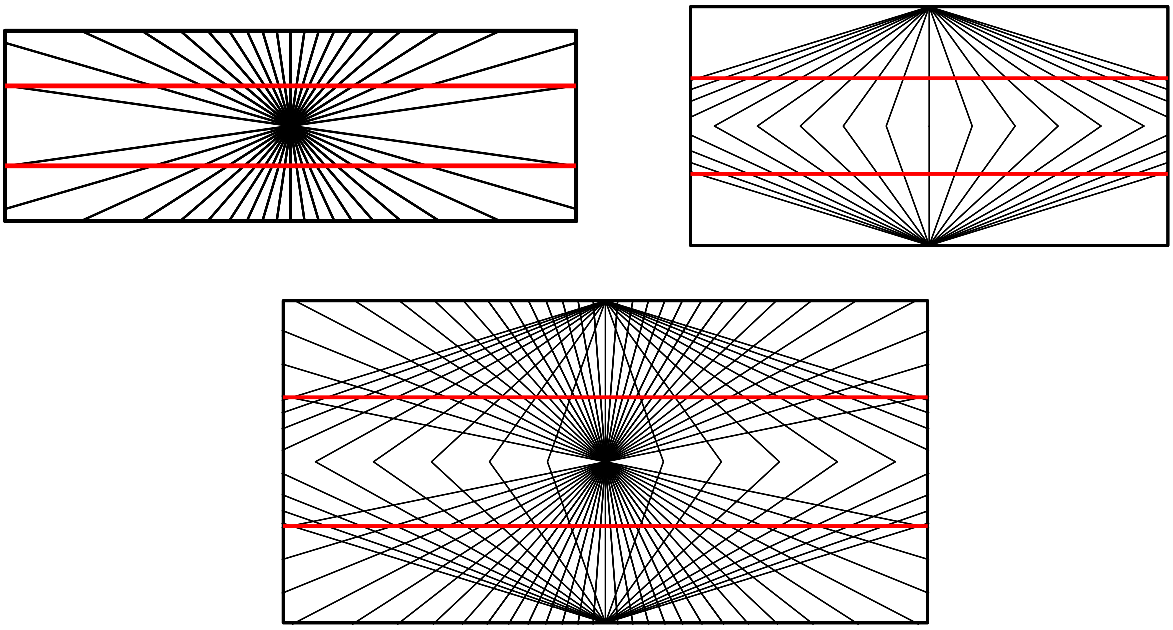

Figure 6.17: The addition of the Hering and Wundt illusions nullify the illusion, suggesting that the effect of the modifiers is linear and depends directly on the angle of intersection.

It turns out that we are good at spotting both vertical or horizontal lines, as when we try to orient a painting on the wall. However, we make systematic mistakes for any other angle. This misjudgement of angles is called *Brentano's Law*, the visual tendency to overestimate acute angles (as larger than they actually are), and underestimate obtuse angles (as smaller than they are in reality), as shown in Figure 6.18.

In terms of effective forces, the net effect of this error can be thought as a local effective torque, the *Brentano torque*, acting on a line. But we know from our study of illusory contours that our perception of the line is that it tends to minimise its curvature. Therefore, we have two opposite effects, the minimisation of the curvature that makes us perceived a line as straight and the Brentano torque that tends to bend it locally. The resolution between these two effects is a slightly bent line.

To test this explanation, we need to go one step further and see if we can actually create such a deformation. This is where we go from a description of a mechanism to an actual predictive model. Mathematically, the balance between the two effects is simply expressed by

$$\theta''(s) + \epsilon(s) = 0, \tag{6.3}$$

where $\epsilon = \epsilon(s)$ is *Brentano's torque*. It is a function of the arc length since at different position on the curve, the angle $\alpha(s)$ between the modifiers and the object changes in the Hering and Wundt illusion.

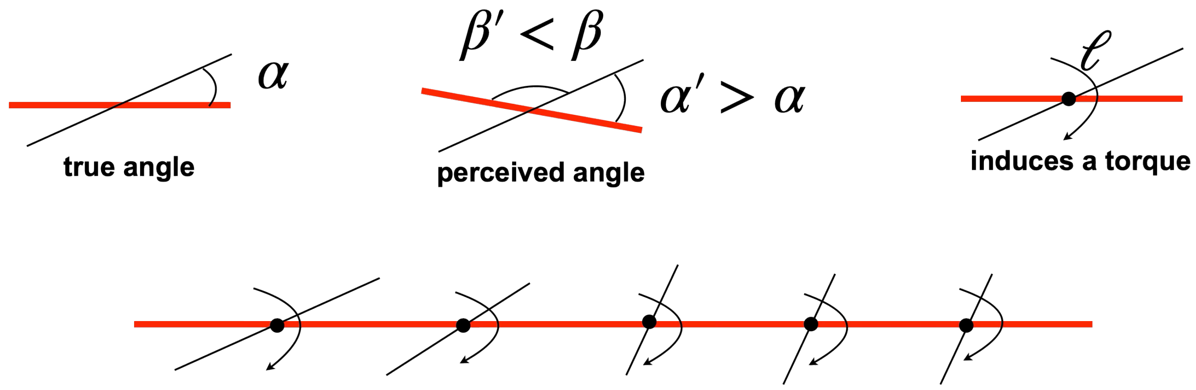

Figure 6.18: Brentano's law and its effect on vision. When we see an acute angle, we tend to overestimate it as shown above. The local effect of such an interaction can be seen as a local torque acting on a line element (top right). The effect of multiple such local torques is to bend the line.

How do we choose $.e(s)$? The simplest way is to do experiments where observers are asked to place virtually a stick at an angle that matches a given angle seen in a different part of the screen as shown in Figure. 6.19. By varying the angle, one can obtain the systematic errors made by the observers. This is exactly what the team of Dale Purves from Duke University did starting in the late 90's [15, 4]. We can now use their experimental data and fit it with a simple trigonometric function as shown in the same figure. The details of the curves are not important apart from the fact that it must go through zero at 0° and 90° (horizontal and vertical) and has a maximum at 30° where the error is maximal. Since we do not know the *magnitude* of such an effect, we multiply this function by an arbitrary constant b that will need to be adjusted for each person. The net result is the Brentano torque:

$$.e(s) = b\,(6 \sin 2a + \sin 4a)\,. \tag{6.4}$$

We can now test our idea on the Hering illusion for which the angles between the object and any modifier positioned at distance s from the middle of the line is $a = \arctan(h/s)$. Hence the equation to solve is

$$\theta^{tt}(s) + b\,(6 \sin 2a + \sin 4a) = 0, \quad a = \arctan(h/s). \tag{6.5}$$

It is easy to solve this equation numerically and by varying the parameter b, I obtain bent curves that look at first sight very much like the ones I see in the Hering illusion. But, how can I be sure, and how do we choose the last remaining parameter b? Clearly it will depend on the observer, but we cannot measure it directly since the deformation only occurs in our brain. A neat trick borrowed from Rubens and Munch is to look at curves with different values of b but with the modifiers, and vary b until the object looks straight. We will have then corrected the illusion will be looking at a bent curve that looks straight. Lacking a pool of subject and the expertise to conduct accurate psychological experiments, I resorted to test it on myself and found that a value of $b \approx 0.1$ works well for me in the conditions I was using (note that the value does not only depend on the subject but also the multiple details of the experiment, such as line colour, number of line, line thickness, brightness, and so on).

Now that I have the parameter b, I can repeat the same procedure for different illusions of the same type (leading to different functions $a(s)$) or by generalising the same process for other curves such as a circle. I show in Figure 6.21 the result of these corrections. Even knowing that these curves are not straight or circular, I cannot see them bent.

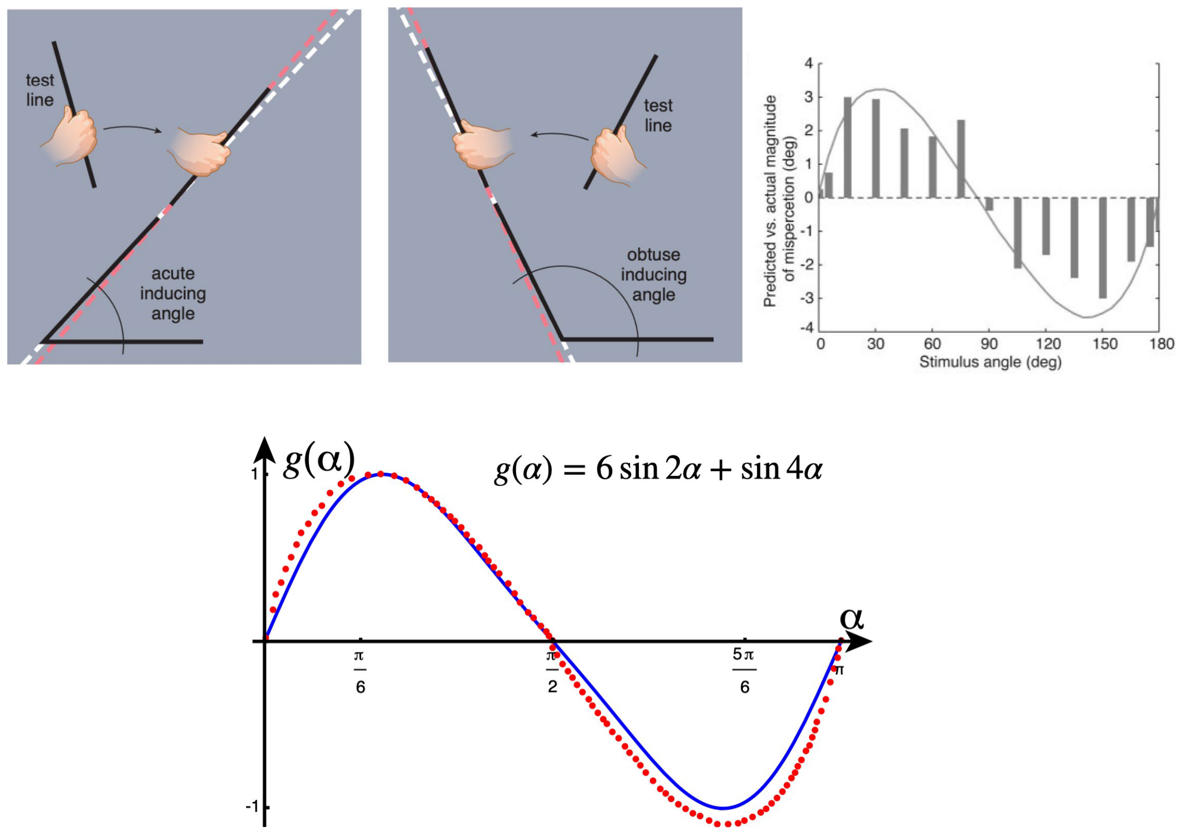

Figure 6.19: Simple experiments show that we make errors in our estimation of angles and systematically overestimate acute angles. Purves and his team tested systematically such errors by asking subjects to place a stick at a given angle. The net result is the graph shown on the top right with a max at around 30° showing an error of about 3°. I use the data fit from this experiment (red dots) to obtain a simple fitting of the normalised torque $g(s)$ in terms of trigonometric function (shown in blue).

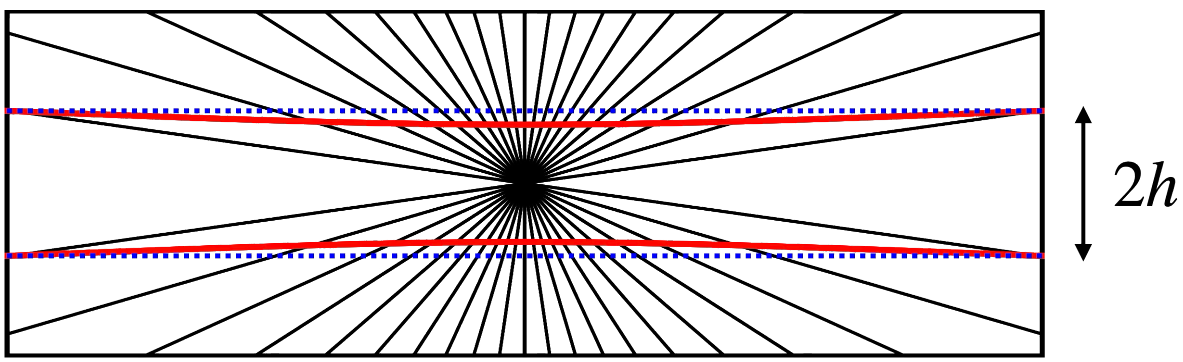

Figure 6.20: The corrected Hering illusion. The red line is bent downward but appears straight to me. The blue dotted line is the original straight line.

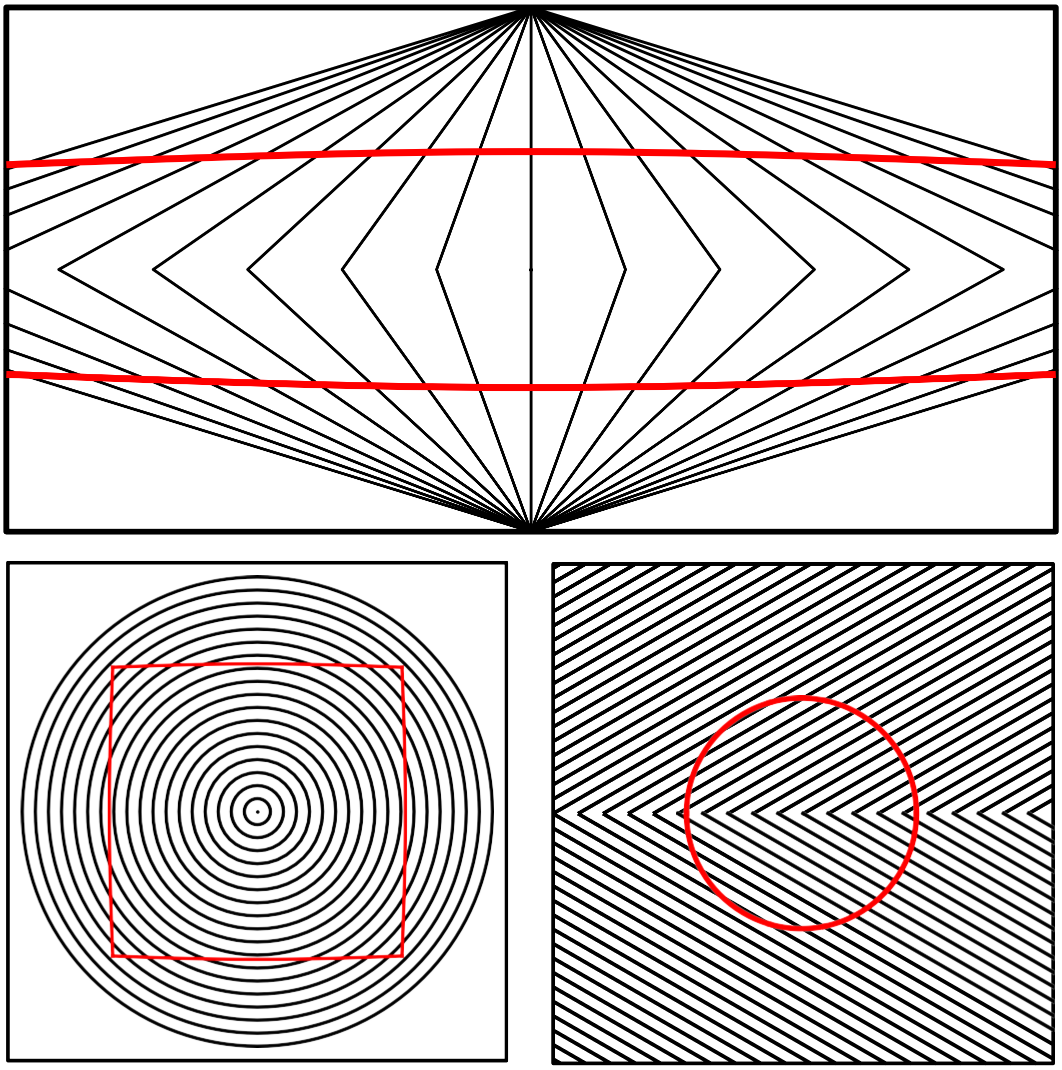

Figure 6.21: The corrected Wundt and Orbinson illusion. Here, I have corrected the lines, square, and circle (the last two are the so-called *Orbinson illusions* [17]) so that they look again straight, squarish, and circular, by using the same value of $b \approx 0.1$.

6.5 Size illusions

We are now interested in understanding how additional features like the wings of the Müller-Lyer illusion, seen in Figure 6.8, change our perception of relative size. To simplify the problem, we start with a simpler configuration of one line and two dots shown in Figure 6.22 and ask the simple question: how is the ***object*** in red affected by the presence of the ***two dots***? The main idea is that the effect of the dots on our perception is to pull the object towards the centre of mass (indicated by an x). Indeed, we know from psychology experiments that when subjects

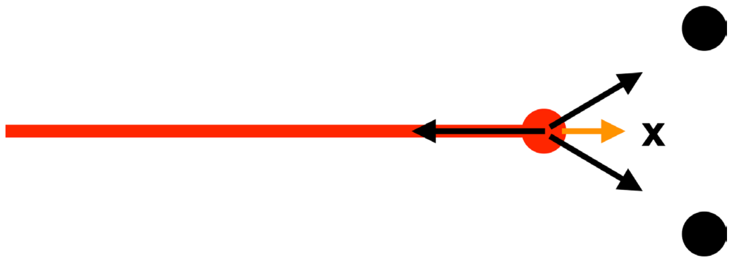

Figure 6.22: The way we perceived the interaction of the two dots with the red object is that it is pulled towards the centre of mass of the two dots, but resists this motion due to its own extension.

are asked to align a red dot in the middle of a cloud of dots, they tend to make an error towards the centre of mass.

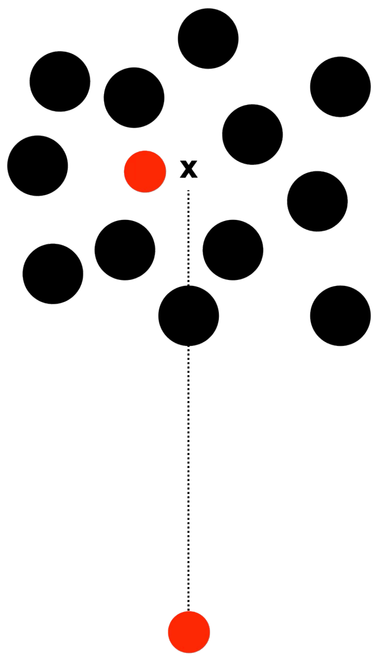

Figure 6.23: When people are asked to move horizontally a red dot (below) so that it aligns with the red dot in a cloud of points, they have a tendency to move it closer to the centre of mass of the cloud as if the black points were pulling the red ones.

Let me briefly remind you what the centre of mass is: Imagine you have several objects (say, balls) placed on a weightless plank. Assume each ball has a mass $m_1, m_2, \ldots, m_n$, and each is sitting at a certain position along the line, $x_1, x_2, \ldots x_n$ measured from any origin point of your choosing on the same splank. The centre of mass, x^*, measured from the same origin, is the point on the line where you would place a fulcrum the plank so that the system is perfectly balanced. With two objects, it is exactly a see-saw with unequal length to balance the mass: heavier objects pull more strongly, so the centre of mass will be closer to the heavier ones. A simple formula for the centre of mass is:

$$x^* = \frac{m_1 x_1 + m_2 x_2 + \cdots + m_n x_n}{m_1 + m_2 + \cdots + m_n} = \frac{M_1}{M_0}, \tag{6.6}$$

where M_1 is the *first moment of area* (the sum of masses times distances) and M_0 is the *total mass* of the system.

This idea can be made much more precise by looking at how cells interact with each other in our visual system. In particular the signal from the retina goes through the lateral geniculate nucleus and it is relayed by surround-centre cells that have the property of lateral inhibition, enhancing the signal at a particular location but inhibiting signals just in their surroundings. This property combined with the fact that the effect of a point away from our focal attention decreases with the distance is enough to explain the effective force pulling away at the red dot. I then wrote a model for the effective force pulling it. To a good approximation, one can show that the error is

$$\text{error} \approx \frac{M_1^d(\text{modifier})}{M_0^d(\text{object}) + M_0^d(\text{modifier})}, \tag{6.7}$$

where the mass and first moment are weighted with the distance (that is, the effective mass decreases away from the focal point–hence, masses that are very far away have no effect). This model has a few hidden parameters that can be adjusted from one subject to the next. For instance, consider the data from actual experiments by Bulatov and his group [1] where subject reported the size of a Müller-Lyer illusion with dots where the dots are moved away at a distance w at a given angle as shown in Figure 6.24. We observe that the error increases at first as the centre of mass moves away from the ends of the line, hence pulling it more. However, since the amplitude of this effect depends on the distance to the centre of mass, it starts decreasing until the dots are sufficiently far away that they don't disturb the object. Data for two different subjects show the subjective aspect of the effect but also that it can be accommodated by finding the relevant parameters for each subject.

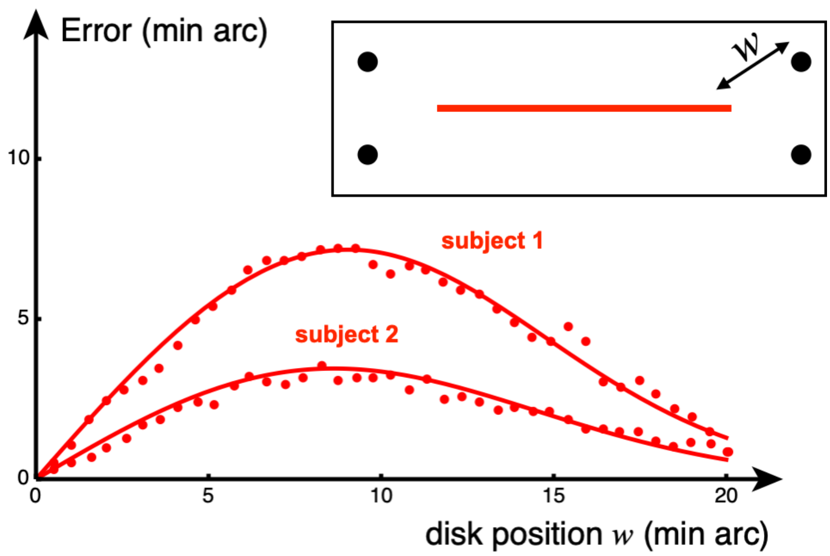

Figure 6.24: The error observed (dots) and fitted by the error formula (solid curve) in an experiment about the size of the red line where the dots are gradually moved apart.

Going back to the original illusion, we can compute the effect of an increasing wing size. We see in Figure 6.25, that in this case the error reaches a plateau due to the fact that increasing wings away from the focal point have diminishing effect on the object, as expected.

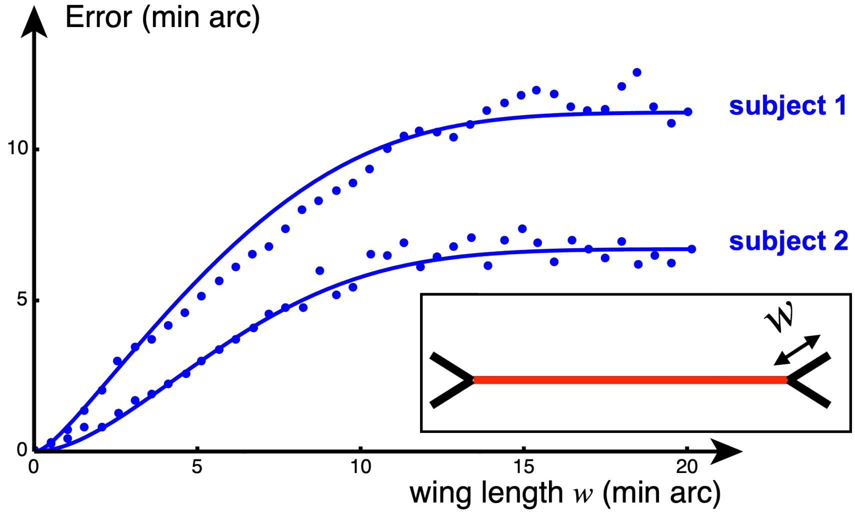

Figure 6.25: The error observed (dots) and fitted by the error formula (solid curve) in an experiment about the size of the red line where the wing size is gradually increased.

6.6 Conclusions

The geometric illusions that we studied are the simplest known illusions. They only involve one type of distortion and can be connected directly to particular processing modality of the visual system. This is what allowed me to develop simple models that capture the important features of the illusion at a local level. Once this mechanism is understood, these models can be applied to a multitude of similar illusions. Indeed I have shown that dozens of named illusions can be captured by the models we discussed.

Yet, it is important to remember that we have only scratched the surface. The world of illusions is huge [20] and we have not discussed illusions of contrast, colours, or motion, among many others. I believe that the same framework can be applied to these illusions but it would require more work to pin down the principal mechanisms responsible for such illusions and fully develop mathematical models to test these mechanisms. It is also important to realise that there are illusions that have not been properly explained, even at a qualitative level. For instance, the sun/moon illusion, where the sun or the moon at the horizon appears much bigger than when they are high in the sky, does not have a satisfactory explanation.

Illusions are delightful tricks of the mind. They compel us to face an important truth: our visual perception is deeply subjective. The world we see is not the same as the physical world that can be measured, it is a reconstruction, shaped by our brain from limited and often ambiguous information.

And yet, the visual system performs astonishingly well. It processes vast amounts of data quickly and efficiently, allowing us to navigate the world with ease. But it is at the limits of this process, where information is sparse or ambiguous, that illusions emerge. In this sense, illusions are not failures; they are by-products of the brain's best guess about what is out there, taking the statistics of the world into account.

In fact, a world perceived with perfect accuracy would be rather dull. If our perception were strictly veridical, we might lose our capacity to enjoy much of what makes art, imagination, and storytelling so powerful. A simple line drawing, for example, would be nothing more than ink on paper—flat, meaningless. After all, lines do not truly exist in the real world; what we think of as lines are merely contours where light intensities shift. Without illusions, the world would be as described by Jean-Paul Sartre in *La Nausée*: "*"Les choses sont uniquement ce qu'elles paraissent être ; derrière elles. . . il n'y a rien."*[2] and we would suffer from endless existentialist crises.

It is precisely our susceptibility to being fooled that makes art possible. Illusions do not trick us, they invite us to dream.

6.7 Further Reading

There are many books on illusions, from the simple show-and-tell that are often rather dull and repetitive, from the sophisticated neuroscience and art analyses.

- In *Trick Eyes: Magical Illusions That Will Activate the Brain* (2005), Akiyoshi Kitaoka, the grand master of illusions, presents over 100 of his own optical illusions, including his renowned "Rotating Snakes" and other motion illusions. It serves both as a visual spectacle and a scientific exploration into how our brains interpret visual information [9].
- In *Mind Sights: Original Visual Illusions, Ambiguities, and Other Anomalies*(1990), Roger Shepard explores the nature of mental imagery, perception, and meaning through a blend

[2] *"Things are entirely what they appear to be and behind them...there is nothing.*"

of visual illusions, scientific reflection, and philosophical insight. The book is richly illustrated and presents Shepard's personal and professional thoughts and drawings on how the mind creates and transforms visual experience [21].

- *Perceiving Geometry: Geometrical Illusions Explained by Natural Scene Statistics* (2005), by Catherine Q. Howe and Dale Purves. This book explores how the human visual system interprets geometric illusions by analyzing the statistical relationships between retinal images and their real-world sources. It provides an empirical framework for understanding why our perceptions often deviate from physical measurements. [4]
- *Vision and Art: The Biology of Seeing* (2002), by Harvard neurobiologist Margaret Livingstone is a beautifully illustrated book that explores how our visual system interprets art, revealing how artists have historically manipulated colour, light, and form to evoke emotional responses and convey meaning [12].
- *Splendors and Miseries of the Brain: Love, Creativity, and the Quest for Human Happiness*(2008) by the famous neurobiologist Semir Zeki explores how the brain's architecture shapes our experiences of love, creativity, and the pursuit of happiness. By examining art, literature, and neuroscience, Zeki offers a personal view into how our neural structures influence our perceptions and emotional lives [24].

References and Further Reading

[1] Aleksandr Bulatov, Algis Bertulis, Natalja Bulatova, and Yelena Loginovich. Centroid extraction and illusions of extent with different contextual flanks. *Acta Neurobiologiae Experimentalis*, 69(4):504–525, 2009.

[2] Hermann von Helmholtz, Edmund Atkinson, et al. Popular lectures on scientific subjects. second series. 1881.

[3] Ewald Hering. *Beiträge zur Physiologie. I. Zur Lehre vom Ortssinne der Netzhaut*. Engelmann, Leipzig, 1861.

[4] Catherine Q. Howe and Dale Purves. *Perceiving Geometry: Geometrical Illusions Explained by Natural Scene Statistics*. Springer Science & Business Media, New York, NY, 2005.

[5] David H. Hubel and Torsten N. Wiesel. Receptive fields of single neurones in the cat's striate cortex. *The Journal of Physiology*, 148(3):574–591, 1959.

[6] Wassily Kandinsky. *Point and Line to Plane*. Solomon R. Guggenheim Foundation, New York, 1947. Translated by Howard Dearstyne and Hilla Rebay. Originally published in 1926.

[7] Gaetano Kanizsa. Margini quasi-percettivi in campi con stimulazione omogenea. *Riv. Psicol.*, 49:7–30, 1955.

[8] Gaetano Kanizsa. Quasi-perceptual margins in homogeneously stimulated fields. In S. Petry and G. E. Meyer, editors, *The Perception of Illusory Contours*, pages 40–49. Springer, New York, 1987. English translation of Kanizsa (1955).

[9] Akiyoshi Kitaoka. *Trick Eyes: Magical Illusions That Will Activate the Brain*. Barnes & Noble, New York, 2005.

[10] Paul Klee. *The Thinking Eye: The Notebooks of Paul Klee, Volume I*. Lund Humphries, London, 1961.

[11] Kurt Koffka. *Principles of Gestalt Psychology*. Harcourt, Brace and Company, New York, 1935.

[12] Margaret S. Livingstone. *Vision and Art: The Biology of Seeing*. Harry N. Abrams, New York, 2002.

[13] Franz Carl Müller-Lyer. Optische urteilstäuschungen. *Archiv für Anatomie und Physiologie, Physiologische Abteilung*, Supplement:263–270, 1889.

[14] David Mumford. Elastica and computer vision. *Algebraic Geometry and Its Applications*, pages 491–506, 1992.

[15] Surajit Nundy, Beau Lotto, David Coppola, Amita Shimpi, and Dale Purves. Why are angles misperceived? *Proceedings of the National Academy of Sciences*, 97(10):5592–5597, 2000.

[16] Johann Joseph Oppel. Uber geometrisch-optische tauschungen. *Jahresbericht des physikalischen Vereins zu Frankfurt am Main*, (37–47), 1855.

[17] William D. Orbison. Shape as a function of the vector field. *The American Journal of Psychology*, 52(1):31–45, 1939.

[18] Johann Christian Poggendorff. Zusatz des herausgebers. *Annalen der Physik und Chemie*, 186(7):500–525, 1860. Published as an editorial note in response to Zöllner's submission.

[19] Friedrich Schumann. Beiträge zur analyse der gesichtswahrnehmungen. erste abhandlung: Einige beobachtungen über die zusammenfassung von gesichtseindrücken zu einheiten. *Zeitschrift für Psychologie und Physiologie der Sinnesorgane*, 23:1–32, 1900.

[20] Arthur G. Shapiro and Dejan Todorović, editors. *The Oxford Compendium of Visual Illusions*. Oxford University Press, New York, 2017.

[21] Roger N Shepard. *Mind Sights: Original visual Illusions, Ambiguities, and other Anomalies, with a Commentary on the play of mind in Perception and art.* WH Freeman/Times Books/Henry Holt & Co, 1990.

[22] David R. Topper. The poggendorff illusion in descent from the cross by rubens. *Perception*, 13(6):655–658, 1984.

[23] Wilhelm Wundt. Die geometrisch-optischen täuschungen. *Abhandlungen der Mathematisch-Physischen Classe der Königlich Sächsischen Gesellschaft der Wissenschaften*, 24:55–178, 1898.

[24] Semir Zeki. *Splendors and Miseries of the Brain: Love, Creativity, and the Quest for Human Happiness*. Wiley-Blackwell, Chichester, UK, 2008.

[25] Johann Karl Friedrich Zöllner. Ueber eine neue art von pseudoskopie und ihre beziehungen zu den von plateau und oppel beschriebenen bewegungsphänomenen. *Annalen der Physik*, 186(7):500–525, 1860.